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CONTINUOUS NORMS AND ABSOLUTELY CONTINUOUS NORMS IN BANACH FUNCTION SPACES ARE NOT THE SAME

Abstract

It is known that the concepts of continuous norm and of absolutely
continuous norm do not coincide. There exists a space in which all func-
tions possess continuous norm but not all functions possess absolutely
continuous norm. In this paper we construct an extremal example of a
Banach function space in which all functions have continuous norm but
only the zero function has absolutely continuous norm.

1 Introduction

The concept of absolutely continuous norm plays a very important role in
characterization of classes of reflexive Banach function spaces and of separable
Banach function spaces. Let us recall two assertions presented by C. Bennett
and R. Sharpley in [1]. The first one [1, Corollary 4.4] shows that a Banach
function space X is reflexive if and only if both X and its associate space X'

have absolutely continuous norms (see also [7] and [8]). In the second one [1, Corollary 5.6] it is proved that a Banach function space (X, μ) is separable if and only if X has absolutely continuous norm and μ is a separable measure. The concept of continuous norm was introduced by Q. Lai and L. Pick in [5]. They proved that the Hardy operator $Tf(x) = \int_0^x f(t)dt$ is a compact mapping from a Banach function space (X, v) into L_∞ if and only if the function $1/v$ has continuous norm in the associate space (X', v). This result raised the question, if in every Banach function space X, the set of all functions with absolutely continuous norm is different from the set of all functions with continuous norm. In [6] two Banach function spaces are constructed which demonstrate that the concepts of continuous norm and absolutely continuous norm are not identical. The first space contains a function with continuous but not absolutely continuous norm and another function with non-continuous norm. In the other space, every function has continuous norm and there is a function with non-absolutely continuous norm.

D. E. Edmunds, J. Lang and A. Nekvinda [2] investigated the notions of continuous norm and absolutely continuous norm in the scale of spaces $L^{p(x)}$ of functions integrable with variable power $p(x)$. These spaces possess many non-standard properties (see [4], [3]). Nevertheless, the set of functions with continuous norm and the set of functions with absolutely continuous norm in these spaces coincide (see [2]).

In the present paper an extreme example of Banach function space is found in the sense that every function has continuous norm and only the zero function has absolutely continuous norm. The paper is organized in the following way. Section 2 has a preparatory character and brings basic notations, definitions and auxiliary assertions. In Section 3 we construct the Banach function space and in Section 4 we prove that it has the desired properties.

2 Preliminaries

Let Ω be a non-empty open subset of $\mathbb{R}$ and let $\mathcal{M}(\Omega)$ be the set of all real measurable functions defined on Ω. Denote by $|E|$ the Lebesgue measure of any measurable subset E of Ω and by χ_E the characteristic function of E. The unit function will be denoted 1_Ω; i.e., $1_\Omega(x) = 1$ for all $x \in \Omega$.

Definition 2.1. A normed linear space $(X, \| \cdot \|)$ is called a Banach function space (abbreviated BFS) if the following conditions are satisfied:

(2.1) the norm $\|f\|$ is defined for all $f \in \mathcal{M}(\Omega)$, and $f \in X$ if and only if $\|f\| < \infty$;

(2.2) $\|f\| = \| |f| \|$ for every $f \in \mathcal{M}(\Omega)$;

(2.3) if $0 \le f_n \nearrow f$ a.e. in Ω, then $\|f_n\| \nearrow \|f\|$;

(2.4) if $E \subset \Omega$, $|E| < \infty$, then $\chi_E \in X$;

(2.5) for every set $E \subset \Omega$ with $|E| < \infty$, there exists a positive constant C_E such that $\int_E |f(x)| dx \le C_E \|f\|$.

Recall that the Fatou property (2.3) immediately yields

(2.6) if $0 \le f \le g$, then $\|f\| \le \|g\|$.

The above definition was introduced in [1]. In [8] another definition of Banach function spaces is given which—requiring properties (2.1), (2.2), (2.6) and the completeness of the space—specifies a wider class of spaces. The space we construct in this paper is a Banach function space in the sense of both definitions.

Definition 2.2. Let $X = (X(\Omega), \|\,.\,\|)$ be a Banach function space and let $f \in X$ be an arbitrary function. We say that the function f has absolutely continuous norm in X if for any sequence of open sets G_n with $G_n \supset G_{n+1}$ for $n = 1, 2, \ldots$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$, the norms $\|f\chi_{G_n}\|$ tend to zero as $n \to \infty$. The set of all functions with absolutely continuous norms is denoted by X_a.

We say that f has continuous norm in X if $\lim_{r \to 0+} \|f\chi_{(x-r,x+r)\cap\Omega}\| = 0$ for every $x \in \overline{\Omega}$ and $\lim_{r \to \infty} \|f\chi_{\Omega\setminus(-r,r)}\| = 0$. The set of all functions with continuous norm is denoted by X_c.

Recall that $\|\|\,.\,\|\| : \mathcal{M}(\Omega) \to [-\infty, \infty]$ is a seminorm on $\mathcal{M}(\Omega)$ if every $f, g \in \mathcal{M}(\Omega)$ and $a \in \mathbb{R}$ satisfy $\|\|f\|\| \ge 0$, $\|\|af\|\| = |a|\,\|\|f\|\|$, $\|\|f + g\|\| \le \|\|f\|\| + \|\|g\|\|$.

We will be particularly concerned with seminorms which satisfy conditions

(2.7) $\|\|f\|\| = \|\|\,|f|\,\|\|$ for all $f \in \mathcal{M}(\Omega)$]

(2.8) if $f_n, f \in \mathcal{M}(\Omega)$ and $0 \le f_n \nearrow f$ a.e. in Ω, then $\|\|f_n\|\| \nearrow \|\|f\|\|$.

Evidently, (2.8) implies

(2.9) if $0 \le f \le g$, then $\|\|f\|\| \le \|\|g\|\|$.

We will need the following three assertions. The easy proofs are omitted.

Proposition 2.3. *Let Ω_i, $i \in I$, be a system of non-empty open subsets of Ω such that $\Omega = \bigcup_{i\in I} \Omega_i \cup M$, where $|M| = 0$. Let $\|\|\,.\,\|\|_i$ be a system of seminorms on Ω_i satisfying conditions (2.7) and (2.8). Define the seminorm $\|\|\,.\,\|\|$ by $\|\|f\|\| = \sup_{i\in I} \|\|f_i\|\|_i$, where f_i denotes the restriction of f on Ω_i. Then $\|\|\,.\,\|\|$ satisfies (2.7) and (2.8).*

Proposition 2.4. *Let Ω_n be a sequence of non-empty open subsets of Ω and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n \cup M$, where $|M| = 0$. Let $\VERT \,.\, \VERT_n$ be a system of seminorms satisfying* (2.7) *and* (2.8). *Define the seminorm $\VERT f \VERT = \sum_{n=1}^{\infty} \VERT f_n \VERT_n$ for f_n being the restriction of f on Ω_n. Then $\VERT \,.\, \VERT$ satisfies* (2.7) *and* (2.8).

