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SUMS OF QUASICONTINUOUS FUNCTIONS WITH CLOSED GRAPHS*

Abstract

We prove that every real-valued $\mathcal{B}_1^*$ function f defined on a separable metric space X is the sum of three quasicontinuous functions with closed graphs, and there is a $\mathcal{B}_1^*$ function which is not the sum of two quasicontinuous functions with closed graphs. Consequently, if X is a separable metric space which is a Baire space in the strong sense, then the next three properties are equivalent: (1) f is a $\mathcal{B}_1^*$ function, (2) f is the sum of (at least) three quasicontinuous functions with closed graphs, and (3) f is a piecewise continuous function.

1 Introduction

Let X be a topological space. A function $f : X \to \mathbb{R}$ is said to be quasicontinuous (cliquish) at a point $x \in X$ if for every neighborhood U of x and every $\varepsilon > 0$ there is an open set $G \subseteq U$ such that $|f(x) - f(y)| < \varepsilon$ for each $y \in G$ ($|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$). A function f is quasicontinuous (cliquish) if it is such at each point. A function $f : X \to \mathbb{R}$ has closed graph if the set $\{(x, f(x)) : x \in X\}$ is a closed subset of $X \times \mathbb{R}$. A function $f : X \to \mathbb{R}$ is piecewise continuous if there are closed sets $X_n \subseteq X$, $n \in \mathbb{N}$ such that $X = \bigcup_{n=0}^{\infty} X_n$ and the restriction $f{\restriction}X_n$ is continuous for each $n \in \mathbb{N}$. A function $f : X \to \mathbb{R}$ is a function of the class $\mathcal{B}_1^*$ (Baire-one-star function)

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if for every closed set $F \subseteq X$ there exists an open set $G \subseteq X$ such that $F \cap G \neq \emptyset$ and $f\restriction(F \cap G)$ is continuous ([5]).

If $\mathcal{F} \subseteq \mathbb{R}^X$ is a family of real functions, then we denote by $\mathcal{G}(\mathcal{F})$ the group generated by $\mathcal{F}$. Further, denote by $\mathcal{Q}$, $\mathcal{U}$, $\mathcal{B}_1^*$, and $\mathcal{P}$ the families of all quasicontinuous functions, closed graph functions, Baire-one-star functions, and piecewise continuous functions (in $\mathbb{R}^X$), respectively.

Evidently, the sum of two quasicontinuous functions with closed graph need not be such. In this paper we will characterize the group generated by real quasicontinuous functions with closed graph. More precisely, we shall show that

$$\mathcal{G}(\mathcal{QU}) = \mathcal{B}_1^* = \mathcal{P} = \mathcal{QU} + \mathcal{QU} + \mathcal{QU}$$

for separable metric spaces which are Baire spaces in the strong sense, in particular, for complete separable metric spaces. Further, we shall show that $\mathcal{QU} + \mathcal{QU} \neq \mathcal{B}_1^*$ (in spite of the facts that every $\mathcal{B}_1^*$ function on a metric space is the sum of two functions with closed graphs, [3], and that every cliquish function, and thus also every $\mathcal{B}_1^*$ function, is the sum of two quasicontinuous functions, [1]).

Recall that X is a Baire space in the strong sense (or totally nonmeager) if every nonempty closed subspace of X is a Baire space ([4]).

We use the following notation in the paper. Let X be a metric space with a metric function $d : X \times X \to [0, \infty)$. For $x \in X$, $A, B \subseteq X$ and $\varepsilon > 0$ we define

$$\begin{aligned}
\operatorname{diam}(A) &= \sup\{d(a,b) : a, b \in A\},\\
\operatorname{dist}(x,A) &= \inf\{d(x,a) : a \in A\},\\
\operatorname{dist}(A,B) &= \inf\{d(a,b) : a \in A \mathbin{\&} b \in B\},\\
S(x,\varepsilon) &= \{y \in X : d(x,y) < \varepsilon\},\\
S(A,\varepsilon) &= \{y \in X : \operatorname{dist}(A,y) < \varepsilon\}.
\end{aligned}$$

For a subset A of X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A, respectively. The letters $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ stand for the set of natural, rational, and real numbers, respectively. For a function $f : X \to \mathbb{R}$, $D(f)$ denotes the set of all discontinuity points of f. The quantifier $\forall^\infty n$ abbreviates the quantifiers $(\exists m)(\forall n > m)$.

2 A Characterization of $\mathcal{G}(\mathcal{QU})$

Lemma 2.1. *Let X be a topological space and let $f : X \to \mathbb{R}$ be a function. The following implications hold:*

(1) *If X is a Baire space, and f is piecewise continuous, then $D(f)$ is nowhere dense.*
(2) *If f is a $\mathcal{B}_1^*$ function, then $D(f)$ is nowhere dense.*
(3) *If f is a quasicontinuous function with closed graph, then $D(f)$ is nowhere dense.*
(4) *f is a $\mathcal{B}_1^*$ function if and only if $D(f\restriction F)$ is nowhere dense in F for every closed set F in X.*

Proof. To obtain a contradiction (in the proofs of conditions (1)–(3)) let us assume that there is an open set $G \neq \emptyset$ such that $D(f)$ is dense in G.

