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Zbigniew Grande,* Institute of Mathematics, Pedagogical University, Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland. e-mail:
`grande@wsp.bydgoszcz.pl`

ON DISCRETE LIMITS OF SEQUENCES OF BILATERALLY QUASICONTINUOUS, BAIRE 1 FUNCTIONS

Abstract

In this article we show that for the discrete limit f of sequence of bilaterally quasicontinuous Baire 1 functions the complement of the set of all points at which f is bilaterally quasicontinuous and has Darboux property, is nowhere dense. Moreover, a construction is given of a bilaterally quasicontinuous function which is the discrete limit of a sequence of Baire 1 functions, but is not the discrete limit of any sequence of bilaterally quasicontinuous Baire 1 functions.

Let $\mathbb{R}$ be the set of all reals. In the article [3] the authors introduced the notion of the discrete convergence of sequences of functions and investigated the discrete limits in different families, for example in the family $\mathcal{C}$ of all continuous functions.

We will say that a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, discretely converges to the limit f $(f = d - \lim_{n\to\infty} f_n)$ if

$$\forall_x \exists_{n(x)} \forall_{n>n(x)} f_n(x) = f(x).$$

For any family $\mathcal{P}$ denote by $B_d(\mathcal{P})$ the family of all discrete limits of sequences of functions from the family $\mathcal{P}$.

In [3] the class $B_d(\mathcal{C})$ is described and the authors observe that every strictly increasing function f whose set of discontinuity points is dense does not belong to the discrete Baire system generated by $\mathcal{C}$ and the discrete convergence.

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A function $f : \mathbb{R} \to \mathbb{R}$ is quasicontinuous (bilaterally quasicontinuous) at a point x if for every positive real η there is a nonempty open set $U \subset (x - \eta, x + \eta)$ (there are open sets $V \subset (x - \eta, x)$ and $W \subset (x, x + \eta)$) such that $f(U) \subset (f(x) - \eta, f(x) + \eta)$ $(f(V \cup W) \subset (f(x) - \eta, f(x) + \eta))$ ([7, 8]).

In [4] it is proved that

(a) A function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of quasicontinuous functions if and only if the set

$$D_q(f) = \{x; f \text{ is not quasicontinuous at } x\}$$

is nowhere dense.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of bilaterally quasicontinuous functions if and only if the set

$$D_{bq}(f) = \{x; f \text{ is not bilaterally quasicontinuous at } x\}$$

is nowhere dense.

Let $\mathcal{D}$ denote the class of all functions $f : \mathbb{R} \to \mathbb{R}$ having Darboux property and let Q (respectively Q_b) be the family of all quasicontinuous (bilaterally quasicontinuous) functions.

In [6] the authors investigate some classes $\mathcal{P}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\mathcal{P} \subset B_d(\mathcal{D} \cap \mathcal{P}).$$

But both of the classes Q and Q_b do not satisfy the hypothesis of that general theorem from [6].

For this observe that

$$Q \cap \mathcal{D} \subset Q_b$$

and that for each continuous from the right and increasing function $f : \mathbb{R} \to \mathbb{R}$ discontinuous on a dense set we have

$$D_q(f) = \emptyset \text{ and the set } D_{bq}(f) \text{ is dense.}$$

Consequently,

$$Q \setminus B_d(\mathcal{D} \cap Q) = Q \setminus B_d(\mathcal{D} \cap Q_b) \neq \emptyset.$$

In article [5] I show two theorems describing the class $B_d(\mathcal{D} \cap Q_b)$. In our considerations we apply the following notations: Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $x \in \mathbb{R}$ be a point. Put

$$K^+(f, x) = \{y : \exists_{(x_n)} x < x_n \to x \text{ and } y = \lim_{n\to\infty} f(x_n)\},$$

$$K^-(f,x) = \{y : \exists_{(x_n)} x > x_n \to x \text{ and } y = \lim_{n\to\infty} f(x_n)\},$$

and recall that x is a Darboux point of a function f if for every positive real r and for all reals $a \in (\min(f(x), \inf(K^+(f,x))), \max(f(x), \sup(K^+(f,x))))$ and $b \in (\min(f(x), \inf(K^-(f,x))), \max(f(x), \sup(K^-(f,x))))$ there are points $c \in (x, x+r)$ and $d \in (x-r, x)$ such that $f(c) = a$ and $f(d) = b$. It is known ([2, 1]) that a function f has the Darboux property if and only if each point x is a Darboux point of f.

