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SUPERDENSE A.E. UNBOUNDED DIVERGENCE IN SOME APPROXIMATION PROCESSES OF ANALYSIS

Abstract

The paper deals with divergence phenomena for various approximation processes of analysis such as Fourier series, Lagrange interpolation, Walsh-Fourier series. We prove the existence of superdense (meaning residual, dense and uncountable) families of functions in appropriate function spaces over an interval $T \subset \mathbb{R}$. One proves that for each function in the family, the corresponding approximation process is unboundedly divergent on a superdense subset of T of full measure.

1 Introduction

A subset T_0 of a topological space T is called *superdense* in T if it is residual (i.e. its complement is of first Baire category) uncountable and dense in T. A general principle of double condensation of singularities proved in [3] was applied to obtain superdense unbounded divergence for superdense families of functions in various approximation processes of analysis as Lagrange interpolation, Fourier series, Walsh-Fourier series, quadrature formulae etc. (see [2], [3], [10]). A good account of convergence and divergence phenomena in these approximation processes is given in the survey paper [11].

The general framework of all of these results is as follows. Let $X = X(T)$ be a Banach space of scalar functions defined on an interval $T \subset \mathbb{R}$. For a

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*Professor Muntean passed away in August 1996, shortly after this work was completed sequence $A_n : X \to X,\ n \in \mathbb{N}$, of continuous linear operators and $x \in X$ let

$$UD(x) = \{t \in T : \sup_n |A_n x(t)| = \infty\}$$

be the *set of unbounded divergence* of the sequence $(A_n x)$. By a *double condensation of singularities* for the sequence (A_n) we understand the existence of a superdense subset X_0 of X such that $UD(x)$ is superdense in T for every $x \in X_0$, meaning unbounded divergence on large subsets of T in the topological sense. There are also well known results emphasizing unbounded divergence on large sets with respect to Lebesgue measure. The first ones of this kind are the famous examples of A. N. Kolmogorov [7], [8] (see also [12]) of Lebesgue integrable functions with Fourier series unboundedly divergent a.e. on T, respectively on the whole $T,\ T = [0, 2\pi]$. The aim of the present paper is to prove a principle of triple condensation of singularities emphasizing the existence of a superdense subset X_0 of X such that $UD(x)$ is superdense in T and of full measure, for every $x \in X_0$.

2 Preliminary Results

Concerning the existence of superdense subsets we mention the following result.

Theorem 2.1. *[1, Th. 2.2] If T is a T_1-separated topological space without isolated points, then every residual subset of T is superdense in T.*

Remark. If T is a Baire space, then every intersection of dense open subsets of T is residual and dense in T. Conversely, every residual subset of T contains a dense $G_\delta-$subset of T.

Now let $(T, \mathcal{A}, \mu)$ be a positive measure space; i.e., T is a nonvoid set, $\mathcal{A}$ is a σ-algebra of subsets of T and $\mu : \mathcal{A} \to [0, \infty]$ a σ-additive measure on $\mathcal{A}$. Denote by $S(T)$ the vector space of all equivalence classes (with respect to equality μ-a.e.) of measurable μ-a.e. finite functions defined on T and taking values in $\overline{K} = \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ or $\overline{K} = \overline{\mathbb{R}} + i\overline{\mathbb{R}}$. For $x, y, z \in S(T)$ denote by the corresponding Greek characters ξ, η, ζ finite functions belonging to these classes, respectively. For a measurable function $\xi : T \to \overline{\mathbb{R}}$ and $\alpha \in \mathbb{R}$ let $\{\xi > \alpha\} = \{t \in T : \xi(t) > \alpha\}$. The sets $\{\xi \geq \alpha\}, \{\xi = \infty\}$ etc. are defined similarly. Since the measures of these sets depend only on the equivalence class x generated by the function ξ we can use the notation $\mu\{x > \alpha\} = \mu\{\xi > \alpha\}$ etc. Let also $|x|$ denote the class generated by the function $|\xi|$.