Proposition 2.5. *Let $\VERT \,.\, \VERT$ be a seminorm satisfying* (2.7) *and* (2.8) *and let $\VERT 1_\Omega \VERT < \infty$. Define the norm $\| \,.\, \|$ by $\|f\| = \VERT f \VERT + \int_\Omega |f|$. Then $\| \,.\, \|$ is a norm in a BFS.*

For the sake of simplicity, in the rest of the paper we assume $\Omega = (0, 1)$.

3　The Construction

Our considerations are based on principles of construction of the Cantor set. Let the symbol $\mathcal{K}$ denote the set of all finite sequences of the numbers 0 and 1 including the empty sequence $\emptyset$. The elements of $\mathcal{K}$ will be called multiindices. The length $\ell(\alpha)$ of a multiindex $\alpha \in \mathcal{K}$, $\alpha = (a_1, a_2, \ldots, a_n)$, is the number of all members of the sequence α; i.e., $\ell(\alpha) = n$. We define a partial ordering $\preceq$ on $\mathcal{K}$ saying that $\alpha \preceq \beta$ for $\alpha = (a_1, a_2, \ldots, a_k)$, $\beta = (b_1, b_2, \ldots, b_n)$ if $k \leq n$ and $a_i = b_i$ for $i = 1, 2, \ldots, k$. We shall write $\alpha \npreceq \beta$ if $\alpha \preceq \beta$ does not hold, and $\alpha \nsim \beta$ if $\alpha \npreceq \beta$ and $\beta \npreceq \alpha$. Note, that $\alpha = \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$, and that the relations $\nsim$, $\neq$ are different.

Recall the construction of the Cantor set $\mathfrak{C}$. Define the intervals I_α, $\alpha \in \mathcal{K}$,

by

$$\ldots$$

$$I_{00} = (\tfrac{1}{3^3}, \tfrac{2}{3^3})$$

$$\ldots$$

$$I_0 = (\tfrac{1}{3^2}, \tfrac{2}{3^2})$$

$$\ldots$$

$$I_{01} = (\tfrac{7}{3^3}, \tfrac{8}{3^3})$$

$$\ldots$$

$$I_\emptyset = (\tfrac{1}{3}, \tfrac{2}{3})$$

$$\ldots$$

$$I_{10} = (\tfrac{19}{3^3}, \tfrac{20}{3^3})$$

$$\ldots$$

$$I_1 = (\tfrac{7}{3^2}, \tfrac{8}{3^2})$$

$$\ldots$$

$$I_{11} = (\tfrac{25}{3^3}, \tfrac{26}{3^3})$$

$$\ldots$$

Note that $|I_\alpha| = 3^{-\ell(\alpha)-1}$. The Cantor set is given by $\mathfrak{C} = [0,1] \setminus \bigcup_{\alpha \in \mathcal{K}} I_\alpha$. If $A, B \subset \mathbb{R}$ we say that A is to the left (right) of B if $\sup A < \inf B$ ($\inf A > \sup B$), respectively. Let $\alpha, \beta \in \mathcal{K}$ and let I_α, I_β be the corresponding intervals from the above construction. We say that α is to the left (right) of β if I_α is to the left (right) of I_β, respectively. If α is to the left of γ and γ is to the left of β, we say that γ is between α and β. We denote by $\mathcal{F}$ the system of all bounded measurable sets $M \subset \mathbb{R}$ such that $|M \cap (x - t, x + t)| > 0$ for every $x \in M$ and $t > 0$, $|M| = 3^{-n}$ for certain positive integer n, and, moreover, neither $\inf M$ nor $\sup M$ belong to M. Let $\mathcal{P}$ be the family of all mappings $P : \mathcal{K} \to \mathcal{F}$ such that $\bigcup_{\alpha \in \mathcal{K}} P(\alpha)$ is bounded,

$$|P(\alpha)| = \frac{1}{3^{\ell(\alpha)+1}} \tag{3.1}$$

and

$$\text{if } \alpha \text{ is to the left of } \beta, \text{ then } P(\alpha) \text{ is to the left of } P(\beta). \tag{3.2}$$

By $\mathcal{E}$ we denote the class of all measurable sets $E \subset \mathbb{R}$ such that $E = \bigcup_{\alpha \in \mathcal{K}} P(\alpha)$ for some $P \in \mathcal{P}$. It is easy to see that for every $E \in \mathcal{E}$ the corresponding mapping $P \in \mathcal{P}$ is unique and $|E| = 1$. We set $I_\alpha^E = P(\alpha)$, $\widetilde{I}_\alpha^E = (\inf I_\alpha^E, \sup I_\alpha^E)$.

Note that the set $[0, 1] \setminus \mathfrak{C}$ belongs to $\mathcal{E}$.

Let $E \in \mathcal{E}$. Let $P \in \mathcal{P}$ be the corresponding mapping and $I_\alpha^E = P(\alpha)$, $\alpha \in \mathcal{K}$. For $\alpha \in \mathcal{K}$ we define the closed intervals J_α^E by

$$\cdots$$

$$J_{00}^E = [\inf E, \inf I_0^E]$$

$$\cdots$$

$$J_0^E = [\inf E, \inf I_\emptyset^E]$$

$$\cdots$$

$$J_{01}^E = [\sup I_0^E, \inf I_\emptyset^E]$$

$$\cdots$$

$$J_\emptyset^E = [\inf E, \sup E]$$

$$\cdots$$

$$J_{10}^E = [\sup I_\emptyset^E, \inf I_1^E]$$

$$\cdots$$

$$J_1^E = [\sup I_\emptyset^E, \sup E]$$

$$\cdots$$

$$J_{11}^E = [\sup I_1^E, \sup E]$$

$$\cdots$$

The generalized Cantor set $\mathfrak{C}^E$ corresponding to the set E is defined by

$$\mathfrak{C}^E = \bigcap_{n=1}^\infty \bigcup_{\ell(\alpha)=n} J_\alpha^E.$$

It is not difficult to show the following properties:

(3.3) $|J_\alpha^E| \geq |J_\alpha^E \cap E| = \sum_{\alpha \preceq \beta} |I_\beta^E| = 3^{-\ell(\alpha)}$;

(3.4) the families $\{I_\alpha^E\}_\alpha$ and $\{\widetilde{I}_\alpha^E\}_\alpha$ are pairwise disjoint;

(3.5) $\alpha \preceq \beta$ if and only if $J_\beta^E \subset J_\alpha^E$ and $\alpha \nsim \beta$ if and only if $J_\alpha^E \cap J_\beta^E = \emptyset$;

(3.6) $\alpha \preceq \beta$ if and only if $\widetilde{I}_\beta^E \subset J_\alpha^E$ and $\alpha \npreceq \beta$ if and only if $\widetilde{I}_\beta^E \cap J_\alpha^E = \emptyset$;

(3.7) if $\alpha \in \mathcal{K}$ and if k is an integer, $k \geq \ell(\alpha)$, then the number of all multi-indices β such that $\alpha \preceq \beta$ and $\ell(\beta) = k$, is equal to $2^{k-\ell(\alpha)}$;

(3.8) for every $x \in \mathfrak{C}^E$ there exists a unique sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_1 \preceq \alpha_2 \preceq \dots$, $\ell(\alpha_n) = n$ and $x \in \bigcap_{n=1}^\infty J_{\alpha_n}^E$.