Let

$$Dar(f) = \{x : x \text{ is not a Darboux point of } f\}.$$

The following two theorems are proved in [5]:

Theorem 1. *Let a function $f : \mathbb{R} \to \mathbb{R}$ be such that the set $Dar(f) \cup D_{bq}(f)$ is nowhere dense. Then f is the discrete limit of a sequence of Darboux bilaterally quasicontinuous functions.*

Theorem 2. *There is a function $f : \mathbb{R} \to \mathbb{R}$ belonging to $B_d(\mathcal{D} \cap Q_b)$ such that the set $Dar(f)$ is dense.*

In connection with these theorems in this article I prove the following theorem:

Theorem 3. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of bilaterally quasicontinuous Baire 1 functions f_n, $n = 1, 2, \ldots$, then the set $D_{bq}(f) \cup Dar(f)$ is nowhere dense.*

Proof. By (b) the set $D_{bq}(f)$ is nowhere dense. Assume, to the contrary that there is an open interval I in which the set $Dar(f)$ is dense. Without loss of the generality we can suppose that $I \cap D_{bq}(f) = \emptyset$. For $n = 1, 2, \ldots$ let

$$A_n = \{x \in I; f_k(x) = f(x) \text{ for } k \geq n\}.$$

Since

$$I = \bigcup_{n=1}^{\infty} A_n,$$

there is a positive integer m for which the set A_m is of the second category. So, there is an open interval $J \subset I$ in which the set A_m is dense. Since $f|J$ does not have the property of Darboux, there is a real

$$s \in (\inf_J f, \sup_J f) \setminus f(J).$$

Let

$$A = \{x \in J; f(x) < s\} \text{ and } B = \{x \in J; f(x) > s\}.$$

Since the restricted function f/J is bilaterally quasicontinuous, the set $E = J \setminus (Int(A) \cup Int(B))$ is nonempty and perfect in J. For $n = 1, 2, \ldots$ let

$$E_n = \{x \in E; |f(x) - s| \geq \frac{1}{2^n}\}.$$

There is a positive integer $k > m$ such that the set $E_k \cap A_k$ is of the second category in E. There is an open interval $L \subset J$ such that

$$L \cap E \neq \emptyset \text{ and } E \cap L \subset cl(E_k \cap A_k \cap L).$$

Since the function f_k is of the first class of Baire, there is a point $u \in L \cap E$ at which the restricted function f_k/E is continuous. Consequently, there is a positive real r such that

$$(u - r, u + r) \subset L \text{ and } |f_m(t) - f_m(u)| < \frac{1}{4^k} \text{ for } t \in E \cap (u - r, u + r).$$

Denote by G the set

$$\{x \in J; f_j \text{ is continuous at } x \text{ for } j \geq m\}.$$

Evidently the set G is dense in J and $f = f_j = f_m$ on G for $j > m$.

Observe that for $j \geq m$ and for a point x which is an endpoint of some component of $Int(A)$ (of $Int(B)$) we have $f_j(x) \leq s$ ($f_j(x) \geq s$).

Since $E \subset cl(A) \cap cl(B)$ and f_k is bilaterally quasicontinuous, we obtain

$$f_k(u) = s.$$

There is a point $w \in (u - r, u + r) \cap E_k \cap A_k$. Evidently,

$$|f(w) - s| \geq \frac{1}{2^k} > \frac{1}{4^k}$$

and

$$|f(w) - s| = |f_k(w) - f_k(u)| < \frac{1}{4^k}.$$

□

Theorem 4. *There is a bilaterally quasicontinuous function $f : \mathbb{R} \to \mathbb{R}$ which is the discrete limit of a sequence of functions belonging to the first class of Baire but is not the discrete limit of any sequence of bilaterally quasicontinuous Baire 1 functions.*

Proof. Let (I_n) be an enumeration of all open intervals with rational endpoints. We will construct our function f by induction. Let $C_1 \subset I_1$ be a nonempty nowhere dense perfect set and let $J_1 = [\min(C_1), \max(C_1)]$. Then

$$J_1 \setminus C_1 = \bigcup_{i=1}^{\infty} (a_{1,i}, b_{1,i}),$$

where $(a_{1,i}, b_{1,i})$ are components of the set $J_1 \setminus C_1$.

Put

$$f_1(x) = \begin{cases} 1 & \text{for} & x = a_{1,i},\ i = 1, 2, \dots \\ 0 & \text{for} & x \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} [a_{1,i}, b_{1,i}) \\ & \text{linear on the intervals} & [a_{1,i}, b_{1,i}],\ i = 1, 2, \dots . \end{cases}$$

Evidently the function f_1 is bilaterally quasicontinuous.