A sequence (x_n) in $S(T)$ is called μ-*convergent* (or *convergent in measure*) to $x \in S(T)$ provided $\lim_n \mu\{|x_n - x| \geq \epsilon\} = 0$ for every $\epsilon > 0$.

A mapping B from a metric space X to $S(T)$ is called *μ-continuous* at $x \in X$ if (Bx_n) is μ-convergent to Bx for every sequence (x_n) in X converging to x. The mapping B is called *μ-continuous* on X if it is continuous at each $x \in X$.

An example of a μ-continuous linear operator is given in the following proposition. As usually, for $1 \le p < \infty$, denote by $L^p(T) = L^p(T, \mathcal{A}, \mu)$ the Banach space of all equivalence classes of $p-$integrable functions, normed by

$$\|x\|_p = \left(\int_T |\xi(t)|^p d\mu(t) \right)^{1/p}$$

for $x \in L^p(T)$ and $\xi \in x$. Since any $p-$integrable function is μ-a.e. finite, it follows that $L^p(T)$ is a subspace of $S(T)$. Moreover we have the following.

Proposition 2.2. *([9, Th. 0.18]) The canonical embedding operator $J : L^p(T) \to S(T)$, defined by $Jx = x$, is linear and μ-continuous.*

PROOF. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $L^p(T)$ converging to $x \in L^p(T)$ with respect to the p-norm. Choose finite functions $\xi_k \in x_k,\ k \in \mathbb{N}$, and $\xi \in x$ and let

$$T_\epsilon^k = \{|\xi_k - \xi| \ge \epsilon\} = \{|\xi_k - \xi|^p \ge \epsilon^p\}.$$

Then

$$\|x_k - x\|_p^p \ge \int_{T_\epsilon^k} |\xi_k(t) - \xi(t)|^p d\mu(t) \ge \epsilon^p \cdot \mu(T_\epsilon^k),$$

implying that $\lim_{k\to\infty} \mu(T_\epsilon^k) = 0$, for every $\epsilon > 0$; i.e., the sequence (Jx_k) converges in measure to Jx. □

Remark. Usually, measurable functions and classes of measurable functions are identified. This identification causes no measure theoretical troubles, but the topological properties of sets defined by measurable functions may change when passing from a function to a μ-equivalent one. For this reason we have to distinct functions from classes of measurable functions. The same caution is taken in [9], too.

If (y_n) is a sequence in $S(T)$, then $y^* = \sup_n |y_n|$ is defined as the class generated by the function $\eta^* = \sup_n |\eta_n|$, where $\eta_n \in y_n,\ n \in \mathbb{N},\ \mathbb{N} = \{1, 2, \dots\}$. Note that the $\mathcal{A}$-measurable function η^* may not be μ-a.e. finite, but its equivalence class does not depend on the particular choice of the functions η_n. For a sequence $A_n : X \to S(T),\ n \in \mathbb{N}$, of μ-continuous mappings, define by $A^*x = \sup_n |A_n x|$, the *maximal operator* associated to the sequence $(A_n x)$.

The following result of A. del Junco and J. Rosenblatt [6] will be essential in the proof of our main result (Theorem 3.1 below).

Theorem 2.3. *([6, Th. 1.1]) Let X be a real normed space which is a Baire space and B_X its closed unit ball. Also let $(T, \mathcal{A}, \mu)$ be a positive measure space. Suppose that $A_n : X \to S(T)$, $n \in \mathbb{N}$, is a sequence of μ-continuous linear mappings satisfying*

$$(JR) \quad \forall \epsilon > 0 \quad \forall M > 0 \quad \exists x_0 \in B_X \text{ such that } \mu\{A^* x > M\} \geq \mu(T) - \epsilon.$$

Then there exists a superdense subset X_0 of X such that for every $x \in X_0$

$$\mu\{A^* x = \infty\} = \mu(T). \tag{2.1}$$

We shall need also the following result of W. Orlicz [13] (also see [17]).

