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ON RELATIONS AMONG VARIOUS CLASSES OF $\mathcal{I}$ A. E. CONTINUOUS DARBOUX FUNCTIONS

Abstract

This paper is devoted to relationships among various classes of $\mathcal{I}$-a.e. continuous functions (i.e., of functions whose sets of discontinuity points belong to certain σ-ideals $\mathcal{I}$ consisting of boundary sets). For instance, if $\mathcal{K}$ is the σ-ideal of first category sets and $\mathcal{I}$ denotes the σ-ideal of all sets that are: of Lebesque measure zero, σ-porous, or countable, then the set of $\mathcal{I}$-a.e. continuous functions is uniformly porous in the space of all $\mathcal{K}$-a.e. continuous Darboux functions from $\mathbb{R}^2$ into $\mathbb{R}^2$ equipped with the metric of uniform convergence. As a tool in the proofs, symmetric Cantor sets in $\mathbb{R}^2$ are used.

1 Introduction

The classes of functions connected with various σ-ideals are the subject of many papers. They were investigated, for example, by Pawlak [4], Semadeni [6], Mauldin [2, 3], and Ciesielski et al. in the monograph [1]. The present paper is devoted to the classes of functions whose sets of discontinuity points belong to certain σ-ideals consisting of boundary sets only. Such functions will be called continuous $\mathcal{I}$-almost everywhere ($\mathcal{I}$-a.e.) with respect to a specified σ-ideal. More precisely, in this paper we shall investigate mutual relations between classes of functions which are continuous $\mathcal{I}$-a.e. with respect to various σ-ideals. From [4, theorem 1.4] it follows that, in some spaces, every $\mathcal{I}$-a.e. continuous function is continuous $\mathcal{K}$-a.e. with respect to the σ-ideal $\mathcal{K}$ of first category subsets. It appears, for instance, that the set of functions continuous $\mathcal{L}$-a.e. (which are conventionally called functions continuous a.e.) with respect to the σ-ideal $\mathcal{L}$ of subsets of Lebesgue measure zero is topologically

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small in the space of continuous $\mathcal{K}$-a.e. functions, namely, they form a uniformly porous set. This property is proved in the space of Darboux functions mapping $\mathbb{R}^2$ into $\mathbb{R}^2$ and it is the main result of Section 3. To obtain it, we make use of certain sets called in this paper Cantor-like sets in $\mathbb{R}^2$ and so Section 2 is devoted to the description of their properties.

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N} = 1, 2, \ldots$, $\mathbf{I} = [0,1]$. In this paper we shall use the Euclidean metric d in the space $\mathbb{R}^2$ and the metric of uniform convergence ρ in the space of Darboux functions. In these spaces, we shall consider open balls $B(x,r)$ of radius r centered at x. By $C_f(D_f)$ we shall denote the set of continuity (discontinuity) points of a function f. The symbols $\operatorname{int}(A)$, $\operatorname{cl}(A)$, $\operatorname{diam}(A)$ stand for the interior, the closure, and the diameter of the set $A \subset \mathbb{R}^2$, respectively. If A is a Lebesgue measurable subset of $\mathbb{R}^2$, we denote its measure by $m(A)$. Let $x = (x^1, x^2) \in \mathbb{R}^2$, $c > 0$, and define $K(x,c) = (x^1 - c, x^1 + c) \times (x^2 - c, x^2 + c)$. The set $K(x,c)$ will be called a two-dimensional open cube with center $x = (x^1, x^2)$ and side $2c$, and its boundary will be denoted by $F(x,c)$. We shall use the symbol $xy^{\rightarrow}$ to denote a half-line starting at x and passing through y.