Moreover, if α is to the left of β and $\alpha \nsim \beta$, then there exists γ between α and β such that $\gamma \preceq \alpha$ and $\gamma \preceq \beta$.

Now, we are ready to construct the desired norm in a BFS. Consider the class $\mathcal{M} = \mathcal{M}(0,1)$. We will identify the functions $f \in \mathcal{M}$ with their extensions by zero outside $(0,1)$. For $E \in \mathcal{E}$ we define the seminorm $\| . \|_E$ by

$$
\begin{aligned}
\|f\|_E &= \sum_{k=0}^\infty 2^{-k} \max_{\ell(\alpha)=k} \sup_{n \ge k} \left| \bigcup I_\beta^E \right|^{-1} \int_{\bigcup I_\beta^E} |f(x)|\, dx \\
&= \sum_{k=0}^\infty \max_{\ell(\alpha)=k} \sup_{n \ge k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f(x)|\, dx, \quad f \in \mathcal{M},
\end{aligned}
\tag{3.9}
$$

where the union is taken over all β such that $\ell(\beta) = n$ and $\alpha \preceq \beta$; the second equality follows from (3.3).

Definition 3.1. The space X is the set of all functions $f \in \mathcal{M}$ with $\|f\| < \infty$ where $\|f\| = \sup_{E \in \mathcal{E}} \|f\|_E$.

Theorem 3.2. $(X, \| . \|)$ *is a BFS.*

PROOF. Set $\|f\|_0 = \|f\| + \int_0^1 |f|$. Considering the only term with $k = n = 0$ on the right hand side of (3.9) we obtain $\|f\|_E \ge 3 \int_{I_\emptyset^E} |f(x)|\, dx$ for every $E \in \mathcal{E}$. If $E_1, E_2, E_3 \in \mathcal{E}$ are such that $I_\emptyset^{E_1} = (0, 1/3)$, $I_\emptyset^{E_2} = (1/3, 2/3)$, $I_\emptyset^{E_3} = (2/3, 1)$, then there exists $i \in \{1, 2, 3\}$ such that $3 \int_{I_\emptyset^{E_i}} |f| \ge \int_0^1 |f|$. Hence

$$
\|f\| \ge \|f\|_{E_i} \ge 3 \int_{I_\emptyset^{E_i}} |f| \ge \int_0^1 |f|,
$$

and so, $2\|f\| \ge \|f\|_0 \ge \|f\|$. It suffices to show that $\| . \|_0$ is a norm in a BFS.

According to Propositions 2.3 and 2.4 every seminorm $\| . \|_E$, $E \in \mathcal{E}$, satisfies conditions (2.7) and (2.8) on the interval $(0,1)$. Repeated application of Proposition 2.3 yields that the seminorm $\|f\| = \sup_{E \in \mathcal{E}} \|f\|_E$ satisfies (2.7) and (2.8). By Proposition 2.5, it remains to verify that the unit function 1_Ω satisfies $\|1_\Omega\| < \infty$ which is seen from (3.9). $\|1_\Omega\|_E \le \sum_{k=0}^\infty 2^{-k} = 2$ for every $E \in \mathcal{E}$; i.e., $\|1_\Omega\| \le 2 < \infty$. $\qquad\square$

4   Properties of the Space X

In this section we prove two theorems. The first one claims that only the zero function in X has absolutely continuous norm; i.e., $X_a = \{0\}$, while the other claims that every function from X has continuous norm; i.e., $X_c = X$.

Theorem 4.1. $X_a = \{0\}$.

PROOF. Let $f \in X$, $f \neq 0$. Then there exist a measurable set M, $|M| > 0$, and a real number $\eta > 0$ such that $|f| \geq \eta > 0$ on M. Set

$$M' = \{x \in M : |(x - t, x + t) \cap M| > 0 \text{ for every } t > 0\}.$$

Clearly, $|M'| = |M|$. Take the integer n_0 such that $3^{-n_0-1} \leq |M'| < 3^{-n_0}$ and the real number $b < 1$ such that $|[\inf M', b] \cap M'| = 3^{-n_0-1}$. Set

$$F_1 = [\inf M', b] \cap M', \; F_2 = [1, 2 - 3^{-n_0-1}], \text{ and } F = F_1 \cup F_2.$$

For every $\alpha \in \mathcal{K}$ we denote by $p(\alpha)$, $q(\alpha)$ the least real numbers such that $|(\inf F, p(\alpha)) \cap F| = \inf I_\alpha$, $|(p(\alpha), q(\alpha)) \cap F| = |I_\alpha|$, and we set $P(\alpha) = (p(\alpha), q(\alpha)) \cap F$. It is easy to see that $P \in \mathcal{P}$ and so the set $E = \bigcup_{\alpha \in \mathcal{K}} P(\alpha)$ belongs to $\mathcal{E}$. Set $E_1 = E \cap F_1$, $E_2 = E \cap F_2$ and $\alpha_0 = (0, 0, \ldots, 0)$, $\ell(\alpha_0) = n_0$. Since E_1 is to the left of E_2 and $|E_1| = 3^{-n_0-1}$, we have $I_\beta^E \cap F_1 = I_\beta^E$ if β is to the left of α_0 and $I_\beta^E \cap F_1 = \emptyset$ in other cases. For every non-negative integer N we set $G_N = \bigcup_{\ell(\alpha) \geq N} \widetilde{I}_\alpha^E$. Obviously, G_N form a decreasing sequence of open sets with empty intersection. Let us calculate $\||f \chi_{G_N}\||_E$ for a fixed N. Taking only the first summand in (3.9), using (3.4) and the fact that $\widetilde{I}_\alpha^E \cap E = I_\alpha^E$,

we obtain

$$
\begin{aligned}
\||f\chi_{G_N}\||_E &\geq \sup_{n\geq 0} \frac{3^{n+1}}{2^n} \sum_{\ell(\beta)=n} \int_{I_\beta^E} |f\chi_{G_N}| \\
&= \sup_{n\geq 0} \frac{3^{n+1}}{2^n} \sum_{\ell(\beta)=n} \int_{I_\beta^E} |f| \sum_{\ell(\alpha)\geq N} \chi_{\widetilde{I}_\alpha^E} \\
&\geq \sup_{n\geq 0} \frac{3^{n+1}}{2^n} \sum_{\ell(\beta)=n} \int_{I_\beta^E} |f| \sum_{\ell(\alpha)\geq N} \chi_{\widetilde{I}_\alpha^E \cap E} \\
&= \sup_{n\geq 0} \frac{3^{n+1}}{2^n} \sum_{\ell(\beta)=n} \sum_{\ell(\alpha)\geq N} \int_{I_\beta^E} |f\chi_{I_\alpha^E}| \\
&\geq \sup_{n\geq N} \frac{3^{n+1}}{2^n} \sum_{\ell(\beta)=n} \sum_{\ell(\alpha)=n} \int_{I_\beta^E} |f\chi_{I_\alpha^E}| \\
&\geq \sup_{n\geq N} \frac{3^{n+1}}{2^n} \sum_{\ell(\beta)=n} \int_{I_\beta^E \cap E_1} |f|.
\end{aligned}
$$

By (3.7) and the inclusion $E_1 \subset M$ we arrive at

$$
\||f\chi_{G_N}\||_E \geq \sup_{n\geq N} \frac{3^{n+1}}{2^n} \int_{\cup I_\beta^E} |f| \geq \sup_{n\geq N} \frac{3^{n+1}}{2^n} \frac{2^{n-\ell(\alpha_0)-1}}{3^{n+1}} \eta = \frac{\eta}{2^{n_0+1}},
$$

where the union is taken over all β left of α_0 such that $\ell(\beta) = n$. Thus,

$$
\|f\chi_{G_N}\| = \sup_{E\in\mathcal{E}} \||f\chi_{G_N}\||_E \geq \frac{\eta}{2^{n_0+1}},
$$

and therefore f does not have absolutely continuous norm. $\square$

The proof of $X_c = X$ is rather technical. The core is contained in the following five lemmas.