Theorem 2.4. *([13, Hilfsatz 1]). Let T be a topological space which is of second Baire category and let $f_n : T \to [0, \infty)$, $n \in \mathbb{N}$, be a sequence of positive continuous functions. If there exists a dense subset T_0 of T such that $\sup_n f_n(t) = \infty$ for every $t \in T_0$, then $S = \{t \in T : \sup_n f_n(t) = \infty\}$ is residual, contains T_0 and, of course, is dense in T.*

3 Triple Condensation of Singularities

First we prove a general principle of triple condensation of singularities, from which we shall derive triple condensation of singularities results in concrete situations.

For a topological space T, denote by $C(T)$ the space of all scalar (meaning real or complex) valued continuous functions on T and by $\mathcal{B}(T)$ the σ-algebra of Borel subsets of T, i.e. the σ-algebra generated by the open subsets of T.

Let $(T, \mathcal{A}, \mu)$ be a finite measure space such that $\mathcal{A} \supset \mathcal{B}(T)$ and let X be a normed space. Further let $A_n : X \to S(T)$, $n \in \mathbb{N}$, be a sequence of μ-continuous mappings such that

$$C(T) \cap A_n x \neq \emptyset \tag{3.1}$$

for every $x \in X$ and every $n \in \mathbb{N}$. For $x \in X$ put

$$UD(x) = \{t \in T : \sup_n |\xi_n(t)| = \infty\}$$

where ξ_n stands for the unique element in $C(T) \cap A_n x$.

The main result of this paper is the following theorem.

Theorem 3.1. *Let T be a T_1-separated Baire topological space without isolated points. Let $\mathcal{A} \supset \mathcal{B}(T)$ be a σ-algebra of subsets of T and let $\mu : \mathcal{A} \to [0, \infty)$ be a finite positive measure on $\mathcal{A}$ such that*

$$\mu(U) > 0 \tag{3.2}$$

for every nonvoid open subset U of T.

Further let X be a Baire normed space and let $A_n : X \to S(T)$, $n \in \mathbb{N}$, be a sequence of μ-continuous linear operators satisfying (3.1). *If there exists an element $y_0 \in X$ such that*

$$\mu(UD(y_0)) = \mu(T), \tag{3.3}$$

then there exists a superdense subset X_0 of X such that for every $x \in X_0$

$$\mu(UD(x)) = \mu(T), \text{ and} \tag{3.4a}$$

$$UD(x) \text{ is superdense in } T. \tag{3.4b}$$

PROOF. We intend to apply Theorem 2.3. In order to show that condition (JR) of this theorem is satisfied let $\epsilon > 0$ and $M > 0$ be given and let $x_0 = \delta y_0$, where $\delta = 1/\|y_0\|$. (Observe that, by (3.3), $y_0 \neq 0$.) Let $\eta_n \in C(T) \cap A_n y_0$, $\xi_n = \delta\eta_n$, $n \in \mathbb{N}$, $\xi^* = \sup_n |\xi_n|$ and $\eta^* = \sup_n |\eta_n|$. It follows that $\xi^* = \delta\eta^*$ and $\{\eta^* = \infty\} = \{\xi^* = \infty\} \subset \{\xi^* > M\}$ which, by (3.3), gives

$$\begin{aligned}\mu\{A^* x_0 > M\} &= \mu\{\xi^* > M\} \geq \mu\{\xi^* = \infty\} \\ &= \mu\{\eta^* = \infty\} = \mu\{A^* y_0 = \infty\} = \mu(T) > \mu(T) - \epsilon.\end{aligned}$$

It follows that there exists a superdense subset X_0 of X such that (3.4a) holds, for every $x \in X_0$.