In the second step we find a nonempty nowhere dense perfect set $C_2 \subset I_2 \setminus C_1$ contained in one component of the set $I_2 \setminus C_1$; denote by J_2 the closed interval $[\min(C_2), \max(C_2)]$ and by $(a_{2,i}, b_{2,i})$, $i = 1, 2, \dots$, the components of the set $J_2 \setminus C_2$. Let

$$g_2(x) = \begin{cases} \frac{1}{5} & \text{for} & x = a_{2,i},\ i = 1, 2, \dots \\ 0 & \text{for} & x \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} [a_{2,i}, b_{2,i}) \\ & \text{linear on the intervals} & [a_{2,i}, b_{2,i}],\ i = 1, 2, \dots , \end{cases}$$

and let

$$f_2 = f_1 + g_2.$$

In general step n $(n > 1)$ we find a nonempty nowhere dense perfect set C_n contained in one component of the difference

$$I_n \setminus \bigcup_{j<n} C_j$$

and denote by J_n the closed interval $[\min(C_n), \max(C_n)]$. Let $(a_{n,i}, b_{n,i})$, $i = 1, 2, \dots$, be the components of the set $J_n \setminus C_n$ and let

$$g_n(x) = \begin{cases} \frac{1}{5^{n-1}} & \text{for} & x = a_{n,i},\ i = 1, 2, \dots \\ 0 & \text{for} & x \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} [a_{n,i}, b_{n,i}) \\ & \text{linear on the intervals} & [a_{n,i}, b_{n,i}],\ i = 1, 2, \dots , \end{cases}$$

and let

$$f_n = f_{n-1} + g_n.$$

Since for $n \geq 1$ the functions g_n are bilaterally quasicontinuous and for $n > 1$ the restricted functions f_{n-1}/J_n are continuous, the functions f_n are bilaterally quasicontinuous. Observe that

$$\forall_{n\geq 1}|f_{n+1} - f_n| \leq \frac{1}{5^{n-1}}.$$

So, the sequence (f_n) uniformly converges to some function f, which must be bilaterally quasicontinuous.

The function f is continuous at each point

$$x \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} C_n,$$

continuous from the right at each point $a_{n,i}$, continuous from the left at each point $b_{n,i}$, $n, i \geq 1$, and

$$f(x) = 0 \text{ for } x \in \bigcup_{n=1}^{\infty} C_n \setminus \{a_{n,i}; n, i \geq 1\}.$$

Observe that the function

$$g(x) = \begin{cases} 0 & \text{for} & x \in \bigcup_n C_n \\ f(x) & \text{otherwise on} & \mathbb{R} \end{cases}$$

is of the first class of Baire as the uniform limit of the sequence of functions

$$h_n(x) = \begin{cases} 0 & \text{for} & x \in \bigcup_{i\leq n} C_i \\ f_n(x) & \text{otherwise on} & \mathbb{R}, \end{cases}$$

being evidently of the first Baire class.

Now, let (u_j) be an enumeration of all points $a_{n,i}$, $n, i = 1, 2, \ldots$ and for $n = 1, 2, \ldots,$ define

$$\phi_n(x) = \begin{cases} f(x) & \text{for} & x = u_j,\ j \leq n \\ g(x) & \text{otherwise on} & \mathbb{R}. \end{cases}$$

The functions ϕ_n, $n \geq 1$, are of Baire 1 class and evidently

$$f = d - \lim_{n\to\infty} \phi_n,$$

so f is the discrete limit of a sequence of functions of Baire 1 class.

Now assume to the contrary that the function f is the discrete limit of a sequence of bilaterally quasicontinuous function ψ_n of Baire 1 class. For $n \geq 1$ let

$$A_n = \{x; \psi_k(x) = \psi_n(x) = f(x) \text{ for } k \geq n\}.$$

There is a positive integer j such that the set A_j is of the second category. Let I be an open interval contained in $Int(cl(A_j))$. There is an integer $m > j$ with $I_m \subset I$. Since f and ψ_j are equal on $I_m \cap A_j$ and ψ_j is bilaterally quasicontinuous, then

$$f(a_{j,i}) = \psi_j(a_{j,i}) \text{ and } f(b_{j,i}) = \psi_j(b_{j,i}) \text{ for } i = 1, 2, \dots .$$

So, the restricted function ψ_j / C_j is discontinuous at each point $x \in C_j$ and it is not of Baire 1 class. □

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