(1) Let $X = \bigcup_{n=0}^{\infty} X_n$ where X_n is closed and $f\restriction X_n$ is continuous for each $n \in \mathbb{N}$. Since $G = \bigcup_{n=0}^{\infty} G \cap X_n$ is not meager, there is $m \in \mathbb{N}$ such that the set $G \cap X_m$ has nonempty interior H. Then f is continuous at every point $x \in H$ which is a contradiction because $H \cap D(f) \neq \emptyset$.

(2) Let $F = \mathrm{Cl}(G)$. Since f is a $\mathcal{B}_1^*$ function, there is an open set $H \subseteq X$ such that $F \cap H \neq \emptyset$ and $f\restriction(F \cap H)$ is continuous. Then also $G \cap H \neq \emptyset$ and $(G \cap H) \cap D(f) = \emptyset$ which is a contradiction.

(3) Let us fix $x \in G$. Since f is quasicontinuous, there is an open set $H \subseteq G$ such that $|f(x) - f(y)| < 1$ for each $y \in H$. Then $f\restriction H$ is bounded and has closed graph (in $H \times \mathbb{R}$). Therefore $f\restriction H$ is continuous and f is continuous at every point $x \in H$ which contradicts the choice of the set G.

(4) If f is a $\mathcal{B}_1^*$ function and F is closed in X, then $f\restriction F$ is a $\mathcal{B}_1^*$ function in F and by condition (2) $D(f\restriction F)$ is nowhere dense in F. Conversely, if f is not a $\mathcal{B}_1^*$ function, then by the definition there exists a closed set F such that $D(f\restriction F)$ is dense in F. $\square$

Condition (3) in the previous lemma can be easily proved for cliquish functions with closed graphs.

A natural question is what is the relation between these three generalized continuity properties of functions. We can make several simple observations.

Lemma 2.2. *If X is a topological space and $f : X \to \mathbb{R}$ has closed graph, then f is piecewise continuous.*

Proof. The inverse images of compact subsets of $\mathbb{R}$ are closed subsets of X and hence it is enough to take $X_n = f^{-1}([-n, n])$. $\square$

In [5] it is shown that the inclusion $\mathcal{B}_1^* \subseteq \mathcal{P}$ holds for metric spaces and the inclusion $\mathcal{P} \subseteq \mathcal{B}_1^*$ holds for all complete metric spaces. Next we will see that the equality holds for Baire metric spaces in the strong sense while this is not true for all Baire metric spaces.

Lemma 2.3. *If X is a Baire space in the strong sense, then every piecewise continuous function $f : X \to \mathbb{R}$ is a $\mathcal{B}_1^*$ function.*

PROOF. Let $X = \bigcup_{n=0}^{\infty} X_n$ where X_n is closed and $f \restriction X_n$ is continuous for each $n \in \mathbb{N}$. Let $F \subseteq X$ be a closed set. Then $F = \bigcup_{n=0}^{\infty}(F \cap X_n)$ and since F is a Baire space there is an open set G in X and $n \in \mathbb{N}$ such that $\emptyset \neq F \cap G \subseteq X_n$ and $f \restriction (F \cap G)$ is continuous. □

Remark 2.4. *In spite of Lemma* 2.1(3) *and Lemma* 2.3*, it is not true that in a Baire space every quasicontinuous function with closed graph is a $\mathcal{B}_1^*$ function.*

PROOF. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ and let $A = \{x_{n,m} : n, m \in \mathbb{N}\}$ be a sequence of distinct irrational numbers. We define a metric d on $X = \mathbb{Q} \cup A$ by

$$\begin{aligned} d(x,y) &= |x-y| \text{ for } x, y \in \mathbb{Q}, \\ d(x_{n',m}, r_n) &= 1/(m+1) + |r_{n'} - r_n| \text{ and} \\ d(x_{n_1,m_1}, x_{n_2,m_2}) &= 1/(m_1+1) + 1/(m_2+1) + |r_{n_1} - r_{n_2}|. \end{aligned}$$

Notice that $\{x_{n,m}\}$ is an open set in X, A is a discrete open dense subset of X, and X is a Baire space. We define $f : X \to \mathbb{R}$ by $f(r_n) = f(x_{n,m}) = n$. Then f is a quasicontinuous function with closed graph and f is not a $\mathcal{B}_1^*$ function because $\mathbb{Q}$ is closed in X and $f \restriction (\mathbb{Q} \cap G)$ is not continuous for any open set G in X such that $\mathbb{Q} \cap G \neq \emptyset$. □

The aim of the paper is the proof of the next theorem.

Theorem 2.5. *Let X be a separable metric space which is a Baire space in the strong sense and let $f : X \to \mathbb{R}$. The following conditions are equivalent:*

(1) *f is the sum of three quasicontinuous functions with closed graphs.*
(2) *f is the sum of at least three quasicontinuous functions with closed graphs.*
(3) *f is piecewise continuous.*
(4) *f is of the class $\mathcal{B}_1^*$.*

PROOF. The implication (1) → (2) is trivial, the implication (2) → (3) is a consequence of Lemma 2.2, and the equality $\mathcal{P}+\mathcal{P} = \mathcal{P}$. The implication (3) → (4) is Lemma 2.3, and the implication (4) → (1) is Theorem 4.1. □

Remark 2.6. *The assumption X is a Baire space in the strong sense can be neither removed nor replaced by the assumption that X is a Baire space, see Remark* 2.4.