To prove (3.4b) we shall apply Theorem 2.4 to the functions $f_n = |\xi_n|$, where $\xi_n \in C(T) \cap A_n x$, for $n \in \mathbb{N}$ and $x \in X_0$. Put $T_0 = UD(x)$ and show that T_0 is dense in T. Indeed, the existence of an element $t_0 \in T \setminus \overline{T_0}$ would imply the existence of an open neighborhood U of t_0 such that $U \cap T_0 = \emptyset$. But then , by (3.2), $\mu(U) > 0$ and

$$\mu(T) \geq \mu(T_0 \cup U) = \mu(T_0) + \mu(U) > \mu(T_0),$$

contrary to $\mu(T) = \mu(T_0)$. Now, by Theorem 2.3, the set T_0 is residual in T and, by Theorem 2.1, it is superdense in T. □

Remark. Condition (2.1) is equivalent to the condition $\operatorname{supp}(\mu) = T$, where $\operatorname{supp}(\mu)$ denotes the *support* of the measure μ, given by

$$\operatorname{supp}(\mu) = T \setminus \bigcup\{U : U \subset T \text{ open and } \mu(U) = 0\}$$

(see [14, p. 57]).

4 Applications

4.1 Fourier Series

Let $T = [0, 1]$ and let $\{e_k : k \in \mathbb{Z}\}$ be the trigonometric orthonormal system on T given by $e_k(t) = \exp(2\pi ikt),\ t \in T,\ k \in \mathbb{Z},\ (i^2 = -1)$. For $x \in L^1(T)$ and $\xi \in x$, let $c_k = \int_T \xi(t)\overline{e_k(t)}dt,\ k \in \mathbb{Z}$, denote the Fourier coefficients of x and let $F_n : L^1(T) \to C(T)$ denote the Fourier partial sum operator defined by

$$F_n x = \sum_{k=-n}^{n} c_k e_k, \quad n \in \mathbb{Z}_+.$$

Finally, let

$$F^* x = \sup_n |F_n x|$$

be the Fourier maximal operator. The triple condensation of singularities for Fourier series in L^1 has the following form.

Theorem 4.1. *There exists a superdense subset X_0 of $L^1(T)$ such that*

$$\lambda\{F^* x = \infty\} = 1, \text{ and } \{F^* x = \infty\} \text{ is superdense in } T,$$

for every $x \in X_0$, where λ denotes the Lebesgue measure on T.

Proof. For $x \in L^1(T),\ \xi \in x$ and $s \in T$, we have

$$\begin{aligned}|F_n x(s)| &\le \sum_{k=-n}^{n} \Big(\int_T |\xi(t)||\overline{e_k(t)}|dt\Big) \cdot |e_k(s)| \\ &= (2n+1)\int_T |\xi(t)|dt = (2n+1)\|x\|_{L^1},\end{aligned}$$

implying $\|F_n x\|_C \le (2n+1)\|x\|_{L^1}$, which is equivalent to the continuity of the linear operator$F_n : L^1(T) \to C(T)$. It follows that $(F_n x_k)_{k\in\mathbb{N}}$ is uniformly convergent to $F_n x$, for every sequence (x_k) converging in $L^1(T)$ to x. Since the uniform convergence of sequences of measurable functions implies convergence in measure, it follows that the operator $A_n := j \circ F_n : L^1(T) \to S(T)$, is μ-continuous, where j denotes the canonical embedding operator of $C(T)$ into $S(T)$. Now, appealing to Kolmogorov's example [7], there is an element $y_0 \in L^1(T)$ such that $\lambda\{F^* y_0 = \infty\} = 1$. But then, for $A^* y_0 = \sup_n |A_n y_0|$, we have $\lambda\{A^* y_0 = \infty\} = \lambda\{F^* y_0 = \infty\} = 1$. Since Lebesgue measure satisfies the condition (3.2), we can apply Theorem 3.1 to obtain the conclusions of the theorem. □

4.2 Lagrange Interpolation

For a triangular matrix $\mathcal{T}$ of nodes $t_n^1 < \cdots < t_n^n$, $n \in \mathbb{N}$, in the interval $T = [-1, 1]$ and $x \in C(T)$ denote by $L_n x$ the Lagrange interpolation polynomial given by $L_n x = \sum_{k=1}^n x(t_n^k) l_n^k$ where $l_n^k(t) = \omega_n(t)/[(t - t_n^k)\omega_n'(t_n^k)]$, and $\omega_n(t) = (t - t_n^1) \cdots (t - t_n^n)$. Also let $L^* x = \sup_n |L_n x|$ be the Lagrange maximal operator and let

$$UD_L(x) = \{t \in T : L^* x(t) = \infty\}$$

denote the set of unbounded divergence of the sequence $(L_n x)_{n \in \mathbb{N}}$. In this case, the triple condensation of singularities takes the following form.

