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COMPARISON OF ψ-DENSITY TOPOLOGIES

Abstract

The paper includes a necessary and sufficient condition under which two ψ-density topologies generated by two functions ψ_1 and ψ_2 are equal. The condition is formulated in terms of the behavior of two sequences of sets: $A_k^+ = \{x \in \mathbb{R}_+ : \psi_1(2x) < \frac{1}{k}\psi_2(2x)\}$ and $B_k^+ = \{x \in \mathbb{R}_+ : \psi_2(2x) < \frac{1}{k}\psi_1(2x)\}$.

The Lebesgue Density Theorem (LDT) plays a central role in real analysis. It constitutes a basis for the construction of the density topology. The proof of the expression Int $A = A \cap \Phi(B)$, where B is any measurable kernel of A and $\Phi(B)$ is the set of all density points of B, depends essentially on LDT.

The interesting phenomenon that the density topology is included in a σ-algebra of Lebesgue measurable sets is also due to LDT together with the countable chain condition for the Lebesgue measure. The interest concerning the density topology has grown up after the appearance of the papers of C. Goffman, C. Neugebauer and T. Nishiura [GNN] and C. Goffman and D. Waterman [GW] explaining the strict connection between approximate continuity and the density topology. The class of approximate continuous functions was introduced by Denjoy in his work on derivatives [D] and was studied, among others, by I. Maximoff [M1], [M2] and Z. Zahorski [Z].

S. J. Taylor in [T] considered the possibility of the improvement of the Lebesgue Density Theorem. His idea was applied by M. Terepeta and E. Wagner-Bojakowska, which introduced in [TW-B] the notions of ψ-density point and ψ-density topology $\mathcal{T}_\psi$, analogously to the classical density topology (see [O]). In the second section in [TW-B] there were studied relationships between topologies generated by different functions. The purpose of this paper is to find some necessary and sufficient condition under which two ψ-density topologies generated by two functions ψ_1 and ψ_2 are equal.

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Recall the basic notions and denotations from [T] and [TW-B]. Throughout the paper $\mathbb{N}$ will denote the set of positive integers, $\mathbb{R}$ ($\mathbb{R}_+$) — the set of real (positive real) numbers, S — the σ-algebra of Lebesgue measurable sets and m — the Lebesgue measure on the real line. Let A' stand for $\mathbb{R} \setminus A$, $\overline{A}$ — the closure of A in the Euclidean topology and $-A = \{-a : a \in A\}$.

Denote by $\mathcal{C}$ the family of all continuous non-decreasing functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to 0^+} \psi(x) = 0$.

Let $\psi \in \mathcal{C}$.

Definition 1. [see [TW-B]]. We say that $x \in \mathbb{R}$ is a ψ-dispersion point of a set $A \in S$ if and only if

$$\lim_{h\to 0^+} \frac{m(A \cap [x-h,\, x+h])}{2h\psi(2h)} = 0.$$

We say that $x \in \mathbb{R}$ is a ψ-density point of a measurable set A if and only if it is a ψ-dispersion point of a set A'.

Let $\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}$ for $A \in S$. The family

$$\mathcal{T}_\psi = \{A \in S : A \subset \Phi_\psi(A)\}$$

is a topology on the real line, essentially stronger than the Euclidean topology and essentially weaker than the density topology ([TW-B], Th. 1.4).

Let $\psi_1, \psi_2 \in \mathcal{C}$.

It is easy to see that if for arbitrary set $A \in S$ from the fact that x is a ψ_1-dispersion point of A it follows that x is a ψ_2-dispersion point of A, then $\mathcal{T}_{\psi_1} \subset \mathcal{T}_{\psi_2}$. So the last inclusion holds, for example, if $\psi_1(x) \le \psi_2(x)$ or, more generally, if $\psi_1(x) \le k\psi_2(x)$ for some $k \in \mathbb{R}_+$ and for each $x \in \mathbb{R}_+$.

Clearly, if $\psi_1, \psi_2 \in \mathcal{C}$ and $\limsup_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} < \infty$, then $\mathcal{T}_{\psi_1} \subset \mathcal{T}_{\psi_2}$ ([TW-B], Th. 2.2). There also is a proof that if $\psi_1, \psi_2 \in \mathcal{C}$, $\liminf_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} > 0$ and $\limsup_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} < \infty$, then $\mathcal{T}_{\psi_1} = \mathcal{T}_{\psi_2}$ ([TW-B], Th. 2.4). It appears that these two inequalities form a sufficient, but not necessary, condition for the equality $\mathcal{T}_{\psi_1} = \mathcal{T}_{\psi_2}$. There exist two functions $\psi_1, \psi_2 \in \mathcal{C}$ such that $\liminf_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} = 0$ and $0 < \limsup_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} < \infty$, for which $\mathcal{T}_{\psi_1} = \mathcal{T}_{\psi_2}$ ([TW-B], Th. 2.5). But if $\lim_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} = 0$, then there exists a measurable set $A \subset \mathbb{R}_+$ such that 0 is a ψ_2-dispersion point of A, but it is not a ψ_1-dispersion point of A ([TW-B], Th. 2.6). It is easy to see that the complement A' of the set A constructed in the proof of Th. 2.6 is open in the topology $\mathcal{T}_{\psi_2}$, but it is not $\mathcal{T}_{\psi_1}$-open, so $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_2}$.