Remark 2.7. *The separability assumption is necessary for the proof of the implication* (4) $\to$ (1)*, namely for the proof of Lemma* 3.3*.*

Remark 2.8. *The sum of three functions cannot be replaced by the sum of two functions in condition* (1) *of Theorem* 2.5*. Namely, the function f defined by $f(x) = 1$, if $x = 0$, and $f(x) = 0$, if $x \neq 0$, is a $\mathcal{B}_1^*$ function, but it is not the sum of two quasicontinuous functions with closed graphs.*

PROOF. Assume that $f = f_1 + f_2$ where f_1, f_2 are quasicontinuous functions with closed graphs. Then $f_1(x) = -f_2(x)$ for every $x \neq 0$. As f_1 is quasicontinuous at 0, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ convergent to 0 such that $\lim_{n\to\infty} f_1(x_n) = f_1(0)$ and then $\lim_{n\to\infty} f_2(x_n) = -f_1(0)$. Then, as f_2 has closed graph, $f_2(0) = -f_1(0)$ and we have this contradiction: $1 = f(0) = f_1(0) + f_2(0) = 0$. $\square$

Problem 2.9. *Characterize the family $\mathcal{QU} + \mathcal{QU}$.*

We will need the next easy property of quasicontinuous functions.

Lemma 2.10. *If $f : X \to \mathbb{R}$, K is an open set, and $x \in \mathrm{Cl}(K)$ is such that $f \restriction \mathrm{Cl}(K)$ is quasicontinuous at x, then f is quasicontinuous at x.*

3 Systems of Closed Nowhere Dense Sets

Lemma 3.1 ([1], Lemma 3.1)**.** *Let X be a metric space, $F \subseteq X$ be a closed nowhere dense set and let $G \subseteq X$ be an open set such that $F \subseteq \mathrm{Cl}(G)$. Then there is a family $\mathcal{K} = \bigcup_{n=0}^{\infty} \mathcal{K}_n$ of nonempty open subsets of X such that the following conditions hold:*

(i) *The set $E_n = \bigcup\{\mathrm{Cl}(K) : K \in \mathcal{K}_n\}$ is closed and $E_n \subseteq S(F, 2/n) \cap G \setminus F$ for every $n \in \mathbb{N}$.*
(ii) $(\forall x \in X \setminus F)(\exists V$ *a neighburhood of* $x)$ $|\{K \in \mathcal{K} : V \cap \mathrm{Cl}(K) \neq \emptyset\}| \leq 1$.
(iii) $(\forall x \in F)(\forall V$ *a neighborhood of* $x)(\forall^{\infty} n)(\exists K \in \mathcal{K}_n)$ $\mathrm{Cl}(K) \subseteq V$.

In particular, $\mathrm{Cl}(\bigcup_{n=0}^{\infty} E_n) = \bigcup_{n=0}^{\infty} E_n \cup F$.

PROOF. The construction is by induction on $n \in \mathbb{N}$. Assume that $\mathcal{K}_i$, $i < n$ have been constructed. Put

$$T_n = G \cap S(F, 1/n) \setminus (F \cup \textstyle\bigcup_{i<n} E_i),$$
$$\alpha_n(x) = \frac{1}{4} \cdot \mathrm{dist}(x, F \cup (X \setminus G) \cup \textstyle\bigcup_{i<n} E_i).$$

Let $S_n \subseteq T_n$ be a maximal set with the property that $d(x, y) > 1/n$ for $x \neq y$ in S_n. We set $\mathcal{K}_n = \{S(x, \alpha_n(x)) : x \in S_n\}$. $\square$

The more detailed proof of the above lemma can be found in [1] and the same arguments we use in the proof of Lemma 3.3 below. We need this result for the next consequence.

Lemma 3.2. *Let H_0 be a nowhere dense subset of a metric space X and let $\mathcal{H}$ be a countable family of closed subsets of H_0 linearly ordered by inclusion $\subseteq$. Then there exists an order isomorphism φ from $\mathcal{H}$ onto a family of open subsets of $X \setminus H_0$ such that $H_0 \cap \mathrm{Cl}(\varphi(H)) = H$ for every $H \in \mathcal{H}$.*