Theorem 4.2. *For any triangular matrix $\mathcal{T}$ of nodes in the interval $T = [-1, 1]$ there exists a superdense subset X_0 of $C(T)$ such that for every $x \in X_0$*

$$\lambda(UD_L(x)) = 2, \text{ and } UD_L(x) \text{ is superdense in } T.$$

Proof. Denote again by j the canonical embedding operator of $C(T)$ in $S(T)$ and let $A_n := j \circ L_n : C(T) \to S(T)$. For any $x \in C(T)$ we have

$$|L_n x(t)| \leq \sum_{k=1}^n |l_n^k(t)||x(t_n^k)| \leq \lambda_n \|x\|_C,$$

where $\|x\|_C = \sup_{t \in T} |x(t)|$ is the uniform norm in $C(T)$ and

$$\lambda_n = \sup_{t \in T} \sum_{k=1}^n |l_n^k(t)|$$

denotes the Lebesgue constant. Consequently $\|L_n x\|_C \leq \lambda_n \|x\|_C$, implying the continuity of $L_n : C(T) \to C(T)$. Reasoning as in the proof of Theorem 4.1 (uniform convergence implies convergence in measure) we infer that the operator $A_n : C(T) \to S(T)$ is linear and μ-continuous, for each $n \in \mathbb{N}$. By a result of P. Erdős and P. Vértesi [4], there exists a function $y_0 \in C(T)$ such that $\lambda(UD_L(x)) = 2 = \lambda(T)$. The theorem is an immediate consequence of Theorem 3.1. □

4.3 Walsh-Fourier Series

For a detailed and thorough presentation of Walsh harmonic analysis (also called dyadic harmonic analysis) we recommend the treatise [15], which we shall follow in the sequel.

Let r be the function defined on $[0, 1)$ by $r(t) = 1$ for $t \in [0, 1/2)$ and by $r(t) = -1$ for $t \in [1/2, 1)$, and extended by periodicity of period 1 to the whole of $\mathbb{R}$. The *Rademacher system* $\mathbf{r} = \{r_n : n \in \mathbb{Z}_+\}$ is defined by $r_n(t) = r(2^n t)$ for $t \in \mathbb{R}$ and $n \in \mathbb{Z}_+$. (Recall that $\mathbb{Z}_+ = \{0, 1 \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$). For $n \in \mathbb{Z}_+$ let $n = \sum_{k=0}^{\infty} n_k 2^k$ be the *binary expansion* of n. The numbers $n_k \in \{0, 1\}$ are called the *binary coefficients* of n. Obviously $n_k = 0$ for $2^k > n$. The Rademacher system is orthonormal (with respect to Lebesgue measure) but not complete in $L^2[0, 1)$. Starting from the Rademacher system several complete orthonormal systems were obtained by J. L. Walsh in 1923, R. E. A. C. Paley in 1932 and A. Schneider in 1948 (see [15]). All these systems contain the same functions and differ only by their enumeration.

The *Walsh-Paley system* $\mathbf{w} = \{w_n : n \in \mathbb{Z}_+\}$ is given by

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k}, \quad n \in \mathbb{Z}_+ \tag{4.1}$$

where n_k are the binary coefficients of $n \in \mathbb{Z}_+$. Obviously $w_{2^n} = r_n$ and $\mathbf{w}$ is closed under finite products ([15]).

The *Walsh-Kaczmarz system* $\mathbf{v} = \{v_n : n \in \mathbb{Z}_+\}$, defined by A. Schneider [16] (see also [15, p. 2]) is given by $v_0 = 1$ and

$$v_n = r_n \prod_{k=0}^{\infty} r_k^{n_{m-k-1}}, \quad n \in \mathbb{N}, \tag{4.2}$$

where, for $n \in \mathbb{N}$, the number $m \in \mathbb{N}$ satisfies $2^m \le n < 2^{m+1}$.