Let $\psi_1, \psi_2 \in \mathcal{C}$. In the whole paper we shall use the following notation:

$$A_k^+ = \{x \in \mathbb{R}_+ : \psi_1(2x) < \frac{1}{k}\psi_2(2x)\},$$
$$B_k^+ = \{x \in \mathbb{R}_+ : \psi_2(2x) < \frac{1}{k}\psi_1(2x)\},$$

$A_k = A_k^+ \cup (-A_k^+)$, $B_k = B_k^+ \cup (-B_k^+)$ for $k \in \mathbb{N}$.

Lemma 1. *Let $E \in S$ and $k \in \mathbb{N}$. If $\lim_{x\to 0^+} \frac{m(E\cap[-x,x])}{2x\psi_2(2x)} = 0$, then*

$$\limsup_{x\to 0^+} \frac{m(E \cap [-x, x])}{2x\psi_1(2x)} = \limsup_{x\to 0^+} \frac{m(E \cap A_k \cap [-x, x])}{2x\psi_1(2x)}.$$

PROOF. It suffices to prove that

$$\limsup_{x\to 0^+} \frac{m(E \cap [-x, x])}{2x\psi_1(2x)} \leq \limsup_{x\to 0^+} \frac{m(E \cap A_k \cap [-x, x])}{2x\psi_1(2x)}.$$

We have $E = (E \cap A_k) \cup (E \setminus A_k)$, so

$$\limsup_{x\to 0^+} \frac{m(E \cap [-x, x])}{2x\psi_1(2x)}$$
$$\leq \limsup_{x\to 0^+} \frac{m(E \cap A_k \cap [-x, x])}{2x\psi_1(2x)} + \limsup_{x\to 0^+} \frac{m((E \setminus A_k) \cap [-x, x])}{2x\psi_1(2x)}.$$

We shall prove that the second term is equal to zero. Let $x > 0$. Consider two cases:

1^0 $x \in A_k$.

Put $t(x) = \max([0, x] \cap A_k')$. Obviously, $t(x) \notin A_k$, $\lim_{x\to 0^+} t(x) = 0$ and

$$(E \setminus A_k) \cap [-x, x] = (E \setminus A_k) \cap [-t(x), t(x)],$$

so

$$\frac{m((E \setminus A_k) \cap [-x, x])}{2x\psi_1(2x)} \leq \frac{m((E \setminus A_k) \cap [-t(x), t(x)])}{2t(x)\psi_1(2t(x))}$$
$$\leq \frac{k\, m((E \setminus A_k) \cap [-t(x), t(x)])}{2t(x)\psi_2(2t(x))}$$
$$\leq \frac{k\, m(E \cap [-t(x), t(x)])}{2t(x)\psi_2(2t(x))}$$

because $t(x) \notin A_k$, and consequently, $\psi_1(2t(x)) \geq \psi_2(2t(x)) / k$.

2^0 $x \notin A_k$. Then

$$\frac{m((E \setminus A_k) \cap [-x,\, x])}{2x\psi_1(2x)} \leq \frac{k\, m((E \setminus A_k) \cap [-x,\, x])}{2x\psi_2(2x)} \leq \frac{k\, m(E \cap [-x,\, x])}{2x\psi_2(2x)}.$$

Consequently, from our assumption we have

$$\lim_{x \to 0^+} \frac{m((E \setminus A_k) \cap [-x,\, x])}{2x\psi_1(2x)} = 0. \qquad \square$$

Corollary 2. *Under the assumptions of Lemma* 1 *we have*

$$\limsup_{x \to 0^+} \frac{m(E \cap [-x,\, x])}{2x\psi_1(2x)} \leq \limsup_{x \to 0^+} \frac{m(A_k \cap [-x,\, x])}{2x\psi_1(2x)}.$$

Theorem 3. *Let $\psi_1, \psi_2 \in \mathcal{C}$ and $\varepsilon_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x,\, x])}{2x\psi_1(2x)}$ for $k \in \mathbb{N}$. If $\lim_{k \to \infty} \varepsilon_k = 0$ and 0 is a ψ_2-dispersion point of a set $E \in S$, then 0 is a ψ_1-dispersion point of E.*

Proof. By Corollary 2 for each $k \in \mathbb{N}$ we have

$$0 \leq \limsup_{x \to 0^+} \frac{m(E \cap [-x,\, x])}{2x\psi_1(2x)} \leq \varepsilon_k.$$

Consequently, 0 is a ψ_1-dispersion point of E. $\square$

Corollary 4. *Under the assumptions of Theorem* 3 $\mathcal{T}_{\psi_2} \subset \mathcal{T}_{\psi_1}$.