Lemma 4.2. *Let $E \in \mathcal{E}$, let $\alpha_1 \preceq \alpha_2 \preceq \cdots \preceq \alpha_j \preceq \ldots$ be a sequence of multiindices with $\ell(\alpha_j) = j$ and let $\||f\||_E < \infty$. Then*

$$
\lim_{j\to\infty} \||f\chi_{J_{\alpha_j}^E}\||_E = 0. \tag{4.1}
$$

PROOF. It follows from (3.5) and (2.9) that $\||f\chi_{J_{\alpha_j}^E}\||_E$ is non-increasing. Assume that (4.1) does not hold. Then there exists $\eta > 0$ such that for every j the inequality

$$
\||f\chi_{J_{\alpha_j}^E}\||_E \geq \eta \tag{4.2}
$$

holds. Let us fix j and calculate $\||f\chi_{J^E_{\alpha_j}}\||_E$. According to (3.9) we have

$$
\begin{aligned}
\||f\chi_{J^E_{\alpha_j}}\||_E = &\sum_{k=0}^{j-1} \max_{\ell(\alpha)=k} \sup_{n\geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha\preceq\beta}} \int_{I^E_\beta \cap J^E_{\alpha_j}} |f| \\
&+ \sum_{k=j}^{\infty} \max_{\ell(\alpha)=k} \sup_{n\geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha\preceq\beta}} \int_{I^E_\beta \cap J^E_{\alpha_j}} |f| \\
= &\, A_j + B_j.
\end{aligned} \tag{4.3}
$$

Let $0 \leq k \leq j-1$. Then $\alpha_k \preceq \alpha_j$. If $\alpha \preceq \beta$, $\ell(\alpha)=k$ and $\alpha \nsim \alpha_k$, then $\beta \nsim \alpha_j$ and, according to (3.6), $I^E_\beta \cap J^E_{\alpha_j} = \emptyset$. Therefore,

$$
\max_{\substack{\ell(\alpha)=k \\ \alpha\nsim\alpha_k}} \sup_{n\geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha\preceq\beta}} \int_{I^E_\beta \cap J^E_{\alpha_j}} |f| = 0
$$

and we can rewrite A_j in the following form.

$$
A_j = \sum_{k=0}^{j-1} \sup_{n\geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha_k\preceq\beta}} \int_{I^E_\beta \cap J^E_{\alpha_j}} |f|.
$$

If $k \leq n \leq j-1$, then the inequalities $\ell(\beta) \leq j-1 < j = \ell(\alpha_j)$ imply $\alpha_j \npreceq \beta$ and (3.6) yields $I^E_\beta \cap J^E_{\alpha_j} = \emptyset$. Thus, we can write

$$
A_j = \sum_{k=0}^{j-1} \sup_{n\geq j} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha_k\preceq\beta}} \int_{I^E_\beta \cap J^E_{\alpha_j}} |f|.
$$

If $n \geq j$, then again (3.6) implies that $I^E_\beta \cap J^E_{\alpha_j} = \emptyset$ if $\alpha_j \npreceq \beta$ and $I^E_\beta \cap J^E_{\alpha_j} = I^E_\beta$ if $\alpha_j \preceq \beta$. Hence, we can further reduce the expression of A_j.

$$
A_j = \sum_{k=0}^{j-1} \sup_{n\geq j} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha_j\preceq\beta}} \int_{I^E_\beta} |f| = j \sup_{n\geq j} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha_j\preceq\beta}} \int_{I^E_\beta} |f|. \tag{4.4}
$$

Since

$$
B_j \leq \sum_{k=j}^{\infty} \max_{\ell(\alpha)=k} \sup_{n\geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha\preceq\beta}} \int_{I^E_\beta} |f| \leq \||f\||_E < \infty,
$$

we have $\lim_{j\to\infty} B_j = 0$, and so, according to (4.2), (4.3), there exists j_0 such that $A_j \geq \eta/2$ for $j \geq j_0$ and, by (4.4),

$$\sup_{n \geq j} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha_j \preceq \beta}} \int_{I_\beta^E} |f| \geq \frac{\eta}{2j}, \quad j \geq j_0.$$

Thus, using the assumption $\ell(\alpha_k) = k$, we obtain

$$\|f\|_E \geq \sum_{k=j_0}^{\infty} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha_k \preceq \beta}} \int_{I_\beta^E} |f| \geq \frac{\eta}{2} \sum_{k=j_0}^{\infty} \frac{1}{k} = \infty,$$

which contradicts the assumption.   □

Lemma 4.3. *Let $E \in \mathcal{E}$ and $\|f\|_E < \infty$. Then $\lim_{t\to 0+} \|f\chi_{(x-t,x+t)}\|_E = 0$ for all $x \in \mathbb{R}$.*

PROOF. We can assume that $x \in [\inf E, \sup E]$.

First of all, consider the case $x \notin \mathfrak{C}^E$. By (3.4), there exists a unique multiindex $\gamma \in \mathcal{K}$ such that $(x-t, x+t) \subset \widetilde{I}_\gamma^E$ for sufficiently small $t > 0$, and so

$$\|f\chi_{(x-t,x+t)}\|_E = \|f\chi_{(x-t,x+t)\cap\widetilde{I}_\gamma^E}\|_E$$

$$= \sum_{k=0}^{\infty} \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E \cap \widetilde{I}_\gamma^E} |f\chi_{(x-t,x+t)}|.$$

Property (3.4) yields

$$\sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E \cap \widetilde{I}_\gamma^E} |f\chi_{(x-t,x+t)}| = \begin{cases} \frac{3^{\ell(\gamma)+1}}{2^{\ell(\gamma)}} \int_{I_\gamma^E} |f\chi_{(x-t,x+t)}| & \text{if } \alpha \preceq \gamma, \\ 0 & \text{if } \alpha \npreceq \gamma. \end{cases}$$

Hence

$$\|f\chi_{(x-t,x+t)}\|_E = \sum_{k=0}^{\ell(\gamma)} \frac{3^{\ell(\gamma)+1}}{2^{\ell(\gamma)}} \int_{I_\gamma^E \cap (x-t,x+t)} |f| \to 0 \text{ as } t \to 0+,$$

since $\int_{I_\gamma^E} |f| \leq \|f\|_E < \infty$.