The *dyadic topology* on $[0, 1)$ is the topology generated by the dyadic intervals

$$I(p, n) = [p2^{-n}, (p + 1)2^{-n}), \quad 0 \le p < 2^n,\ p, n \in \mathbb{Z}_+. \tag{4.3}$$

This topology is generated by a metric defined as follows. Let

$$x = \sum_{k=0}^{\infty} x_k 2^{-k} \tag{4.4}$$

be the dyadic expansion of $x \in [0, 1)$, where $x_k \in \{0, 1\}$ for $k \in \mathbb{Z}_+$. For $x \in \mathbb{Q}_2 := \{p2^{-n} : 0 \le p < 2^n,\ n \in \mathbb{Z}_+\}-$ the set of dyadic rationals in $[0, 1)$, by its dyadic expansion we mean the expansion of the form (4.4) which terminates in zeros.

The *dyadic addition* of two numbers $x, y \in [0, 1)$ is defined by

$$x \oplus y = \sum_{k=0}^{\infty} |x_k - y_k|\, 2^{-k-1}.$$

It is easily seen that $\rho(x,y) = x \oplus y,\ x,y \in [0,1)$ is a metric on $[0,1)$ generating the dyadic topology (see [15, p. 11]). The dyadic intervals (4.3) are both open and closed with respect to this topology, so that the dyadic topology differs from the usual topology of $[0,1)$. Some of its basic properties are listed in the next proposition.

Proposition 4.3. *1. For all $x,y \in [0,1)$ we have $|x-y| \le x \oplus y$.*

2. The metric space $([0,1),\oplus)$ is Baire but not complete.

3. The metric space $[0,1)$ is separable and has no isolated points.

4. If λ denote the Lebesgue measure on $[0,1)$, then $\lambda(U) > 0$ for every nonvoid open subset U of $([0,1),\oplus)$.

5. The Walsh-Kaczmarz functions (4.2) are continuous with respect to the dyadic topology of $[0,1)$.

Proof. Assertions 1 to 3 were proved in [2, Th. 6.1]. Assertion 4 follows immediately from the equality $\lambda(I(p,n)) = 2^{-n} > 0$ and the fact that every nonvoid open subset U of $([0,1),\oplus)$ contains a dyadic interval. (In fact, it is the union of a countable family of such intervals.)

Since each Walsh function w_n is a linear combination of characteristic functions of dyadic intervals (see [15, p. 11]) and each dyadic interval is open and closed with respect to the dyadic topology, each Walsh function is continuous with respect to this topology. As the Walsh-Kaczmarz system $\mathbf{v}$ is a rearrangement of the Walsh-Paley system $\mathbf{w}$, the Walsh-Kaczmarz functions are continuous with respect to the dyadic topology, too. □

Denote by $C_w[0,1)$ the space of all real-valued functions on $[0,1)$ which are continuous with respect to the dyadic topology and let W be the linear subspace of $C_w[0,1)$ spanned by the Walsh-Kaczmarz system $\mathbf{v}$. For $x \in L^1[0,1),\ \xi \in x$ and $k \in \mathbb{Z}_+$, let $c_k = \int_0^1 \xi(t) v_k(t)dt, \quad k \in \mathbb{Z}_+$, denote the Walsh-Kaczmarz coefficients of the function x and let $U_n : L^1[0,1) \to W$ be the Walsh-Kaczmarz partial sum operator defined by $U_n x = \sum_{k=0}^n c_k v_k$.

Let also $V_n := j \circ U_n : L^1[0,1) \to L^1[0,1)$, where j denotes the canonical embedding operator of W in $L^1[0,1)$. Finally, for $x \in L^1[0,1)$, let

$$UD(x) = \{t \in T : \sup_n |U_n x(t)| = \infty\}$$

the set of unbounded divergence of the sequence $(U_n x)$.

Now we can state the triple condensation of singularities theorem for Walsh-Kaczmarz series.