Proof. Without loss of generality we can assume that $\emptyset, H_0 \in \mathcal{H}$. Let $\mathfrak{n}$ be the cardinality of $\mathcal{H}$; i.e., either $\mathfrak{n}$ is an integer or $\mathfrak{n} = \omega$, and let $\{H_n : n < \mathfrak{n}\}$ be an enumeration of $\mathcal{H}$ such that $H_1 = \emptyset$ (and H_0 is the given nowhere dense set). We define $\varphi(H_n)$ by induction for $n < \mathfrak{n}$. Set $\varphi(H_0) = X \setminus H_0$, $\varphi(H_1) = \emptyset$ and let us assume that $n > 1$ and the open sets $\varphi(H_i)$ are defined for $i < n$. There are $j, k < n$ such that $H_j \subseteq H_n \subseteq H_k$ and for every $i < n$ either $H_i \subseteq H_j$ or $H_k \subseteq H_i$. Set $F = H_n$ and $G = \varphi(H_k)$ in Lemma 3.1 and let $\mathcal{K}$ be the obtained system of open sets. Let $V = \bigcup \mathcal{K}$. Then $V \subseteq \varphi(H_k)$ and $\mathrm{Cl}(V) = \bigcup_{i=0}^{\infty} E_i \cup H_n$ while $\bigcup_{i=0}^{\infty} E_i \subseteq \varphi(H_k) \subseteq X \setminus H_0$. Therefore $H_0 \cap \mathrm{Cl}(V) = H_n$. Finally, $\mathrm{Cl}(V \cup \varphi(H_j)) = \mathrm{Cl}(V) \cup \mathrm{Cl}(\varphi(H_j))$ and $H_0 \cap \mathrm{Cl}(\varphi(H_j)) = H_j \subseteq H_n$. Therefore we can set $\varphi(H_n) = V \cup \varphi(H_j)$. $\square$

Lemma 3.3. *Let X be a metric space and let ξ be a countable ordinal number. Let $\{F_\alpha\}_{\alpha \le \xi}$ be a sequence of closed nowhere dense sets such that $F_\xi = \emptyset$ and $F_\beta \subsetneqq F_\alpha$ for $\alpha < \beta \le \xi$. There exists a system $\mathcal{L} = \bigcup_{\alpha<\xi} \mathcal{L}^\alpha$ of disjoint nonempty open sets, where $\mathcal{L}^\alpha = \bigcup_{n=0}^{\infty} \mathcal{L}^\alpha_n$ is a disjoint union, such that*

1. $(\forall K \in \mathcal{L})\ \mathrm{Cl}(K) \cap F_0 = \emptyset$.
2. $(\forall x \in X \setminus F_0)(\exists V$ *a neighbourhood of* $x)\ |\{K \in \mathcal{L} : V \cap \mathrm{Cl}(K) \neq \emptyset\}| \le 1$.
3. $(\forall x \in F_0 \setminus F_\alpha)(\exists V$ *a neighbourhood of* $x)(\forall K \in \bigcup_{\beta\ge\alpha} \mathcal{L}^\beta)\ V \cap \mathrm{Cl}(K) = \emptyset$.
4. $(\forall x \in F_\alpha)(\forall V$ *a neighbourhood of* $x)(\forall^\infty n)(\exists K \in \mathcal{L}^\alpha_n)\ \mathrm{Cl}(K) \subseteq V$.
5. $(\forall x \in F_0)(\forall K \in \mathcal{L})\ \mathrm{Cl}(K) \subseteq S(x, 2\,\mathrm{dist}(x, \mathrm{Cl}(K)))$.

Proof. By Lemma 3.2 let us fix a descending sequence of open sets $\{V_\alpha\}_{\alpha\le\xi}$ such that $V_\alpha \cap F_0 = \emptyset$ and $F_\alpha = F_0 \cap \mathrm{Cl}(V_\alpha)$. Let $(\lambda, \rho) : \omega \to \xi \times \omega$ be a bijection such that

$$n \le m \text{ and } \lambda(n) = \lambda(m) \text{ implies } \rho(n) \le \rho(m).$$

By induction we define the families $\mathcal{L}^{\lambda(n)}_{\rho(n)}$ so that the following conditions hold:

(i) $\mathrm{diam}(K) < 1/(2n)$ for $K \in \mathcal{L}^{\lambda(n)}_{\rho(n)}$.

(ii) The set $E_n = \bigcup\{\mathrm{Cl}(K) : K \in \mathcal{L}^{\lambda(n)}_{\rho(n)}\}$ is closed and even every set $E_n(\mathcal{A}) = \bigcup\{\mathrm{Cl}(K) : K \in \mathcal{A}\}$ is closed for $\mathcal{A} \subseteq \mathcal{L}^{\lambda(n)}_{\rho(n)}$.

(iii) $E_n \subseteq S(F_{\lambda(n)}, 2/n) \cap V_{\lambda(n)}$.

(iv) $(\forall x \in F_{\lambda(n)})(\exists K \in \mathcal{L}^{\lambda(n)}_{\rho(n)})\ K \subseteq S(x, 2/n)$.

(v) $(\forall x \in F_0)(\forall K \in \mathcal{L}^{\lambda(n)}_{\rho(n)})\ \mathrm{Cl}(K) \subseteq S(x, 2\,\mathrm{dist}(x, \mathrm{Cl}(K)))$.

Assume that the families $\mathcal{L}^{\lambda(i)}_{\rho(i)}$, $i < n$ have been constructed. Put

$$T_n = S(F_{\lambda(n)}, 1/n) \cap V_{\lambda(n)} \setminus (F_0 \cup \textstyle\bigcup_{i<n} E_i),$$
$$\alpha_n(x) = \frac{1}{4} \cdot \mathrm{dist}(x, F_0 \cup (X \setminus V_{\lambda(n)}) \cup \textstyle\bigcup_{i<n} E_i).$$

Let $S_n \subseteq T_n$ be a maximal set with the property that $d(x,y) > 1/n$ for $x \neq y$ in S_n. We set

$$\mathcal{L}^{\lambda(n)}_{\rho(n)} = \{S(x, \alpha_n(x)) : x \in S_n\}.$$

Now we verify conditions (i)–(v).