Corollary 5. *Let $\psi_1, \psi_2 \in \mathcal{C}$,*

$$\varepsilon_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x,\, x])}{2x\psi_1(2x)} \text{ and } \eta_k = \limsup_{x \to 0^+} \frac{m(B_k \cap [-x,\, x])}{2x\psi_2(2x)}.$$

If $\lim_{k \to \infty} \varepsilon_k = \lim_{k \to \infty} \eta_k = 0$, then $\mathcal{T}_{\psi_1} = \mathcal{T}_{\psi_2}$.

The proof follows immediately from Corollary 4. $\square$

Obviously, $A^+_{k+1} \subset A^+_k$ and $A_{k+1} \subset A_k$, so $\varepsilon_{k+1} \leq \varepsilon_k$ for $k \in \mathbb{N}$. Consequently, the sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ decreasingly tends to zero or to some positive number.

Theorem 6. *Let $\psi_1, \psi_2 \in \mathcal{C}$ and $\varepsilon_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x,\, x])}{2x\psi_1(2x)}$ for $k \in \mathbb{N}$. If $\lim_{k \to \infty} \varepsilon_k > 0$, then there exists a measurable set $E \subset \mathbb{R}$ such that 0 is a ψ_2-dispersion point of E, but it is not a ψ_1-dispersion point of E.*

PROOF. If $\lim_{x\to 0^+} \psi_1(x) / \psi_2(x) = 0$, then the existence of a measurable set $E \subset \mathbb{R}$ fulfilling our theorem follows from Theorem 2.6 in [TW-B]. So, we can assume now that $\limsup_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} > 0$. Hence there exists $k_0 \in \mathbb{N}$ such that for each $\delta > 0$ there exists a point $x \in (0, \delta)$ such that $\frac{\psi_1(2x)}{\psi_2(2x)} \geq \frac{1}{k_0}$. Clearly, $x \in (0, \delta) \setminus A_{k_0}$. The sequence $\{A_k\}_{k\in\mathbb{N}}$ is decreasing, so there exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ and each $\delta > 0$ there exists a point x such that

$$x \in (0, \delta) \setminus A_k. \tag{1}$$

Put $a = \lim_{i\to\infty} \varepsilon_i / 2$. Then $a > 0$ and $\varepsilon_i \geq 2a$ for $i \in \mathbb{N}$; i.e.

$$\limsup_{x\to 0^+} \frac{m(A_i \cap [-x, x])}{2x\psi_1(2x)} \geq 2a$$

for $i \in \mathbb{N}$. Hence for each $i \in \mathbb{N}$ there exists a sequence of points $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ such that $x_n^{(i)} \searrow_{n\to\infty} 0$ and

$$\frac{m(A_{k_0+i} \cap [-x_n^{(i)}, x_n^{(i)}])}{2x_n^{(i)}\psi_1(2x_n^{(i)})} > a \tag{2}$$

for $i \in \mathbb{N}$.

Consider the interval $[0, x_1^{(1)}]$. From (1) it follows that A_{k_0+1} does not contain any right-hand neighborhood of 0. Let $y_1 \in (0, x_1^{(1)})$ be the left end-point of some component interval of A_{k_0+1} such that $\frac{m(A_{k_0+1}\cap[y_1, x_1^{(1)}])}{x_1^{(1)}\psi_1(2x_1^{(1)})} > a$. Put $f(t) = \frac{m(A_{k_0+1}\cap[y_1, t])}{t\psi_1(2t)}$ for $t \in [y_1, x_1^{(1)}]$. Clearly, f is continuous on $[y_1, x_1^{(1)}]$, $f(y_1) = 0$ and $f(x_1^{(1)}) > a$. Let $t_1 = \inf([y_1, x_1^{(1)}] \cap f^{-1}(\{a\}))$. Obviously, $f(t) < a$ for $t \in [y_1, t_1)$ and $f(t_1) = a$.

Put $E_1 = A_{k_0+1} \cap [y_1, t_1]$. Suppose now that the sets $E_1, E_2, \ldots, E_{i-1}$ are defined. Let $z_{i-1} \in \mathbb{R}_+$ be a number such that

$$z_{i-1} < \min(y_{i-1}, a\, y_{i-1}\psi_1(2y_{i-1})). \tag{3}$$

Consider the sequence $\{x_n^{(i)}\}_{n\in\mathbb{N}}$. There exists $n_i \in \mathbb{N}$ such that $x_{n_i}^{(i)} < z_{i-1}$. Obviously, by (2) we have $\frac{m(A_{k_0+i}\cap[-x_{n_i}^{(i)}, x_{n_i}^{(i)}])}{2x_{n_i}^{(i)}\psi_1(2x_{n_i}^{(i)})} > a$. From (1) it follows that A_{k_0+i} does not contain any right-hand neighborhood of 0. Let $y_i \in (0, x_{n_i}^{(i)})$ be the left end-point of some component interval of A_{k_0+i} such that

$$\frac{m(A_{k_0+i} \cap [y_i, x_{n_i}^{(i)}])}{x_{n_i}^{(i)}\psi_1(2x_{n_i}^{(i)})} > a.$$