Now, assume that $x \in \mathfrak{C}^E$. Let $\{\alpha_n\}$ be the sequence of multiindices from (3.8). If x is an inner point of all $J_{\alpha_n}^E$, then there is a sequence of positive real numbers t_n such that $(x - t_n, x + t_n) \subset J_{\alpha_n}^E$ and for $0 < t < t_n$ we have

$$\||f\chi_{(x-t,x+t)}\||_E \leq \||f\chi_{J_{\alpha_n}^E}\||_E. \tag{4.5}$$

By Lemma 4.2 the last term tends to 0 as $n \to \infty$. So $\lim\limits_{t \to 0+} \||f\chi_{(x-t,x+t)}\||_E = 0$.

If $x = \inf E$ or $x = \sup E$, then $(x - t, x + t) \cap E$ is $(x, x + t) \cap E$ or $(x - t, x) \cap E$ and we arrive at (4.5) replacing $(x - t, x + t)$ with $(x, x + t)$ or $(x - t, x)$ respectively.

Finally, if there is n such that x is one of the end points of $J_{\alpha_n}^E$ and $x \neq \inf E$, $x \neq \sup E$, then there exists a unique interval $\widetilde{I}_\alpha^E$ such that either $x = \inf \widetilde{I}_\alpha^E$ or $x = \sup \widetilde{I}_\alpha^E$. We will investigate only the case $x = \inf \widetilde{I}_\alpha^E$. The other case is analogous. Then $x = \sup J_{\alpha_j}^E$ for all α_j with $\ell(\alpha_j) \geq \ell(\alpha) + 1$, and we can write

$$\||f\chi_{(x-t,x+t)}\||_E \leq \||f\chi_{(x-t,x)}\||_E + \||f\chi_{(x,x+t)}\||_E$$
$$= \||f\chi_{(x-t,x) \cap J_{\alpha_j}^E}\||_E + \||\chi_{(x,x+t) \cap I_\alpha^E}\||_E$$

for such t that $(x - t, x) \subset J_{\alpha_j}^E$ and $(x, x + t) \subset \widetilde{I}_\alpha^E$. The first term can be estimated in the same way as (4.5). The second term can be treated as the case $x \notin \mathfrak{C}^E$. $\square$

For $r \in \mathbb{R}$ and $M \subset \mathbb{R}$ we define the set $r + M$ by

$$r + M = \{r + x : x \in M\}.$$

Lemma 4.4. *Let $0 < t < 1$, $0 < \tau \leq \eta/4$ and let $f \in X$ satisfy*

$$\sup_{E \in \mathcal{E}} \||f\chi_{(0,t)}\||_E \geq \eta. \tag{4.6}$$

Then there exists $E \in \mathcal{E}$ and $\gamma \in \mathcal{K}$ such that

$$0 < s := \inf I_\gamma^E < t, \tag{4.7}$$

$$\||f\chi_{(s,t)}\||_E \geq \frac{\eta}{2} - 2\tau, \tag{4.8}$$

$$\sup I_\omega^E \leq 4t \quad and \quad |I_\omega^E| \leq 3t \quad whenever \quad I_\omega^E \cap (s, t) \neq \emptyset. \tag{4.9}$$

PROOF. Take $E_0 \in \mathcal{E}$ such that

$$\eta - \tau \leq \||f\chi_{(0,t)}\||_{E_0} \leq \sup_{E \in \mathcal{E}} \||f\chi_{(0,t)}\||_E. \tag{4.10}$$

At first, we will find $E_1 \in \mathcal{E}$ and $\gamma \in \mathcal{K}$ such that (4.7) and (4.8) hold with E_1 in place of E. According to Lemma 4.3, there is a number s_0, $0 < s_0 < t$, such that $\|\|f\chi_{(0,s_0)}\|\|_{E_0} \le \tau$ and so, by (4.10),

$$\|\|f\chi_{(s_0,t)}\|\|_{E_0} \ge \eta - 2\tau.$$

Set $r = \inf \bigcup I_\beta^{E_0}$ where the union is taken over all $\beta \in \mathcal{K}$ such that

$$I_\beta^{E_0} \cap (s_0, t) \neq \emptyset. \tag{4.11}$$

Note, that $r < t$. We will distinguish two cases: $s_0 \le r$ and $r < s_0$. Let $s_0 \le r$. Then $\|\|f\chi_{(s_0,t)}\|\|_{E_0} = \|\|f\chi_{(r,t)}\|\|_{E_0}$. By Lemma 4.3, there exists a number t_1, $r < t_1 < t$, such that $\|\|f\chi_{(r,t_1)}\|\|_{E_0} < \tau$. We take γ such that $r \le \inf I_\gamma^{E_0} < t_1$. Then γ satisfies (4.11). We define the mapping $P : \mathcal{K} \to \mathcal{F}$ by

$$P(\omega) = \begin{cases} I_\omega^{E_0} & \text{if } \omega = \gamma \text{ or if } \omega \text{ is to the right of } \gamma, \\ -1 + I_\omega & \text{if } \omega \text{ is to the left of } \beta, \end{cases}$$

where I_ω are the intervals from the construction of the classical Cantor set in Section 3 and we set $E_1 = \bigcup_{\omega \in \mathcal{K}} P(\omega)$. Clearly, $P \in \mathcal{P}$ and $E_1 \in \mathcal{E}$. Then (4.7) is satisfied and the estimates

$$\|\|f\chi_{(s,t)}\|\|_{E_1} \ge \|\|f\chi_{(t_1,t)}\|\|_{E_0} \ge \|\|f\chi_{(r,t)}\|\|_{E_0} - \|\|f\chi_{(r,t_1)}\|\|_{E_0} \ge \eta - 3\tau$$

yield

$$\|\|f\chi_{(s,t)}\|\|_{E_1} > \frac{\eta}{2} - 2\tau. \tag{4.12}$$

Now, let $r < s_0$. There exists a unique multiindex β satisfying (4.11) and $\inf I_\beta^{E_0} < s_0$. Then $\inf I_\beta^{E_0} = r$ and either

$$r < s_0 < t \le q \tag{4.13}$$

or

$$r < s_0 < q < t, \tag{4.14}$$

where $q = \sup I_\beta^{E_0}$.