(i) For $x \in S_n$ we have $\mathrm{dist}(x, F_{\lambda(n)}) < 1/n$ because $x \in T_n$, and hence $\alpha_n(x) < 1/(4n)$. Therefore $\mathrm{diam}(S(x, \alpha_n(x))) < 1/(2n)$.

(ii) For any $x \neq y$ in S_n, $d(x,y) > 1/n$ and $\alpha_n(x), \alpha_n(y) < 1/(4n)$. Therefore $\mathrm{dist}(S(x, \alpha_n(x)), S(y, \alpha_n(y))) \geq d(x,y) - \alpha_n(x) - \alpha_n(y) > 1/(2n)$.

(iii) By definition of $\alpha_n(x)$ we can see that $\mathrm{Cl}(S(x, \alpha_n(x))) \subseteq V_{\lambda(n)}$ for $x \in S_n$. Moreover, since $x \in S(F_{\lambda(n)}, 1/n)$,

$$\mathrm{Cl}(S(x, \alpha_n(x))) \subseteq S(F_{\lambda(n)}, 1/n + \alpha_n(x)) \subseteq S(F_{\lambda(n)}, 2/n).$$

(iv) Let $x \in F_{\lambda(n)}$. Since $F_{\lambda(n)}$ is nowhere dense and disjoint from $\bigcup_{i<n} E_i$, there is $y \in S(x, 1/(2n)) \setminus (F_{\lambda(n)} \cup \bigcup_{i<n} E_i)$. Notice that

$$S_n \cap S(x, 1/(2n) + 1/n) \neq \emptyset$$

since otherwise y could be added to S_n contradicting the maximality of S_n. Now, for $y \in S_n \cap S(x, 1/(2n) + 1/n)$ we have

$$S(y, \alpha_n(y)) \subseteq S(x, 1/(2n) + 1/n + \alpha_n(y)) \subseteq S(x, 2/n).$$

(v) Let $x \in F_0$ and $K \in \mathcal{L}^{\lambda(n)}_{\rho(n)}$, $K = S(y, \alpha_n(y))$. Then

$$\alpha_n(y) \leq \mathrm{dist}(y, F_0)/4 \leq d(y,x)/4.$$

Therefore $\mathrm{dist}(x, K) \geq d(y,x) - \alpha_n(y) > 2\alpha_n(y)$ and

$$K \subseteq S(x, \mathrm{dist}(x, K) + 2\alpha_n(y)) \subseteq S(x, 2\,\mathrm{dist}(x, K)).$$

Now we show that conditions (1)–(5) are satisfied. Conditions (1), (4) and (5) are consequences of conditions (iii), (iv) and (v), respectively.

(2) Let $x \in X \setminus F_0$, $\operatorname{dist}(x, F_0) > 4/k$ for some k. If $n \geq k$, then by (iii) $E_n \subseteq S(F_{\lambda(n)}, 2/n) \subseteq S(F_0, 2/k)$. It follows that $S(x, 2/k) \cap \bigcup_{n \geq k} E_n = \emptyset$. Now there are two possibilities: Either, $x \in \operatorname{Cl}(K_0)$ for some $K_0 \in \mathcal{L}$ and then the set $V = S(x, 2/k) \setminus \bigcup_{n<k}(E_n \setminus \operatorname{Cl}(K_0))$ is open, $x \in V$, and $V \cap K \neq \emptyset$ if and only if $K = K_0$ for $K \in \mathcal{L}$. Or, $x \notin \bigcup_{n<k} E_n$, and then $V = S(x, 2/k) \setminus \bigcup_{n<k} E_n$ is open, $x \in V$, and $V \cap K = \emptyset$ for every $K \in \mathcal{L}$.

(3) Let $\alpha \leq \beta \leq \xi$ and let $x \in F_0 \setminus F_\alpha$. Then, by the choice of the set V_α, $x \in X \setminus \operatorname{Cl}(V_\alpha) \subseteq X \setminus \operatorname{Cl}(V_\beta)$, and by (iii), $\operatorname{Cl}(K) \subseteq V_\beta$ for every $K \in \mathcal{L}^\beta$. □

4 Main Result

Theorem 4.1. *Let X be a separable metric space. Then every $\mathcal{B}_1^*$ function $f : X \to \mathbb{R}$ is the sum of three quasicontinuous functions with closed graphs.*

Remark 4.2. *As we already mentioned every $\mathcal{B}_1^*$ function is piecewise continuous. However in Theorem 4.1 $\mathcal{B}_1^*$ can't be replaced by piecewise continuous because for $X = \mathbb{Q}$ every function is piecewise continuous while there exists a function which is not cliquish and hence not every function is a sum of quasicontinuous functions.*

PROOF. Let $f \in \mathcal{B}_1^*$. Let us introduce the following notation:

$$f^+ = \max\{f, 0\}, \qquad f^- = \min\{f, 0\},$$

and for a closed set $A \subseteq X$ let $h_A : X \setminus A \to \mathbb{R}$ be defined by

$$h_A(x) = 1/\operatorname{dist}(x, A), \text{ if } A \neq \emptyset, \text{ and } h_A(x) = 0, \text{ if } A = \emptyset.$$