Analogously as earlier there exists a point $t_i \in (y_i,\, x^{(i)}_{n_i})$ such that

$$\frac{m(A_{k_0+i} \cap [y_i,\, t])}{t\psi_1(2t)} < a$$

for $t \in [y_i,\, t_i)$ and

$$\frac{m(A_{k_0+i} \cap [y_i,\, t_i])}{t_i\psi_1(2t_i)} = a. \tag{4}$$

Put $E_i = A_{k_0+i} \cap [y_i,\, t_i]$. In such a way we have defined the sequence of measurable sets $\{E_i\}_{i\in\mathbb{N}}$. Now let $E = \bigcup_{i=1}^{\infty}(E_i \cup (-E_i))$. Observe that 0 is not a ψ_1-dispersion point of E since from (4) we obtain

$$\frac{m(E \cap [-t_i,\, t_i])}{2t_i\psi_1(2t_i)} \geq \frac{2m(A_{k_0+i} \cap [y_i,\, t_i])}{2t_i\psi_1(2t_i)} = a$$

for $i \in \mathbb{N}$.

Now we shall prove that 0 is a ψ_2-dispersion point of E. For this purpose we shall show that

$$\frac{m(E \cap [-x,\, x])}{2x\psi_1(2x)} < 2a \tag{5}$$

for $x \in (0,\, y_1)$.

Let $x \in (0,\, y_1)$. Then there exists $i \in \mathbb{N}$ such that $x \in [y_i,\, y_{i-1})$. Obviously, $t_{i+1} < x^{(i+1)}_{n_{i+1}} < z_i$, so by virtue of (3) and (4) we obtain

$$\begin{aligned}
\frac{m(E \cap [-x,\, x])}{2x\psi_1(2x)} &= \frac{m\big(\bigcup_{p=i+1}^{\infty}(E_p \cup (-E_p)) \cup (E_i \cup (-E_i)) \cap [-x,\, x]\big)}{2x\psi_1(2x)} \\
&\leq \frac{2m(0,\, t_{i+1}])}{2x\psi_1(2x)} + \frac{2m(E_i \cap [-x,\, x])}{2x\psi_1(2x)} \\
&\leq \frac{2z_i}{2y_i\psi_1(2y_i)} + \frac{2m(A_{k_0+i} \cap [y_i,\, t_i] \cap [y_i,\, x])}{2x\psi_1(2x)} < 2a.
\end{aligned}$$

Let $x \in [y_i,\, y_{i-1})$. Consider two cases:

1^0 $x \in \overline{A}_{k_0+i}$.

Then $\frac{\psi_1(2x)}{\psi_2(2x)} \leq \frac{1}{k_0+i}$, so by virtue of (5) we obtain

$$\frac{m(E \cap [-x,\, x])}{2x\psi_2(2x)} = \frac{m(E \cap [-x,\, x])}{2x\psi_1(2x)} \cdot \frac{\psi_1(2x)}{\psi_2(2x)} < \frac{2a}{k_0+i}.$$

2^0 $x \notin \overline{A}_{k_0+i}$.

Put $t(x) = \sup(\overline{A}_{k_0+i} \cap [y_i, x])$. Then $t(x) \in \overline{A}_{k_0+i}$, so by 1^0 we have

$$\frac{m(E \cap [-x, x])}{2x\psi_2(2x)} \le \frac{m(E \cap [-t(x), t(x)])}{2t(x)\psi_2(2t(x))} < \frac{2a}{k_0 + i}.$$

If $x \to 0^+$, then $i \to \infty$. Consequently, $\lim_{x\to 0^+} \frac{m(E\cap[-x, x])}{2x\psi_2(2x)} = 0$. □

Corollary 7. *Under the assumptions of the last theorem* $\mathcal{T}_{\psi_2} \setminus \mathcal{T}_{\psi_1} \neq \emptyset$.

Proof. It is clear that if we put $F_i = \overline{A}_{k_0+i} \cap [y_i, t_i]$ for $i \in \mathbb{N}$ (where A_{k_0+i} and $[y_i, t_i]$ are the same as in the proof of Theorem 6), then the sets F_i, $i \in \mathbb{N}$, are closed in the Euclidean topology and $m(E_i \cap I) = m(F_i \cap I)$ for each interval I. Consequently $\mathbb{R} \setminus \left(\bigcup_{i=1}^{\infty}(F_i \cup (-F_i)\right) \in \mathcal{T}_{\psi_2} \setminus \mathcal{T}_{\psi_1}$. □

Theorem 8. *Let* $\psi_1, \psi_2 \in \mathcal{C}$,

$$\varepsilon_k = \limsup_{x\to 0^+} \frac{m(A_k \cap [-x, x])}{2x\psi_1(2x)} \text{ and } \eta_k = \limsup_{x\to 0^+} \frac{m(B_k \cap [-x, x])}{2x\psi_2(2x)}.$$

The topologies $\mathcal{T}_{\psi_1}$ *and* $\mathcal{T}_{\psi_2}$ *are equal if and only if* $\lim_{k\to\infty} \varepsilon_k = \lim_{k\to\infty} \eta_k = 0$.