2 Symmetric Cantor Sets in $\mathbb{R}^2$

A sketch of the construction of a symmetric Cantor set in $\mathbb{R}$ can be found, for example, in [8]. It is similar to the construction of the Cantor ternary set.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers $\alpha_n \in (0,1)$ and let $\mathbf{J}$ be a closed interval whose length will be denoted by β_1. In the first step of the construction, we remove from $\mathbf{J}$ the concentric open interval (a_1^1, b_1^1) of length $\alpha_1\beta_1$. Let $F^1 = \mathbf{J} \setminus (a_1^1, b_1^1)$. In the n-th ($n > 1$) step of the construction, from the remaining 2^{n-1} closed intervals of length equal to

$$\beta_n = \frac{\beta_1}{2^{n-1}}(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1})$$

we remove the concentric open intervals (a_i^n, b_i^n), $i = 1, 2, \ldots, 2^{n-1}$, of length

$$\alpha_n \cdot \beta_n = \frac{\alpha_n\beta_1}{2^{n-1}}(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1}). \tag{1}$$

In this way we obtain 2^n closed intervals whose union is denoted by F^n. The set $C(\alpha_n) = \bigcap_{n=1}^{\infty} F^n$ is called the symmetric Cantor set with respect to the sequence $(\alpha_n)_{n\in\mathbb{N}}$. Of course, the set $C(\frac{1}{3})$ is the classical Cantor ternary set.

From the construction it is clear that the set $C(\alpha_n)$ is closed, nowhere dense, and uncountable. Other properties are connected with the sequence

$(\alpha_n)_{n\in\mathbb{N}}$. The following facts will be needed to investigate symmetric Cantor sets in $\mathbb{R}^2$.

Lemma 2.1. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers $\alpha_n \in (0,1)$. Then*

$$\sum_{n=1}^{\infty} \alpha_n = \infty \iff \lim_{n\to\infty} (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n) = 0.$$

Lemma 2.2. [7] *The set $C(\alpha_n)$ has the Lebesgue measure zero if and only if $\sum_{n=1}^{\infty} \alpha_n = \infty$.*

Lemma 2.3. [7, 8] *The set $C(\alpha_n)$ is non σ-porous if and only if $\lim_{n\to\infty} \alpha_n = 0$.*

We are now going to define the symmetric Cantor set $F(\alpha_n)$ in $\mathbb{R}^2$. Let $x_0 \in \mathbb{R}^2$ and $0 \in C(\alpha_n) \subset [0,\infty)$. Put $F(\alpha_n) = \bigcup_{c\in C(\alpha_n)} F(x_0,c)$. (We assume that $F(x_0,0) = x_0$.) Of course,

$$F(\alpha_n) = \operatorname{cl} K(x_0,\beta_1) \setminus \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} (K(x_0,b_i^n) \setminus \operatorname{cl} K(x_0,a_i^n)). \tag{2}$$

It appears that $F(\alpha_n)$ has similar properties as the set $C(\alpha_n)$.

Lemma 2.4. *The set $F(\alpha_n)$ is closed and nowhere dense in $\mathbb{R}^2$.*

Proof. From (2) it follows that $F(\alpha_n)$ is a closed set in $\mathbb{R}^2$. Let B be an open set in $\mathbb{R}^2$ and $x_1 \in B \setminus x_0$. Notice that the mapping $\nu : [0,\infty) \to x_0x_1^{\rightarrow}$ given by the formula $\nu(t) = x_0x_1^{\rightarrow} \cap F(x_0,t)$ is a homeomorphism such that $\nu(C(\alpha_n)) = x_0x_1^{\rightarrow} \cap F(\alpha_n)$. Therefore the set $x_0x_1^{\rightarrow} \cap F(\alpha_n)$ is nowhere dense in $x_0x_1^{\rightarrow}$ and so there exists $x \in (x_0x_1^{\rightarrow} \cap B) \setminus (x_0x_1^{\rightarrow} \cap F(\alpha_n)) \subset B \setminus F(\alpha_n)$. Finally, by the arbitrariness of $B \subset \mathbb{R}^2$, we obtain that $F(\alpha_n)$ is nowhere dense in $\mathbb{R}^2$. $\square$

Lemma 2.5. *The set $F(\alpha_n)$ has the Lebesgue measure zero if and only if $\sum_{n=1}^{\infty} \alpha_n = \infty$.*

Proof. Notice that, by the symmetric construction of the set $C(\alpha_n)$, we have $a_i^n + b_{2^{n-1}-i+1}^n = \beta_1$ for $n \in \mathbb{N}$ and $i = 1,2,\ldots,2^{n-1}$; hence

$$\sum_{i=1}^{2^{n-1}} (a_i^n + b_i^n) = \sum_{i=1}^{2^{n-1}} (a_i^n + b_{2^{n-1}-i+1}^n) = 2^{n-1}\cdot\beta_1. \tag{3}$$