By induction we define the following sequence of closed nowhere dense subsets of X:

$$\begin{aligned} F_0 &= \operatorname{Cl}(D(f)), \\ F_{\alpha+1} &= \operatorname{Cl}(D(f \restriction F_\alpha)), \\ F_\alpha &= \bigcap_{\beta<\alpha} F_\beta \text{ for } \alpha \text{ a limit ordinal.} \end{aligned}$$

By Lemma 2.1(4) the set F_0 is nowhere dense and $F_{\alpha+1}$ is nowhere dense in F_α. As X is separable every descending sequence of closed sets in X must be countable. Let $\xi < \omega_1$ be the least ordinal for which $F_\xi = \emptyset$. If $\xi = 0$, then f is continuous and $f = f + 0 + 0$ is the sum of three continuous functions.

Therefore let us assume that $\xi > 0$. By Lemma 3.3 there exists a system $\mathcal{L} = \bigcup_{\alpha<\xi} \mathcal{L}^\alpha$ of disjoint nonempty open sets satisfying conditions (1)–(5). For $\alpha < \xi$ and $K \in \mathcal{L}^\alpha$ let us fix $b_K \in K$ and $a_K \in F_\alpha \setminus F_{\alpha+1}$ such that $0 < d(b_K, a_K) < 2\,\mathrm{dist}(b_K, F_\alpha \setminus F_{\alpha+1})$. This is possible because $\mathrm{Cl}(K) \cap F_0 = \emptyset$. By condition (2) of Lemma 3.3 the set $D = F_0 \cup \bigcup\{\mathrm{Cl}(K) : K \in \mathcal{L}\}$ is closed. We define quasicontinuous functions $f_1, f_2, f_3 : X \to \mathbb{R}$ with closed graphs as follows:

For $x \in F_\alpha \setminus F_{\alpha+1}$, $\alpha < \xi$,

$$\begin{aligned} f_1(x) &= f^+(x) + h_{F_{\alpha+1}}(x), \\ f_2(x) &= h_{F_{\alpha+1}}(x), \\ f_3(x) &= f^-(x) - 2h_{F_{\alpha+1}}(x). \end{aligned}$$

For $x \in \mathrm{Cl}(K)$, $K \in \mathcal{L}^\alpha_{3n}$, $n \in \mathbb{N}$, $\alpha < \xi$,

$$\begin{aligned} f_1(x) &= f^+(a_K) + h_{F_{\alpha+1}}(a_K), \\ f_2(x) &= f^+(x) + h_{F_\alpha}(x), \\ f_3(x) &= f^-(x) - h_{F_\alpha}(x) - f^+(a_K) - h_{F_{\alpha+1}}(a_K). \end{aligned}$$

For $x \in \mathrm{Cl}(K)$, $K \in \mathcal{L}^\alpha_{3n+1}$, $n \in \mathbb{N}$, $\alpha < \xi$,

$$\begin{aligned} f_1(x) &= f^+(x) + h_{F_\alpha}(x), \\ f_2(x) &= h_{F_{\alpha+1}}(a_K), \\ f_3(x) &= f^-(x) - h_{F_\alpha}(x) - h_{F_{\alpha+1}}(a_K). \end{aligned}$$

For $x \in \mathrm{Cl}(K)$, $K \in \mathcal{L}^\alpha_{3n+2}$, $n \in \mathbb{N}$, $\alpha < \xi$,

$$\begin{aligned} f_1(x) &= f^+(x) + h_{F_\alpha}(x) - f^-(a_K) + 2h_{F_{\alpha+1}}(a_K), \\ f_2(x) &= f^-(x) - h_{F_\alpha}(x), \\ f_3(x) &= f^-(a_K) - 2h_{F_{\alpha+1}}(a_K). \end{aligned}$$

For $x \in X \setminus D$,

$$\begin{aligned} f_1(x) &= f^+(x) + h_D(x), \\ f_2(x) &= h_D(x), \\ f_3(x) &= f^-(x) - 2h_D(x). \end{aligned}$$

Easily we can verify that $f = f_1 + f_2 + f_3$. We prove that the functions f_1, f_2, f_3 are quasicontinuous and have closed graphs.

The functions f and h_{F_α} are continuous on the open set $X \setminus F_0$ and hence the functions f^+, f^-, h_{F_α} are continuous at every point of $x \in X \setminus D$ and at every point $x \in \mathrm{Cl}(K)$ for $K \in \mathcal{L}$, $\alpha < \xi$. Further, the function h_D is continuous on the open set $X \setminus D$. The functions f_1, f_2, f_3 are constructed from these functions in such a way that the restrictions $f_i \restriction \mathrm{Cl}(K)$ and $f_i \restriction (X \setminus D)$ for $K \in \mathcal{L}$ and $i = 1, 2, 3$ are continuous. Since $X \setminus F_0 = (X \setminus D) \cup \bigcup_{K \in \mathcal{L}} \mathrm{Cl}(K)$, by Lemma 2.10 it follows that the functions f_1, f_2, f_3, are quasicontinuous at every $x \in X \setminus F_0$.