Theorem 4.4. *There exists a superdense subset X_0 of $L^1[0,1)$ such that for every $x \in X_0$*

$$\lambda(UD(x)) = 1, \text{ and } UD(x) \text{ is superdense in } ([0,1), \oplus) \tag{4.5}$$

Proof. Since $|v_k(t)| = 1$, we have

$$\begin{aligned} \|V_n x\|_{L^1} &\le \int_0^1 \sum_{k=0}^{n} |c_k||v_k(s)|\, ds \\ &\le \sum_{k=0}^{n} \int_0^1 |\xi(t)||v_k(t)|\, dt = (2n+1)\|x\|_{L^1} \end{aligned}$$

for every $x \in L^1[0,1)$ and every $\xi \in x$. It follows that the linear operator $V_n : L^1[0,1) \to L^1[0,1)$ is continuous. By a result of L. A. Balashov (see [15], Ch. 6, Th. 21), for any sequence (ϵ_n) of positive numbers decreasing to zero there exists $y_0 \in L^1[0,1)$ such that

$$\limsup_n \frac{|U_n(y_0(t)|}{\epsilon_n \log(n+2)} = \infty \tag{4.6}$$

a.e. on $[0,1)$.

Since, by Proposition 2.2, the canonical embedding operator $J : L^1[0,1) \to S[0,1)$ is linear and μ-continuous, it follows that the operator $A_n := J \circ V_n : L^1[0,1) \to S[0,1)$ is linear and μ-continuous and $C_w[0,1) \cap A_n x = \{U_n x\}$, for all $x \in L^1[0,1)$ and $n \in \mathbb{Z}_+$. Taking $\epsilon_n = 1/\log(n+2)$ in (4.6), it follows that condition (3.3) of Theorem 3.1 holds, implying that the assertions (4.5) are true. $\square$

Remark. Concerning the pointwise divergence of Walsh-Fourier series (i.e., Fourier series with respect to the Walsh-Paley system (4.1)), there are examples of Lebesgue integrable functions with Walsh-Paley series unboundedly divergent on dense subsets of $[0,1)$ ([15], Ch. 6, Th. 18), a result used in [2] to prove a double condensation of singularities for Walsh-Paley series. On the other hand, there are examples of integrable functions with Walsh-Paley series boundedly divergent a.e. on $[0,1)$, but we are unaware of examples of integrable functions with a.e. unboundedly divergent Walsh-Paley series.

N. J. Fine [5] and N. Ja. Vilenkin [18] (see also [15]) proposed another approach to Walsh analysis, namely as a special case of harmonic analysis on a compact abelian group, in a way we shall briefly describe below.

Denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2; i.e., the set $\{0,1\}$ with addition modulo 2 and discrete topology. The dyadic group is the group

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots$, equipped with the product topology and addition $x + y = (|x_n - y_n|)_{n \in \mathbb{Z}_+}$, for $x, y \in G$. In fact, G is a vector space over the field $\mathbb{Z}_2$ and the formula $|x|_2 = \sum_{k=0}^{\infty} x_k 2^{-k-1}$ for $x = (x_k) \in G$, defines a norm on G generating its topology. The measure μ on G, obtained as the product measure from the discrete measure ν on $\mathbb{Z}_2$ assigning to each singleton the measure $1/2$, is a translation invariant measure (i.e., a Haar measure) on G with $\mu(G) = 1$. Since for $x_i \in \{0, 1\}$, $\mu(\{x_0\} \times \{x_1\} \times \dots \times \{x_n\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots) = 2^{-n-1} > 0$, it follows that $\mu(U) > 0$ for every nonvoid open subset U of G, i.e. condition (3.3) holds.

Now, combining Theorems 9 and 12 from Chapter 4 in [15], we deduce the existence of a function $f_0 \in L^1(G)$ whose Fourier series is unboundedly divergent μ-a.e. on G. Using this result and Theorem 4.1 one can prove the following.

Theorem 4.5. *There exists a superdense subset X_0 of $L^1(G)$ such that for every $f \in X_0$*

$$\mu(UD(f)) = 1 \text{ and } UD(f) \text{ is superdense in } G.$$

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