For simplicity of notation of the proof we can put $\alpha_0 = 0$. Then, from (2), (1), (3) and Lemma 2.1 we have

$$
\begin{aligned}
m(F(\alpha_n)) = 0 &\Longleftrightarrow \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} \left((2b_i^n)^2 - (2a_i^n)^2\right) = 4 \cdot \beta_1^2 \\
&\Longleftrightarrow \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} \left(\left(\frac{\alpha_n \beta_1}{2^{n-1}} \cdot \prod_{j=0}^{n-1} (1-\alpha_j) \right) \cdot (a_i^n + b_i^n) \right) = \beta_1^2 \\
&\Longleftrightarrow \sum_{n=1}^{\infty} \left(\left(\frac{\alpha_n \beta_1}{2^{n-1}} \cdot \prod_{j=0}^{n-1} (1-\alpha_j) \right) \cdot (2^{n-1}\beta_1) \right) = \beta_1^2 \\
&\Longleftrightarrow \sum_{n=1}^{\infty} \alpha_n \cdot (1-\alpha_0) \cdot (1-\alpha_1) \cdots (1-\alpha_{n-1}) = 1 \\
&\Longleftrightarrow \lim_{n\to\infty} ((1-\alpha_1) \cdot (1-\alpha_2) \cdots (1-\alpha_n)) = 0 \\
&\Longleftrightarrow \sum_{n=1}^{\infty} \alpha_n = \infty. \qquad \square
\end{aligned}
$$

Lemma 2.6. *The set $F(\alpha_n)$ is non σ-porous in $\mathbb{R}^2$ if $\lim_{n\to\infty} \alpha_n = 0$.*

Proof. Let $x_0 = (x_0^1, x_0^2)$. From Lemma 2.3 it follows that the set $C(\alpha_n)$ is non σ-porous in $\mathbb{R}$. By [9, theorem 1] we obtain that the set $(x_0^1 + C(\alpha_n)) \times (x_0^2 + C(\alpha_n))$ is non σ-porous in $\mathbb{R}^2$. Since $(x_0^1+C(\alpha_n))\times(x_0^2+C(\alpha_n)) \subset F(\alpha_n)$, we conclude that the set $F(\alpha_n)$ is really non σ-porous in $\mathbb{R}^2$. $\square$

3 $\mathcal{I}$-Almost Everywhere Continuous Darboux Functions

We shall denote by $\mathcal{I}$ a σ-ideal of subsets of any fixed topological space, but we assume that nonempty open sets are excluded as the elements of the σ-ideal under consideration. In particular, we shall consider the following σ-ideals: $\mathcal{L}$ – the σ-ideal of subsets of Lebesgue measure zero, $\mathcal{K}$ – the σ-ideal of first category subsets, $\mathcal{N}$ – the σ-ideal of countable subsets, and $\mathcal{P}$ – the σ-ideal of σ-porous subsets.

Let X, Y be arbitrary topological spaces. We say that a function $f : X \to Y$ is continuous $\mathcal{I}$-almost everywhere ($\mathcal{I}$-a.e. for short) if the set of discontinuity points of the function f belongs to the σ-ideal $\mathcal{I}$.

In this section we shall need the definition of the uniform porosity of a set in a metric space (see [5]). Let P be a metric space, $S \subset P$, $x \in P$, $R > 0$, and $\gamma(x, R, S) = \sup\{r > 0 \colon \exists_{z\in P} B(z, r) \subset B(x, R) \setminus S\}$. The number $p(S, x) = 2 \cdot \limsup_{R\to 0^+} \frac{\gamma(x,R,S)}{R}$ is called the porosity of S at x (see [8]). We say that S is porous at x if $p(S,x) > 0$, and S is uniformly porous if there exists $\alpha \in (0,1]$ such that $p(S,x) \geq \alpha$ for each $x \in S$.

The following theorem is evident.