Let $x \in F_0$, i.e., $x \in F_\alpha \setminus F_{\alpha+1}$ for some $\alpha < \xi$. Let U be a neighbourhood of x and let $\varepsilon > 0$ be arbitrary. Since the functions $f \restriction F_\alpha$ (and also $f^+ \restriction F_\alpha$, $f^- \restriction F_\alpha$) and $h_{F_{\alpha+1}} \restriction F_\alpha$ are continuous at x, there exists $V \subseteq U$, a neighborhood of x, such that

$$\max\{|f^+(x) - f^+(y)|, |f^-(x) - f^-(y)|, |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(y)|\} < \varepsilon/3 \quad (4.1)$$

for each $y \in V \cap F_\alpha$. Let $\delta > 0$ be such that $S(x, 3\delta) \subseteq V$. By condition (4) of Lemma 3.3 there exists $n \in \mathbb{N}$ and $K_1 \in \mathcal{L}^\alpha_{3n}$, $K_2 \in \mathcal{L}^\alpha_{3n+1}$, $K_3 \in \mathcal{L}^\alpha_{3n+2}$ such that $\mathrm{Cl}(K_i) \subseteq S(x, \delta)$ for $i = 1, 2, 3$. Then $\mathrm{dist}(b_{K_i}, F_\alpha \setminus F_{\alpha+1}) < \delta$ and hence $d(b_{K_i}, a_{K_i}) < 2\delta$. It follows that $a_{K_i} \in V \cap F_\alpha$ because $d(x, a_{K_i}) \leq d(x, b_{K_i}) + d(b_{K_i}, a_{K_i}) < \delta + 2\delta = 3\delta$. Now applying the second case in the definition of f_1, the third case in the definition of f_2, and the fourth case in the definition of f_3 we obtain

$$|f_1(x) - f_1(y)| \leq |f^+(x) - f^+(a_{K_1})| + |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(a_{K_1})| \text{ for } y \in K_1,$$
$$|f_2(x) - f_2(y)| \leq |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(a_{K_2})| \text{ for } y \in K_2,$$
$$|f_3(x) - f_3(y)| \leq |f^-(x) - f^-(a_{K_3})| + 2|h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(a_{K_3})| \text{ for } y \in K_3,$$

and since $a_{K_i} \in V \cap F_\alpha$, using the inequality (*) we get $|f_i(x) - f_i(y)| < \varepsilon$ for all $y \in K_i$ and $i = 1, 2, 3$. Therefore, the functions f_1, f_2, f_3 are quasicontinuous also at every $x \in F_0$ and we have proved that these functions are quasicontinuous everywhere.

It remains to prove that graphs of functions f_1, f_2, f_3 are closed subsets of $X \times \mathbb{R}$. It is enough to prove (see [2]) that for every $x \in X$ and $i = 1, 2, 3$

$$C(f_i, x) = \bigcap \{\mathrm{Cl}(f(U)) : U \text{ is a neighbourhood of } x\} = \{f_i(x)\}.$$

For $x \in X \setminus D$, $C(f_i, x) = \{f_i(x)\}$ because f_i's are continuous on $X \setminus D$. For every $x \in D$ we find a sequence V_m, $m \in \mathbb{N}$ of neighborhoods of x such that

$$f_i(y) \in (-\infty, -m) \cup (f_i(x) - 1/m, f_i(x) + 1/m) \cup (m, \infty). \quad (4.2)$$

for each $m \in \mathbb{N}$, $y \in V_m$ and $i = 1, 2, 3$. This will end the proof since

$$\bigcap_{m=0}^{\infty} \mathrm{Cl}(f_i(V_m)) = \{f_i(x)\} = C(f_i, x), \text{ for } i = 1, 2, 3.$$

Let $x \in D$. There are two cases:

Case 1. $x \in \mathrm{Cl}(K)$ for some $K \in \mathcal{L}_n^\alpha$ with $\alpha < \xi$ and $n \in \mathbb{N}$. By condition (2) of Lemma 3.3 there exists a neighborhood V of x such that $\mathrm{Cl}(L) \cap V = \emptyset$ for every $L \in \mathcal{L} \setminus \{K\}$. As $f_i \restriction \mathrm{Cl}(K)$ are continuous, we can find neighborhoods W_m of x for $m \in \mathbb{N}$ such that $|f_i(x) - f_i(y)| < 1/m$ for $y \in W_m \cap \mathrm{Cl}(K)$ and $i = 1,\ 2,\ 3$. Let $r \in \mathbb{N}$ be such that $\mathrm{dist}(x, F_0) > 1/r$. Then the sets $V_m = W_m \cap V \cap S(x, 1/(r+m))$ for $m \in \mathbb{N}$ are neighborhoods of x disjoint from F_0 and

$$V_m = (V_m \cap \mathrm{Cl}(K)) \cup (V_m \setminus D).$$

For $y \in V_m \cap \mathrm{Cl}(K)$ we have $|f_i(x) - f_i(y)| < 1/m$. For $y \in V_m \setminus D$ we have $\mathrm{dist}(y, D) \le \mathrm{dist}(y, x) < 1/(r+m)$ because $x \in D$ and hence $h_D(y) > m$. Since $f^+(y) \ge 0$, $f^-(y) \le 0$, by the fifth cases of the definitions of functions f_1, f_2, f_3 we get $f_1(y) > m$, $f_2(y) > m$, and $f_3(y) < -2m$.