The proof follows immediately from Corollary 5 and Corollary 7. □

It is easy to see that we can consider the arbitrary increasing sequence $\{a_k\}_{k\in\mathbb{N}}$ of positive numbers tending to infinity instead of the sequence of positive integers in the definitions of A_k^+, B_k^+, A_k and B_k.

In the family $\mathcal{A} = \{\mathcal{T}_\psi : \psi \in \mathcal{C}\}$ of all ψ-density topologies generated by functions from $\mathcal{C}$ we can introduce the partial order using the inclusion relation. From the proof of Theorem 2.12 in [TW-B] it follows that for arbitrary sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of functions from $\mathcal{C}$ there exists a function $\psi \in \mathcal{C}$ such that $\lim_{x\to 0^+} \frac{\psi_n(x)}{\psi(x)} = 0$ for $n \in \mathbb{N}$. Consequently, $\mathcal{T}_{\psi_n} \subsetneq \mathcal{T}_\psi$ for $n \in \mathbb{N}$ by Theorem 2.6 in [TW-B]. So for each countable subset of $\mathcal{A}$ there exists the upper bound of this set in $\mathcal{A}$.

We shall prove that this partial order in $\mathcal{A}$ is dense.

Theorem 9. *For arbitrary* $\psi_1, \psi_2 \in \mathcal{C}$ *such that* $\psi_1(x) \le \psi_2(x)$ *for* $x \in \mathbb{R}_+$ *and* $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_2}$ *there exits a function* $\psi_3 \in \mathcal{C}$ *such that* $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_3} \subsetneq \mathcal{T}_{\psi_2}$.

Proof. Let $\psi_1, \psi_2 \in \mathcal{C}$, $\psi_1(x) \le \psi_2(x)$ for $x \in \mathbb{R}_+$ and

$$\varepsilon_k = \limsup_{x\to 0^+} \frac{m(A_k \cap [-x, x])}{2x\psi_1(2x)}$$

for $k \in \mathbb{N}$. From Corollary 4 it follows that $\lim_{k\to\infty} \varepsilon_k > 0$, because $\mathcal{T}_{\psi_2} \not\subset \mathcal{T}_{\psi_1}$. Let $\varepsilon \in (0, \lim_{k\to\infty} \varepsilon_k)$. Then for arbitrary $\eta > 0$ and each $k \in \mathbb{N}$ there exists a point $\overline{x} \in (0, \eta)$ such that

$$\frac{m(A_k \cap [-\overline{x}, \overline{x}])}{2\overline{x}\psi_1(2\overline{x})} > \varepsilon. \tag{6}$$

If $\lim_{x\to0^+} \psi_1(x) / \psi_2(x) = 0$, then put $\psi_3(x) = \sqrt{\psi_1(x)\psi_2(x)}$ for $x \in \mathbb{R}_+$. Clearly, $\psi_1(x) \le \psi_3(x) \le \psi_2(x)$ and $\lim_{x\to0^+} \frac{\psi_1(x)}{\psi_3(x)} = \lim_{x\to0^+} \frac{\psi_3(x)}{\psi_2(x)} = 0$, so from Theorem 2.6 in [TW-B] we have $\mathcal{T}_{\psi_1} \subsetneqq \mathcal{T}_{\psi_3} \subsetneqq \mathcal{T}_{\psi_2}$. Suppose now that $\limsup_{x\to0^+} \frac{\psi_1(x)}{\psi_2(x)} > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $\limsup_{x\to0^+} \frac{\psi_1(x)}{\psi_2(x)} > \frac{1}{k_0}$. Consequently, if $k \ge k_0$, then the open set A_k does not contain any right-hand neighborhood of 0.

Now put $t_0 = \eta = 1$ and $k = k_0$. By (6) there exists a point $a_1 \in (0, \eta)$ such that $\frac{m(A_{k_0}\cap[-a_1, a_1])}{2a_1\psi_1(2a_1)} > \varepsilon$. It is easy to see (analogously as in Theorem 6) that there exists a point $a_2 \in (0, a_1)$ such that

$$\frac{m(A_{k_0} \cap ([-a_1, -a_2] \cup [a_2, a_1]))}{2a_1\psi_1(2a_1)} > \varepsilon. \tag{7}$$

Since $\lim_{x\to0^+} \psi_2(x) = 0$ and $\psi_1(2a_2) \le \psi_2(2a_2)$, so there exists a point $z_1 \in (0, a_2)$ such that $\psi_2(2z_1) = \psi_1(2a_2)$. Now let $\eta = z_1$ and $k = k_0$. From (6) there exists a point $b_1 \in (0, z_1)$ such that $\frac{m(A_{k_0}\cap[-b_1, b_1])}{2b_1\psi_1(2b_1)} > \varepsilon$ and $b_2 \in (0, b_1)$ such that

$$\frac{m(A_{k_0} \cap ([-b_1, -b_2] \cup [b_2, b_1]))}{2b_1\psi_1(2b_1)} > \varepsilon. \tag{8}$$