Theorem 3.1. *Let X and Y be arbitrary metric spaces. If a function $f : X \to Y$ is continuous $\mathcal{N}$-a.e. or $\mathcal{P}$-a.e. then it is continuous $\mathcal{K}$-a.e. (and $\mathcal{L}$-a.e. if $X = \mathbb{R}^n$).*

On the basis of [4, theorem 1.4] we obtain a more general case.

Theorem 3.2. *Let X be an arbitrary topological space and Y be a second countable space possessing a regular neighborhood system. If a function $f : X \to Y$ is continuous $\mathcal{I}$-a.e. with respect to some σ-ideal of subsets of the space X, then it is continuous $\mathcal{K}$-a.e.*

If $\mathcal{I}$ is the σ-ideal of countable, σ-porous, or Lebesgue measure zero sets, then the converse isn't true, because there exist functions continuous $\mathcal{K}$-a.e. which are not continuous $\mathcal{I}$-a.e. with respect to any of the above-mentioned σ-ideals. Moreover, these functions form the set which is not topologically small in the space of functions continuous $\mathcal{K}$-a.e. We have the following theorem.

Theorem 3.3. *Let KD be the space of Darboux $\mathcal{K}$-a.e. continuous functions $f\colon \mathbf{I}^2 \to \mathbf{I}^2$ endowed with the metric of uniform convergence. Then the following subsets of KD are uniformly porous: L – of functions continuous $\mathcal{L}$-a.e., P – of functions continuous $\mathcal{P}$-a.e., and N – of functions continuous $\mathcal{N}$-a.e.*

PROOF. We shall show that L is a uniformly porous set with a constant $\alpha \geq 0.2$. Let $\epsilon > 0$ and $h \in L \cap KD$ be given. Choose an arbitrary $x_0 \in C_h \cap \operatorname{int} \mathbf{I}^2$ and put $h_0 = h(x_0)$. Let δ be a positive real number such that $K(x_0,\delta) \subset \mathbf{I}^2$ and

$$h\left(\mathrm{cl}\ K(x_0,\delta)\right) \subset B\left(h_0, \frac{4\epsilon}{10}\right).$$

We shall construct a Darboux continuous $\mathcal{K}$-a.e. function $f : \mathbf{I}^2 \to \mathbf{I}^2$ satisfying the condition $B\left(f, \frac{\epsilon}{10}\right) \subset B(h,\epsilon)$ and show that the set of discontinuity points of an arbitrary function $f_1 \in B\left(f, \frac{\epsilon}{10}\right)$ has positive Lebesgue measure.

Let us consider the symmetric Cantor set $C = C(1/2^n)$ in the interval $\mathbf{J} = [0,\delta]$. Since $\sum_{n=1}^{\infty} 1/2^n = 1$, we conclude that the set $F = F(1/2^n)$ has positive Lebesgue measure by Lemma 2.5. Moreover, from Lemma 2.4 it follows that the set F is nowhere dense in $\mathbb{R}^2$. Therefore $F \in \mathcal{K} \setminus \mathcal{L}$.

Let us divide the set $C \setminus \{\delta\}$ into continuum subsets which are pairwise disjoint and dense in C. Denote by $\mathcal{U}$ the family of all those subsets. Without loss of generality we may assume that there exists one set $U_0 \in \mathcal{U}$ containing all one-sided accumulation points of the set C. Let $g : \mathcal{U} \to B(h_0, \epsilon/2)$ be a one-to-one function such that $g(U_0) = h_0$. Now, we can define the function $f : \mathbf{I}^2 \to \mathbf{I}^2$ as

$$f(x) = \begin{cases} h(x) & \text{when } x \notin K(x_0, \delta) \\ g(U_{c_x}) & \text{when } x \in F(x_0, c_x) \text{ and } c_x \in C \setminus \{\delta\} \\ h_0 & \text{when } x \in \bigcup_{c \in \mathbf{J} \setminus C} F(x_0, c) \end{cases}$$

where U_{c_x} is the set containing c_x and belonging to the family $\mathcal{U}$. Observe that the function f is continuous at each point of the set $\bigcup_{c \in \mathbf{J} \setminus C} F(x_0, c)$. Therefore $D_f \cap K(x_0, \delta)$ is a first category set. By the definition of f and h, we conclude that $D_f \setminus \mathrm{cl}\ K(x_0, \delta)$ is a first category set, too, and finally, that the function f is continuous $\mathcal{K}$-a.e.