Case 2. $x \in F_0$ and hence $x \in F_\alpha \setminus F_{\alpha+1}$ for some $\alpha < \xi$. The functions $f \restriction F_\alpha$ and $h_{F_{\alpha+1}}$ are continuous at x; so we can find neighborhoods W_m of x such that

$$\max\{|f^+(x) - f^+(y)|, |f^-(x) - f^-(y)|, |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(y)|\} < 1/(3m) \quad (4.3)$$

for each $y \in W_m \cap F_\alpha$ and $m \in \mathbb{N}$. Let $0 < \delta_m < 1/(6m)$ be such that $S(x, 6\delta_m) \subseteq W_m$. By condition (3) of Lemma 3.3 there exists an open neighborhood V of x such that $V \cap \mathrm{Cl}(K) = \emptyset$ for $K \in \bigcup_{\beta > \alpha} \mathcal{L}^\beta$. The sets $V_m = V \cap S(x, \delta_m) \setminus F_{\alpha+1}$ for $m \in \mathbb{N}$ are neighborhoods of x and $V_m = (V_m \setminus D) \cup (V_m \cap F_\alpha \setminus F_{\alpha+1}) \cup (V_m \cap F_0 \setminus F_\alpha) \cup (V_m \cap \bigcup_{\beta \le \alpha} \bigcup_{K \in \mathcal{L}^\beta} \mathrm{Cl}(K))$. So for $y \in V_m$ we have four subcases:

Case 2a. $y \in V_m \setminus D$. Then $h_D(y) > m$, because $x \in D$ and $d(x, y) < 1/m$, and hence $f_1(y) > m$, $f_2(y) > m$, and $f_3(y) < -2m$.

Case 2b. $y \in V_m \cap (F_\alpha \setminus F_{\alpha+1})$. Then by the choice of W_m

$$\begin{aligned}
&|f_1(x) - f_1(y)| \le |f^+(x) - f^+(y)| + |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(y)| < 1/m,\\
&|f_2(x) - f_2(y)| = |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(y)| < 1/m,\\
&|f_3(x) - f_3(y)| \le |f^-(x) - f^-(y)| + 2|h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(y)| < 1/m.
\end{aligned}$$

Case 2c. $y \in V_m \cap (F_0 \setminus F_\alpha)$. There is $\beta < \alpha$ such that $y \in F_\beta \setminus F_{\beta+1}$. As $x \in F_\alpha \subseteq F_{\beta+1}$, we have $\mathrm{dist}(y, F_{\beta+1}) \le d(x, y) < 1/m$ and $h_{F_{\beta+1}}(y) > m$. Therefore $f_1(y) > m$, $f_2(y) > m$, and $f_3(y) < -2m$.

Case 2d. $y \in V_m \cap \mathrm{Cl}(K)$ for some $K \in \mathcal{L}^{\beta}_{3n+j}$, $j \in \{0,1,2\}$, $n \in \mathbb{N}$, and $\beta \leq \alpha$. As $x \in F_\alpha \subseteq F_\beta$, $\mathrm{dist}(y, F_\beta) \leq d(y,x) \leq \delta_m < 1/m$. Therefore $h_{F_\beta}(y) > m$ and hence

$f_1(y) > m$, if $j = 1, 2$, $\quad f_2(y) > m$, if $j = 0$, $\quad f_3(y) < -m$, if $j = 0, 1$.
$f_2(y) < -m$, if $j = 2$,

As $\mathrm{dist}(x, \mathrm{Cl}(K)) \leq d(x,y) \leq \delta_m$ and $\mathrm{Cl}(K) \subseteq S(x, 2\,\mathrm{dist}(x, \mathrm{Cl}(K))) \subseteq S(x, 2\delta_m)$ by condition (5) of Lemma 3.3, we have

$$d(x, a_K) \leq d(x, b_K) + d(b_K, a_K) < 2\delta_m + 4\delta_m = 6\delta_m.$$

So if $\beta = \alpha$, $a_K \in W_m \cap F_\alpha$ and by (4.3),

if $j = 0$, then
$$|f_1(x) - f_1(y)| \leq |f^+(x) - f^+(a_K)| + |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(a_K)| < 1/m,$$
if $j = 1$, then
$$|f_2(x) - f_2(y)| = |h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(a_K)| < 1/m,$$
and if $j = 2$, then
$$|f_3(x) - f_3(y)| \leq |f^-(x) - f^-(a_K)| + 2|h_{F_{\alpha+1}}(x) - h_{F_{\alpha+1}}(a_K)| < 1/m.$$

If $\beta < \alpha$, then $x \in F_\alpha \subseteq F_{\beta+1}$ and hence $\mathrm{dist}(a_K, F_{\beta+1}) \leq d(a_K, x) < 6\delta_m < 1/m$. Then $h_{F_{\beta+1}}(a_K) > m$ and hence

$f_1(y) > m$, if $j = 0$, $\quad f_2(y) > m$, if $j = 1$, $\quad f_3(y) < -2m$, if $j = 2$.

In all cases we have proved the property (4.2) and so the proof of Theorem 4.1 is complete. □

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