Assume first that (4.13) holds. We change the set E_0 pasting the mass of $I_\beta^{E_0} \setminus (s_0, t)$ to the right of t. We define the mapping $P : \mathcal{K} \to \mathcal{F}$ by

$$P(\omega) = \begin{cases} \left[(s_0, t) \cap I_\beta^{E_0}\right] \cup \left(t, t + |I_\beta^{E_0} \setminus (s_0, t)|\right) & \text{if } \omega = \beta, \\ -1 + I_\omega & \text{if } \omega \text{ is to the left of } \beta, \\ t + |I_\beta^{E_0} \setminus (s_0, t)| + I_\omega & \text{if } \omega \text{ is to the right of } \beta, \end{cases}$$

and set $E_1 = \bigcup_{\omega \in \mathcal{K}} P(\omega)$. Clearly, $P \in \mathcal{P}$ and $E_1 \in \mathcal{E}$. We set $\gamma = \beta$. Since $(s_0, t) \cap E_1 = (s_0, t) \cap E_0 = (s_0, t) \cap I_\beta^{E_0} = (s_0, t) \cap I_\beta^{E_1} = (s, t) \cap I_\beta^{E_1}$, we have

$$\|\|f\chi_{(s,t)}\|\|_{E_1} = \|\|f\chi_{(s_0,t)}\|\|_{E_0} \geq \eta - 2\tau > \frac{\eta}{2} - 2\tau.$$

Now, let (4.14) hold. Then

$$\eta - 2\tau \leq \|\|f\chi_{(s_0,t)}\|\|_{E_0} \leq \|\|f\chi_{(s_0,q)}\|\|_{E_0} + \|\|f\chi_{(q,t)}\|\|_{E_0} = A + B$$

and so, either $A \geq \frac{\eta}{2} - \tau$ or $B \geq \frac{\eta}{2} - \tau$. If $A \geq \frac{\eta}{2} - \tau$ we define P by

$$P(\omega) = \begin{cases} -1 + I_\omega & \text{if } \omega \text{ is to the left of } \beta, \\ [(s_0, q) \cap E_0] \cup (q, q + |(r, s_0) \cap E_0|) & \text{if } \omega = \beta, \\ q + |(r, s_0) \cap E_0| + I_\omega & \text{if } \omega \text{ is to the right of } \beta, \end{cases}$$

and set $E_1 = \bigcup_{\omega \in \mathcal{K}} P(\omega)$. As in the previous case, $P \in \mathcal{P}$, setting $\gamma = \beta$ we obtain (4.7), and

$$\|\|f\chi_{(s,t)}\|\|_{E_1} \geq \|\|f\chi_{(s,q)}\|\|_{E_1} = \|\|f\chi_{(s_0,q)}\|\|_{E_0} = A \geq \frac{\eta}{2} - \tau$$

implies (4.12).

If $B \geq \frac{\eta}{2} - \tau$, we set $t_1 = \inf \bigcup I_\omega^{E_0}$ where the infimum is taken over all $\omega \in \mathcal{K}$, which are to the right of β. Then $q < t_1 < t$. By Lemma 4.3, there is a number s_2, $t_1 < s_2 < t$, such that $\|\|f\chi_{(s_2,t)}\|\|_{E_0} \geq \frac{\eta}{2} - 2\tau$. According to the definition of t_1 there is a multiindex β' such that $t_1 \leq \inf I_{\beta'}^{E_0} < s_2$. We define

$$P(\omega) = \begin{cases} -1 + I_\omega & \text{if } \omega \text{ is to the left of } \beta', \\ I_{\beta'}^{E_0} & \text{if } \omega = \beta', \\ I_\omega^{E_0} & \text{if } \omega \text{ is to the right of } \beta'. \end{cases}$$

Then $P \in \mathcal{P}$, and setting $E_1 = \bigcup_{\omega \in \mathcal{K}} P(\omega)$ and $\gamma = \beta'$ we obtain (4.7) and also (4.12) since

$$\|\|f\chi_{(s,t)}\|\|_{E_1} \geq \|\|f\chi_{(s_2,t)}\|\|_{E_0}.$$

Thus, we have $E_1 \in \mathcal{E}$ and $\gamma \in \mathcal{K}$ satisfying (4.7) and (4.8). If E_1 satisfies (4.9), we set $E = E_1$. Assume that E_1 does not satisfy (4.9). We have to change it so that (4.7), (4.8) remain valid and, moreover, (4.9) holds.

If $|I_\omega^{E_1}| \leq 3t$ for all ω such that $I_\omega^{E_1} \cap (s, t) \neq \emptyset$, then there exists $\alpha \in \mathcal{K}$ such that $\inf I_\alpha^{E_1} < t < \sup I_\alpha^{E_1}$. We define

$$P(\omega) = \begin{cases} -1 + I_\omega & \text{if } \omega \text{ is to the left of } \alpha, \\ [I_\alpha^{E_1} \cap (s, t)] \cup (t, t + |I_\alpha^{E_1} \setminus (s, t)|) & \text{if } \omega = \alpha, \\ 5t + I_\omega & \text{if } \omega \text{ is to the right of } \alpha \end{cases}$$

and set $E = \bigcup_{\omega \in \mathcal{K}} P(\omega)$. It is seen that (4.9) holds.

If $|I_\alpha^{E_1}| > 3t$ and $I_\alpha^{E_1} \cap (s,t) \neq \emptyset$ for some $\alpha \in \mathcal{K}$, then this α is determined in a unique way. We take the integer j such that

$$3^j t < |I_\alpha^{E_1}| \leq 3^{j+1} t \tag{4.15}$$

and denote by $\mathcal{L}$ the set of all $\omega \in \mathcal{K}$ such that $I_\omega^{E_1} \cap (s,t) \neq \emptyset$. We denote $\alpha = (a_1, \ldots, a_k)$ and define the multiindex $\alpha' = (a_1, \ldots, a_k, 0, 1, \ldots, 1)$ with $\ell(\alpha') = k + j$. The family $\mathcal{L}$ has the properties

$$\alpha' \notin \mathcal{L},$$
$$\text{if } \omega \text{ is between } \omega_1, \omega_2 \in \mathcal{L}, \text{ then } \omega \in \mathcal{L}, \tag{4.16}$$
$$\text{if } \omega \in \mathcal{L} \setminus \{\alpha\}, \text{ then } \alpha' \preceq \omega, \ \ell(\omega) > \ell(\alpha') \text{ and } \omega \text{ is to the right of } \alpha'.$$

We set $z = 3^{-k-j-1} - |I_\alpha^{E_1} \cap (s,t)|$. Note that (4.15) yields $z > 0$. It follows from (4.16) that the $(k+j+1)$-th component of every $\omega \in \mathcal{L} \setminus \{\alpha\}$ is 1. We consider the injection $\kappa : \mathcal{L} \setminus \{\alpha\} \to \mathcal{K}$ which changes the $(k+j+1)$-th component from 1 to 0 and keeps all other components unchanged. Obviously, κ preserves the relation $\preceq$. We define the mapping $P : \mathcal{K} \to \mathcal{F}$ by

$$P(\omega) = \begin{cases} -1 + I_\omega & \text{if } \omega \text{ is to the left of any element of} \\ & \kappa(\mathcal{L} \setminus \{\alpha\}), \\ I_{\kappa^{-1}(\omega)}^{E_1} & \text{if } \omega \in \kappa(\mathcal{L} \setminus \{\alpha\}), \\ \left[I_\alpha^{E_1} \cap (s,t) \right] \cup (t, t+z) & \text{if } \omega = \alpha', \\ t + z + I_\omega & \text{if } \omega \text{ is to the right of } \alpha'. \end{cases}$$

A simple analysis shows that $P \in \mathcal{P}$. We set $E = \bigcup_{\omega \in \mathcal{K}} P(\omega)$. Then $E \in \mathcal{E}$, (4.7) and (4.9) are satisfied, and it remains to verify the estimate (4.8). According to (3.9), we have

$$\||f\chi_{(s,t)}\||_E = \sum_{k=0}^\infty \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\omega)=n \\ \alpha \preceq \omega}} \int_{I_\omega^E} |f\chi_{(s,t)}|.$$