In order to prove that f is a Darboux function, we shall show that the image of any arc $A \subset \mathbf{I}^2$ is a connected set. We must consider the following cases:

1. $A \subset \mathbf{I}^2 \setminus K(x_0, \delta)$; then $f(A)$ is a connected set because $f|_{\mathbf{I}^2 \setminus K(x_0,\delta)} = h|_{\mathbf{I}^2 \setminus K(x_0,\delta)}$ and h is a Darboux function.
2. $A \subset \mathrm{cl}\, K(x_0, b_i^n) \setminus K(x_0, a_i^n)$ for $n \in \mathbb{N}$ and $i \in \{1, \dots, 2^{n-1}\}$. In this case $f(A) = \{h_0\}$.
3. $A \subset F(x_0, c)$ where $c \in C \setminus \{\delta\}$. Then $f(A) \subset g(U_c)$.
4. $A \subset K(x_0, \delta)$ and there exist $c_1, c_2 \in C \setminus \{\delta\}$ such that $F(x_0, c_i) \cap \cap A \neq \emptyset$, $i = 1, 2$, and at least one of the above two points is a bilateral accumulation point of C. Then $f(A) = B(h_0, \frac{\epsilon}{2})$.
5. $A \cap K(x_0, \delta) \neq \emptyset \neq A \setminus K(x_0, \delta)$. Let $\delta' = \inf\{\delta_1 > 0 : A \cap K(x_0, \delta_1) \neq \emptyset\}$. The interval (δ', δ) contains a continuum of points of the set C. For this reason, there exist $c_1, c_2 \in C \cap (\delta', \delta)$ such that $U_{c_1} \cap U_{c_2} = \emptyset$ and $F(x_0, c_1) \cap A \neq \emptyset \neq F(x_0, c_2) \cap A$. From the previous case it follows that $f(A \cap K(x_0, \delta)) = B(h_0, \epsilon/2)$.

The set $K = A \setminus K(x_0, \delta)$ is closed in $\mathbf{I}^2$, so we can consider the family $\mathcal{S}$ of all components of the set K. Obviously, each component $S \in \mathcal{S}$ is a closed set. Let ζ be a homeomorphism mapping $\mathbf{I}$ onto A and let $S \in \mathcal{S}$. Put $p_1 = \inf \zeta^{-1}(S)$ and $p_2 = \sup \zeta^{-1}(S)$. Notice that $\zeta([p_1, p_2]) = S$. Thus S is an arc or a one-point set. From the above we conclude that $f(S)$ is a connected set because $f|_{\mathbf{I}^2 \setminus K(x_0,\delta)}$ is a Darboux function.

We shall show that $S \cap F(x_0, \delta) \neq \emptyset$. By the assumptions of 5), we have that $p_1 \neq 0$ or $p_2 \neq 1$. Without loss of generality we may assume that $p_1 \neq 0$.

Hence there exists a sequence $\{q_n\}_{n\in\mathbb{N}} \subset [0, p_1)$ such that $\lim_{n\to\infty} q_n = p_1$ and $\zeta(q_n) \in K(x_0, \delta)$ for each $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} \zeta(q_n) = \zeta(p_1) \in S$. Thus $\zeta(p_1) \in F(x_0, \delta)$. Hence $S \cap F(x_0, \delta) \neq \emptyset$ and $f(\zeta(p_1)) \in B(h_0, 4\epsilon/10)$. So we have shown that $f(S)$ is a connected set and $f(S) \cap B(h_0, \epsilon/2) \neq \emptyset$. Consequently, $f(A) = B(h_0, \epsilon/2) \cup \bigcup_{S\in\mathcal{S}} f(S)$ and, finally, $f(A)$ is a connected set. We have proved that f is a Darboux function.

Next, since we have $f|_{\mathbf{I}^2\setminus K(x_0,\delta)} = h|_{\mathbf{I}^2\setminus K(x_0,\delta)}$, $f(K(x_0, \delta)) = B\left(h_0, \frac{\epsilon}{2}\right)$, and $h(\operatorname{cl} K(x_0, \delta)) \subset B\left(h_0, \frac{4\epsilon}{10}\right)$, we can deduce that

$$\rho(f, h) \leq \sup_{x\in K(x_0,\delta)} (d(h_0, f(x)) + d(h_0, h(x))) < \frac{9\epsilon}{10}$$

and $B\left(f, \frac{\epsilon}{10}\right) \subset B(h, \epsilon)$.