Since $\psi_1(x) \le \psi_2(x)$ for $x \in \mathbb{R}_+$, let t_1 be an arbitrary point from the interval $(0, b_2)$. Obviously, $\psi_1(2t_1) \le \psi_2(2b_2)$. Now we are proceeding by induction. Suppose that the points $t_0, a_1, a_2, z_1, b_1, b_2, t_1, \ldots, a_{2i-1}, a_{2i}, z_i, b_{2i-1}, b_{2i}, t_i$ are defined in such a way that

$$\frac{m(A_{k_0+j} \cap ([-a_{2j+1}, -a_{2j+2}] \cup [a_{2j+2}, a_{2j+1}]))}{2a_{2j+1}\psi_1(2a_{2j+1})} > \varepsilon, \tag{9}$$

$$\frac{m(A_{k_0+j} \cap ([-b_{2j+1}, -b_{2j+2}] \cup [b_{2j+2}, b_{2j+1}]))}{2b_{2j+1}\psi_1(2b_{2j+1})} > \varepsilon \tag{10}$$

and $\psi_2(2z_{j+1}) = \psi_1(2a_{2j+2})$ for $j = 0, 1, \ldots, i-1$. Then we also have $\psi_1(2t_{j+1}) \le \psi_2(2b_{2j+2})$ for $j = 0, 1, \ldots, i-1$.

Now put $\eta = t_i$ and $k = k_0 + i$. Using (6), analogously as earlier, we find the points $a_{2i+1}, a_{2i+2}, z_{i+1}, b_{2i+1}, b_{2i+2}, t_{i+1}$ such that

$$0 < t_{i+1} < b_{2i+2} < b_{2i+1} < z_{i+1} < a_{2i+2} < a_{2i+1} < t_i,$$

$$\frac{m(A_{k_0+i} \cap ([-a_{2i+1},\ -a_{2i+2}] \cup [a_{2i+2},\ a_{2i+1}]))}{2a_{2i+1}\psi_1(2a_{2i+1})} > \varepsilon, \tag{11}$$

$$\frac{m(A_{k_0+i} \cap ([-b_{2i+1},\ -b_{2i+2}] \cup [b_{2i+2},\ b_{2i+1}]))}{2b_{2i+1}\psi_1(2b_{2i+1})} > \varepsilon, \tag{12}$$

$$\psi_2(2z_{i+1}) = \psi_1(2a_{2i+2}) \text{ and } \psi_1(2t_{i+1}) \le \psi_2(2b_{2i+2}).$$

Now we define a function ψ_3 as follows:

$$\psi_3(2x) = \begin{cases} \psi_1(2x) & \text{if } 2x \ge 2t_0 = 2 \\ \psi_1(2x) & \text{if } 2x \in [2a_{2i+2}, 2t_i],\ \text{for } i = 0,1,2,\dots \\ \psi_2(2x) & \text{if } 2x \in [2b_{2i+2}, 2z_{i+1}],\ \text{for } i = 0,1,2,\dots \\ \text{linear} & \text{on the intervals } [2z_{i+1}, 2a_{2i+2}] \\ & \text{and on } [2t_{i+1}, 2b_{2i+2}],\ \text{for } i = 0,1,2,\dots. \end{cases}$$

Obviously, $\psi_3 \in \mathcal{C}$.

Put

$$C^+_{k_0+i} = \{x \in \mathbb{R}_+ : \psi_3(2x) < \frac{1}{k_0+i}\psi_2(2x)\},$$
$$D^+_{k_0+i} = \{x \in \mathbb{R}_+ : \psi_1(2x) < \frac{1}{k_0+i}\psi_3(2x)\},$$

$$C_{k_0+i} = C^+_{k_0+i} \cup (-C^+_{k_0+i}), \quad D_{k_0+i} = D^+_{k_0+i} \cup (-D^+_{k_0+i}), \quad i = 0,1,2,\dots.$$

Clearly,

$$C^+_{k_0+i} \supset \bigcup_{j=k_0+1}^{\infty} (A_j \cap [2a_{2i+2}, 2a_{2i+1}])$$
$$D^+_{k_0+i} \supset \bigcup_{j=k_0+1}^{\infty} (A_j \cap [2b_{2i+2}, 2b_{2i+1}])$$

for $i = 0,1,2,\dots$.

So, from (11) for $k \ge k_0$ we have

$$\limsup_{x\to 0^+} \frac{m(C_k \cap [-x,x])}{2x\psi_3(2x)} \ge \limsup_{i\to\infty} \frac{m(A_k \cap [-a_{2i+1}, a_{2i+1}])}{2a_{2i+1}\psi_1(2a_{2i+1})} \ge \varepsilon,$$

since $\psi_3(2a_{2i+1}) = \psi_1(2a_{2i+1})$.