The integrals vanish for all ω except $\omega = \alpha'$ or $\omega \in \kappa(\mathcal{L})$, and

$$\frac{3^{\ell(\omega)+1}}{2^{\ell(\omega)}} \int_{I_\omega^E} |f\chi_{(s,t)}| = \frac{3^{\ell(\omega)+1}}{2^{\ell(\omega)}} \int_{I_{\kappa^{-1}(\omega)}^{E_1}} |f\chi_{(s,t)}| \quad \text{for } \omega \in \kappa(\mathcal{L}),$$

$$\frac{3^{\ell(\alpha')+1}}{2^{\ell(\alpha')}} \int_{I_{\alpha'}^E} |f\chi_{(s,t)}| = \left(\frac{3}{2}\right)^j \frac{3^{\ell(\alpha)+1}}{2^{\ell(\alpha)}} \int_{I_\alpha^{E_1}} |f\chi_{(s,t)}|.$$

This immediately yields the inequality $\|\|f\chi_{(s,t)}\|\|_E \geq \|\|f\chi_{(s,t)}\|\|_{E_1}$ and so, by (4.12), the estimate (4.8) holds. $\square$

Lemma 4.5. *Let $E \in \mathcal{E}$, $\gamma \in \mathcal{K}$, $0 < s < t < 1$ and $0 < \tau < \theta/2$. Let $f \in X$ be such that $\|\|f\chi_{(s,t)}\|\|_E \geq \theta$. Then there exists a positive integer N such that*

$$N \geq \ell(\gamma) \tag{4.17}$$

and

$$\sum_{k=0}^{N} \max_{\ell(\alpha)=k} \sup_{k \leq n \leq N} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}| \geq \theta - 2\tau.$$

PROOF. There exists a positive integer $N_1 \geq \ell(\gamma)$ such that

$$\sum_{k=0}^{N_1} \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}| \geq \theta - \tau.$$

For every α, $\ell(\alpha) = k \leq N_1$, we can find an integer $N_2(k,\alpha) \geq N_1$ such that

$$\sup_{k \leq n \leq N_2(k,\alpha)} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}| \geq \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}| - \frac{\tau}{2^{N_1+1} - 1}.$$

Setting $N = \max\{N_2(\alpha,k) : 0 \leq k \leq N_1, \ell(\alpha) = k\}$ we obtain

$$\sum_{k=0}^{N} \max_{\ell(\alpha)=k} \sup_{k \leq n \leq N} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}|$$

$$\geq \sum_{k=0}^{N_1} \max_{\ell(\alpha)=k} \sup_{k \leq n \leq N_2(\alpha,k)} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}|$$

$$\geq \sum_{k=0}^{N_1} \max_{\ell(\alpha)=k} \left(\sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}| - \frac{\tau}{2^{N_1+1} - 1} \right)$$

$$\geq \sum_{k=0}^{N_1} \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}| - \tau \geq \theta - 2\tau. \qquad \square$$

Lemma 4.6. *Let $f \in X$ be such that*

$$\lim_{t \to 0+} \sup_{E \in \mathcal{E}} \|\|f\chi_{(0,t)}\|\|_E = \eta > 0. \tag{4.18}$$

Let $0 < s < t < 1$, $E \in \mathcal{E}$ and $\gamma \in \mathcal{K}$ satisfy (4.7) and

$$\|\|f\chi_{(s,t)}\|\|_E = \theta > 0. \tag{4.19}$$

Then for every τ, $0 < \tau < \min(\eta/8, \theta/2)$ there exist $F \in \mathcal{E}$, $s_1 \in (0,s)$ and $\gamma_1 \in \mathcal{K}$ such that $s_1 = \inf I_{\gamma_1}^F$ and $\|\|f\chi_{(s_1,t)}\|\|_F \geq \theta + \frac{\eta}{2} - 4\tau$.

PROOF. Let

$$0 < t_1 < \min\{s/7, 3^{-N-2}\} \tag{4.20}$$

where N is the integer from Lemma 4.5. Since (4.6) holds with t_1 in place of t, from Lemma 4.4 we obtain $s_1 \in (0, t_1)$, $\gamma_1 \in \mathcal{K}$ and $E_1 \in \mathcal{E}$ such that

$$0 < s_1 := \inf I_{\gamma_1}^{E_1} < t_1, \tag{4.21}$$

$$\|\|f\chi_{(s_1,t_1)}\|\|_{E_1} \geq \frac{\eta}{2} - 2\tau, \tag{4.22}$$

$$\sup I_\omega^{E_1} \leq 4t_1 \quad \text{and} \quad |I_\omega^{E_1}| \leq 3t_1 \quad \text{whenever} \quad I_\omega^{E_1} \cap (s_1, t_1) \neq \emptyset. \tag{4.23}$$

Set

$$\mathcal{L} = \{\omega : I_\omega^E \cap (s,t) \neq \emptyset\}, \quad \mathcal{L}_1 = \{\omega : I_\omega^{E_1} \cap (s_1, t_1) \neq \emptyset\}$$

and denote $\alpha = (a_1, \ldots, a_k)$ the (unique) minimal element of $\mathcal{L}_1$ with respect to the relation $\preceq$. By (4.23), (4.20) and (4.17), the inequalities

$$3^{-\ell(\alpha)-1} = |I_\alpha^{E_1}| \leq 3t_1 < 3^{-N-1} \leq 3^{-\ell(\gamma)-1}$$

hold. Hence $\ell(\alpha) \geq \ell(\gamma) + 1$. Let $\gamma = (c_1, \ldots, c_j)$. The first k components of every element of $\mathcal{L}_1$ coincide with the components of α and we can define the injection $\kappa : \mathcal{L}_1 \to \mathcal{K}$ by

$$\omega = (a_1, \ldots, a_k, b_1, \ldots, b_n) \mapsto \kappa(\omega) = (c_1, \ldots, c_j, 0, 1, \ldots, 1, b_1, \ldots, b_n),$$
$$\ell(\kappa(\omega)) = \ell(\omega).$$

Every $\kappa(\omega)$ is to the left of every element of $\mathcal{L}$ because its $(j+1)$-th component is zero and, by (4.7), γ is the left most element of $\mathcal{L}$. Denote by $\mathcal{H}_1$ the family of all multiindices which are to the left of every element of $\kappa(\mathcal{L}_1)$, by $\mathcal{H}_2$ the family of multiindices which are between $\kappa(\mathcal{L}_1)$ and $\mathcal{L}$, and by $\mathcal{H}_3$ the set of all multiindices which are to the right of every element of $\mathcal{L}$.