To complete the proof we have to show that $B(f, \frac{\epsilon}{10}) \cap L = \emptyset$. Let us take $f_1 \in B(f, \frac{\epsilon}{10})$, $c_1 \in C$, and $x_1 \in F(x_0, c_1)$. For $\epsilon_1 < \frac{2\epsilon}{10}$ we have

$$\operatorname{diam}\left(B\left(h_0, \frac{\epsilon}{2}\right) \cap \mathbf{I}^2\right) \geq \frac{\sqrt{2}}{2} \cdot \epsilon,$$
$$\operatorname{diam}\left(B\left(f_1(x_1), \epsilon_1 + \frac{\epsilon}{10}\right)\right) \leq 2 \cdot \frac{3\epsilon}{10},$$

and therefore,

$$\operatorname{diam}\left(B\left(h_0, \frac{\epsilon}{2}\right) \cap \mathbf{I}^2\right) > \operatorname{diam}\left(B\left(f_1(x_1), \epsilon_1 + \frac{\epsilon}{10}\right)\right).$$

Hence there exists

$$y_2 \in B\left(h_0, \frac{\epsilon}{2}\right) \setminus B\left(f_1(x_1), \epsilon_1 + \frac{\epsilon}{10}\right). \tag{4}$$

By definition, $g(\mathcal{U}) = B(h_0, \frac{\epsilon}{2})$; so there exists $U_\gamma \in \mathcal{U}$ such that $g(U_\gamma) = y_2$. Let $\delta_1 > 0$, $c_2 \in (c_1 - \delta_1, c_1 + \delta_1) \cap U_\gamma$ and $x_2 \in B(x_1, \delta_1) \cap F(x_0, c_2)$. Then $f(x_2) = y_2$, and since $f_1 \in B(f, \frac{\epsilon}{10})$, we have $f_1(x_2) \in B(y_2, \frac{\epsilon}{10})$. Moreover, from (4) it follows that $f_1(x_2) \notin B(f_1(x_1), \epsilon_1)$. In this way, we have proved that, for an arbitrary $\delta_1 > 0$, $f_1(B(x_1, \delta_1)) \not\subset B(f_1(x_1), \epsilon_1)$. Hence $D_{f_1} \supset F$ and, finally, the function $f_1 \in B(f, \frac{\epsilon}{10})$ is not continuous $\mathcal{L}$-a.e.

We have proved that $B(f, \frac{\epsilon}{10}) \subset B(h, \epsilon) \setminus L$, and consequently the porosity of L at h is not less than 0.2. Finally, by the arbitrariness of $h \in L$ and $\epsilon > 0$, we have shown that the set L is uniformly porous in the space KD.

From Lemma 2.6 it follows that F is not σ-porous in $\mathbb{R}^2$. Of course, F is not countable, therefore the proof of the fact that the sets P and N are uniformly porous in the space KD is analogous to the above proof. $\square$

It remains to investigate relations between the classes of functions continuous $\mathcal{L}$-a.e. and $\mathcal{P}$-a.e.

Theorem 3.4. *In the space LD of Darboux and $\mathcal{L}$-a.e. continuous functions (with the metric of uniform convergence) mapping $\mathbf{I}^2$ into $\mathbf{I}^2$, the set $P \cup N$ of functions continuous $\mathcal{P}$-a.e. or $\mathcal{N}$-a.e. is uniformly porous.*

PROOF. Let us consider the symmetric Cantor set with respect to the sequence with general term $\alpha_n = \frac{1}{n+1}$. Since $\lim_{n\to\infty} \frac{1}{n+1} = 0$ and $\sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$, by Lemmas 2.5 and 2.6, the set $F(\frac{1}{n+1}) \in \mathcal{L} \setminus (\mathcal{P} \cup \mathcal{N})$. The rest of the proof runs analogously to the proof of Theorem 3.3. $\square$

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