Analogously, from (12) for $k \geq k_0$

$$\limsup_{x\to 0^+} \frac{m(D_k \cap [-x,x])}{2x\psi_1(2x)} \geq \limsup_{i\to\infty} \frac{m(A_k \cap [-b_{2i+1}, b_{2i+1}])}{2b_{2i+1}\psi_1(2b_{2i+1})} \geq \varepsilon.$$

Put $\xi_k = \limsup_{x\to 0^+} \frac{m(C_k\cap[-x,x])}{2x\psi_3(2x)}$ for $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \xi_k \geq \varepsilon$, so from Corollary 7 $\mathcal{T}_{\psi_2} \setminus \mathcal{T}_{\psi_3} \neq \emptyset$. Analogously, $\mathcal{T}_{\psi_3} \setminus \mathcal{T}_{\psi_1} \neq \emptyset$. Simultaneously, $\psi_1(x) \leq \psi_3(x) \leq \psi_2(x)$ for $x \in \mathbb{R}_+$, so $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_3} \subsetneq \mathcal{T}_{\psi_2}$. □

Corollary 10. *If $\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_2}$, then there exits a function $\psi_3 \in \mathcal{C}$ such that*

$$\mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_3} \subsetneq \mathcal{T}_{\psi_2}.$$

Proof. Put $\psi_0 = \min(\psi_1, \psi_2)$. Obviously, $\psi_0 \in \mathcal{C}$. We shall prove that $\mathcal{T}_{\psi_0} = \mathcal{T}_{\psi_1}$ Clearly, $\psi_0(x) \leq \psi_1(x)$ for $x \in \mathbb{R}_+$, so $\mathcal{T}_{\psi_0} \subset \mathcal{T}_{\psi_1}$. Now, let

$$C_k^+ = \big\{x \in \mathbb{R}_+ : \psi_0(2x) < \frac{1}{k}\psi_1(2x)\big\},\ C_k = C_k^+ \cup (-C_k^+)$$

and

$$\xi_k = \limsup_{x\to 0^+} \frac{m(C_k \cap [-x,\, x])}{2x\psi_0(2x)}$$

for $k \in \mathbb{N}$, where the sets B_k were defined before Lemma 1. For our purpose it is sufficient to prove that $\lim_{k\to\infty} \xi_k = 0$ (by virtue of Corollary 4).

Obviously, $C_k^+ = B_k^+$ and $C_k = B_k$ for $k \in \mathbb{N}$.

Let $x > 0$. Consider two cases:

1^0 $x \in \overline{C}_k^+$. Then $\psi_0(2x) = \psi_2(2x)$ and

$$\frac{m(C_k \cap [-x,\, x])}{2x\psi_0(2x)} = \frac{m(B_k \cap [-x,\, x])}{2x\psi_2(2x)}.$$

2^0 $x \notin \overline{C}_k^+$. Put $t(x) = \max([0,x] \cap C_k^+)$. Clearly, $t(x) \in \bar{C}_k^+$ and $t(x) < x$, so

$$\frac{m(C_k \cap [-x,\, x])}{2x\psi_0(2x)} \leq \frac{m(B_k \cap [-t(x),\, t(x)])}{2t(x)\psi_2(2t(x))}.$$

Therefore

$$\xi_k = \limsup_{x\to 0^+} \frac{m(C_k \cap [-x,\, x])}{2x\psi_0(2x)} \leq \limsup_{x\to 0^+} \frac{m(B_k \cap [-x,\, x])}{2x\psi_2(2x)} = \eta_k$$

for $k \in \mathbb{N}$. Since $\mathcal{T}_{\psi_1} \subset \mathcal{T}_{\psi_2}$, so $\lim_{k\to\infty} \eta_k = 0$ by Corollary 7. Consequently, $\lim_{k\to\infty} \xi_k = 0$ and by Corollary 4 $\mathcal{T}_{\psi_1} \subset \mathcal{T}_{\psi_0}$.

Simultaneously, $\psi_0(x) \leq \psi_2(x)$ for $x \in \mathbb{R}_+$ and $\mathcal{T}_{\psi_0} = \mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_2}$; so from Theorem 9 it follows that there exists a function $\psi_3 \in \mathcal{C}$ such that $\mathcal{T}_{\psi_0} = \mathcal{T}_{\psi_1} \subsetneq \mathcal{T}_{\psi_3} \subsetneq \mathcal{T}_{\psi_2}$. □

Theorem 11. *There exists subfamily $\mathcal{C}_0 \subset \mathcal{C}$ such that* $\mathrm{card}(\mathcal{C}_0) = \mathfrak{c}$ *and for each $\psi_1, \psi_2 \in \mathcal{C}_0$, $\psi_1 \neq \psi_2$, the topologies $\mathcal{T}_{\psi_1}$ and $\mathcal{T}_{\psi_2}$ are not comparable by inclusion.*