Set $\delta = (c_1, \ldots, c_j, 0, 1, \ldots, 1)$ with $\ell(\delta) = k+1$, $r = \inf \bigcup_{\delta \preceq \beta} I_\beta$ and define the mapping $P : \mathcal{K} \to \mathcal{F}$ by

$$P(\omega) = \begin{cases} -1 + I_\omega & \text{if } \omega \in \mathcal{H}_1, \\ I_{\kappa^{-1}(\omega)}^{E_1} & \text{if } \omega \in \kappa(\mathcal{L}_1), \\ -r + 4t_1 + I_\omega & \text{if } \omega \in \mathcal{H}_2, \\ I_\omega^E & \text{if } \omega \in \mathcal{L}, \\ \sup E + I_\omega & \text{if } \omega \in \mathcal{H}_3, \end{cases}$$

where I_ω are the intervals from the construction of the classical Cantor set in Section 3. Set $F = \bigcup_{\omega \in \mathcal{K}} P(\omega)$. Then P satisfies (3.1), F is bounded and in order to show that $F \in \mathcal{E}$ we have to verify (3.2).

If β is to the left of β' and both β, β' belong to one of the sets $\mathcal{H}_1$, $\kappa(\mathcal{L})$, $\mathcal{H}_2$, $\mathcal{L}$ or $\mathcal{H}_3$, then obviously $P(\beta)$ is to the left of $P(\beta')$. If $\beta \in \mathcal{H}_1$, then $P(\beta) = -1 + I_\omega \subset (-1, 0)$. If $\beta = \kappa(\omega)$ for some $\omega \in \mathcal{L}_1$, then $P(\beta) = I_\omega^{E_1} \subset [s_1, 4t_1]$ according to (4.21) and (4.23). If $\beta \in \mathcal{L}$, then $P(\beta) \subset [s, \sup E]$. If $\beta \in \mathcal{H}_3$, then $P(\beta) \subset (\sup E, \sup E + 1)$. Finally, let $\beta \in \mathcal{H}_2$. Since

$$\mathcal{H}_2 \subset \{\omega \in \mathcal{K} : \delta \preceq \omega\}, \tag{4.24}$$

we have $\inf P(\beta) > 4t_1$. On the other hand, by (4.24) and (4.23),

$$\sup_{\delta \preceq \omega} \bigcup I_\omega - \inf_{\delta \preceq \omega} \bigcup I_\omega = |\bigcup_{\delta \preceq \omega} I_\omega| = 3^{-\ell(\delta)} = 3^{-k-1} = |I_\alpha^{E_1}| \le 3t_1.$$

This yields $\sup P(\beta) = -r + 4t_1 + \sup I_\beta < 7t_1 < s$ by (4.20). All these relations together imply that F satisfies (3.2).

Now, we can write

$$\|f\chi_{(s_1,t)}\|_F \ge \sum_{k=0}^{N} \max_{\ell(\alpha)=k} \sup_{k \le n \le N} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^F} |f\chi_{(s,t)}|$$

$$+ \sum_{k=N+1}^{\infty} \max_{\ell(\alpha)=k} \sup_{n \ge k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^F} |f\chi_{(s_1,t_1)}| = A + B.$$

Since $I_\beta^F \cap (s,t) \neq \emptyset$ if and only if $\beta \in \mathcal{L}$, we have

$$A = \sum_{k=0}^{N} \max_{\ell(\alpha)=k} \sup_{k \leq n \leq N} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^F} |f\chi_{(s,t)}|$$

$$= \sum_{k=0}^{N} \max_{\ell(\alpha)=k} \sup_{k \leq n \leq N} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^E} |f\chi_{(s,t)}|$$

and so, according to the assumption (4.19) and Lemma 4.5, $A \geq \theta - 2\tau$. Using the definitions of $\mathcal{L}_1$, κ and P we obtain

$$B = \sum_{k=N+1}^{\infty} \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^F} |f\chi_{(s_1,t_1)}|$$

$$= \sum_{k=N+1}^{\infty} \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^{E_1}} |f\chi_{(s_1,t_1)}|.$$

According to (4.23) and (4.20), for every $\beta \in \mathcal{L}_1$ we have $3^{-N-1} > 3t_1 \geq |I_\beta^{E_1}| = 3^{-\ell(\beta)-1}$ and so, $\ell(\beta) > N$. Hence, by (4.22),

$$B = \sum_{k=0}^{\infty} \max_{\ell(\alpha)=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{\ell(\beta)=n \\ \alpha \preceq \beta}} \int_{I_\beta^{E_1}} |f\chi_{(s_1,t_1)}| = \||f\chi_{(s_1,t_1)}|\|_{E_1} \geq \frac{\eta}{2} - 2\tau.$$

Thus, $\||f\chi_{(s_1,t)}|\| \geq A + B \geq \theta + \frac{\eta}{2} - 4\tau$. $\qquad\square$

Lemma 4.7. *Every function $f \in X$ has continuous norm in X; i.e., $X_c = X$.*

PROOF. Our aim is to prove that for every $f \in X$, $\lim_{t \to 0+} \|f\chi_{(x-t,x+t)\cap(0,1)}\| = 0$, $x \in [0,1]$ or, equivalently,

$$\lim_{t \to 0+} \|f\chi_{(x,x+t)}\| = 0, \quad x \in [0,1), \qquad (4.25)$$

and $\lim_{t \to 0+} \|f\chi_{(x-t,x)}\| = 0, \quad x \in (0,1]$. We will prove only (4.25); the other relation can be proved in the same way. Without loss of generality we can suppose that $x = 0$ because $\|f\chi_{(x,x+t)}\| = \|g\chi_{(0,t)}\|$ for $g(\xi) = f(\xi + x)$, $0 < \xi < 1 - x$.

Assume, to the contrary, that $f \in X \setminus X_c$, $f \neq 0$; i.e., f satisfies (4.18). Let $0 < \tau < \eta/22$. Then there exist $0 < t < 1$ and $E \in \mathcal{E}$ such that

$$\eta \leq \sup_{E \in \mathcal{E}} \|\!|f\chi_{(0,t)}|\!\|_E < \eta + \tau. \qquad (4.26)$$

Let $s \in (0, t)$, $\gamma \in \mathcal{K}$ and $E \in \mathcal{E}$ be from Lemma 4.4 and let N be the corresponding integer from Lemma 4.5. Then the assumptions of Lemma 4.6 are satisfied with $\theta \geq \frac{\eta}{2} - 2\tau$ and so there exist $E_1 \in \mathcal{E}$, $s_1 \in (0, s)$ and $\gamma_1 \in \mathcal{K}$ such that $s_1 = \inf I_{\gamma_1}^{E_1}$ and $\|\!|f\chi_{(s_1,t)}|\!\|_{E_1} \geq \theta + \frac{\eta}{2} - 4\tau \geq \eta - 6\tau$. Hence the assumptions of Lemma 4.6 are satisfied with s_1, E_1, γ_1 and $\theta \geq \eta - 6\tau$, and there exist $E_2 \in \mathcal{E}$, $s_2 \in (0, s_1)$, $\gamma_2 \in \mathcal{K}$ such that $s_2 = \inf I_{\gamma_2}^{E_2}$ and

$$\|\!|f\chi_{(s_2,t)}|\!\|_{E_2} \geq \theta + \frac{\eta}{2} - 4\tau \geq \frac{3}{2}\eta - 10\tau > \eta + \tau.$$

This contradicts the assumption (4.26). □

Remark. It is possible to repeat the last step in the proof ad libitum to get that $\|f\| = \infty$.

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