Proof. First we construct two auxiliary sequences of intervals $\{(a_k, b_k)\}_{k=2}^{\infty}$ and $\{(c_k, d_k)\}_{k=2}^{\infty}$. Put $b_2 = 1$. For $k \geq 2$ let

$$d_k = \frac{1}{k} b_k,\ c_k = \frac{1}{2} d_k = \frac{1}{2k} b_k,\ a_k = \frac{1}{2k+1} b_k \text{ and } b_{k+1} = \frac{1}{2k+2} b_k.$$

Obviously, $(c_k, d_k) \subset (a_k, b_k)$ and $b_{k+1} < a_k$ for $k \in \mathbb{N}$. Now for $k \in \mathbb{N}$ let

$$\phi_k(2x) = \begin{cases} b_k & \text{if } x \in [c_k, b_k], \\ x & \text{if } x \in (0, a_k] \cup [b_k, \infty), \\ \text{linear} & \text{on the interval } [a_k, c_k]. \end{cases}$$

Let $\{N_t\}_{t \in T}$ be the family of subsets of $\mathbb{N}$ such that

1. $\mathrm{card}(T) = \mathfrak{c}$;
2. $\mathrm{card}(N_t) = \chi_0$ for $t \in T$;
3. $\mathrm{card}(N_{t_1} \cap N_{t_2}) < \chi_0$ if $t_1, t_2 \in T$, $t_1 \neq t_2$.

Such a family does exist (see for example [L], prop. 5.26, p. 193). Usually it is called a family of almost disjoint sets.

For each $t \in T$ we define a function ψ_t as follows:

$$\psi_t(x) = \begin{cases} \varphi_k(x) & \text{if } k \in N_t \text{ and } x \in [a_k, b_k], \\ x & \text{if } x \in \mathbb{R}_+ \setminus \bigcup_{k \in N_t} [a_k, b_k]. \end{cases}$$

Obviously, ψ_t is continuous, nondecreasing and $\lim_{x \to 0^+} \psi_t(x) = 0$, so $\psi_t \in \mathcal{C}$ for $t \in T$. Let $t_1, t_2 \in T$, $t_1 \neq t_2$. We shall prove that the topologies $\mathcal{T}_{\psi_{t_1}}$ and $\mathcal{T}_{\psi_{t_2}}$ are not comparable by inclusion. From the third property of $\{N_t\}_{t \in T}$ we have $\mathrm{card}(N_{t_1} \cap N_{t_2}) < \chi_0$. Simultaneously, $\mathrm{card}(N_{t_1} \setminus N_{t_2}) = \mathrm{card}(N_{t_2} \setminus N_{t_1}) = \chi_0$.

Let $k \in N_{t_1} \setminus N_{t_2}$. Then $\psi_{t_1}(x) = \varphi_k(x) = b_k$ for $x \in [c_k, d_k]$ and $\psi_{t_2}(x) = x < d_k = \frac{1}{k} b_k$ for $x \in (c_k, d_k)$. Put $P_k = \{n \geq k : n \in N_{t_1} \setminus N_{t_2}\}$. Then

$$\left\{x \in \mathbb{R}_+ : \psi_{t_2}(2x) < \frac{1}{k} \psi_{t_1}(2x)\right\} \supset \bigcup_{n \in P_k} \left(\frac{c_n}{2}, \frac{d_n}{2}\right).$$

Let

$$C_k = \left\{x \in \mathbb{R}_+ : \psi_{t_2}(2x) < \frac{1}{k} \psi_{t_1}(2x)\right\}$$

and

$$\xi_k = \limsup_{x\to 0^+} \frac{m(C_k \cap [-x, x])}{2x\psi_{t_2}(2x)}$$

for $k \in \mathbb{N}$. Let $\{n_p\}_{p\in N}$ be the increasing sequence of all numbers of P_k. Then

$$\xi_k \geq \limsup_{l\to\infty} \frac{m\big(\big(\bigcup_{p=l}^{\infty} \big(\frac{c_{n_p}}{2}, \frac{d_{n_p}}{2}\big)\big) \cap \big[-\frac{d_{n_l}}{2}, \frac{d_{n_l}}{2}\big]\big)}{d_{n_l}\psi_{t_2}(d_{n_l})}$$
$$\geq \limsup_{l\to\infty} \frac{d_{n_l} - c_{n_l}}{2d_{n_l}\psi_{t_2}(d_{n_l})} = \limsup_{l\to\infty} \frac{\frac{1}{2}d_{n_l}}{2d_{n_l}\psi_{t_2}(d_{n_l})} = \infty,$$

since $d_{n_l} \underset{l\to\infty}{\longrightarrow} 0$ and $\lim_{x\to 0^+} \psi_{t_2}(x) = 0$. From Corollary 7 $\mathcal{T}_{\psi_{t_1}} \not\subset \mathcal{T}_{\psi_{t_2}}$. Analogously, we can prove that $\mathcal{T}_{\psi_{t_2}} \not\subset \mathcal{T}_{\psi_{t_1}}$. □

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