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ON THE FRACTIONAL PART OF THE SEQUENCE $\{\xi\beta_n - a\}$

Abstract

Let $\{\alpha_n\}_{n=0}^{\infty}$ denote a sequence of positive real numbers and let the sequence $\{\beta_n\}_{n=0}^{\infty}$ be defined by $\beta_0 = 1$ and $\beta_{n+1} = \prod_{j=0}^{n} \alpha_j$. For $0 \leq a < 1$, $0 < t < 1$, and n, a nonnegative integer, the inequality $0 \leq \{\xi\beta_n - a\} \leq t$ is studied, where $\{x\}$ denotes the fractional part of x.

Let $\delta(a, k) = \sup_{m\in\mathbb{Z}}\{a - (a + m)k\}$ for each real number k, where $\mathbb{Z}$ is the set of all integers. If $\alpha_n \geq 1 + \delta(a, \alpha_n)/t$, for each nonnegative integer n, where $0 \leq a < 1$, $0 < t < 1$, and $b = a + t$, then it is proved that there exists a $\xi \in [a + m, b + m]$, for each $m \in \mathbb{Z}$, such that $0 \leq \{\xi\beta_n - a\} \leq t$ holds for all nonnegative integers n. Further, if $t\alpha_n\alpha_{n+1} - (1 + \delta(a, \alpha_n))\alpha_{n+1} - t - \delta(a, \alpha_{n+1}) \geq 0$ for infinitely many nonnegative integers n, then for each $m \in \mathbb{Z}$, there exists a set of $\xi \in [a + m, b + m]$ that has the cardinality of the continuum so that $0 \leq \{\xi\beta_n - a\} \leq t$ is true for all nonnegative integers n.

1 Introduction

In his paper "An unsolved problem on the powers of 3/2", Mahler [12] defines a real number $\alpha, \alpha > 0$, to be a Z-number if $0 \leq \{\alpha(\frac{3}{2})^n\} < 1/2$, for all $n \in W$, where $\{x\}$ denotes the fractional part of x, and W is the set of nonnegative integers. Although Mahler proves that the set of all Z-numbers is at most countable, it is still unknown whether such Z-numbers exist. This problem is now known as Mahler's problem [15]. Some results related to this problem are contained in references [3, 4, 5, 6, 9].

A related problem is the existence of $\xi > 0$ so that $0 \leq \{\xi\beta^n\} \leq t$, for all $n \in W$, given $0 < t < 1$ and $\beta > 1$. Tijdeman [15] has shown that such a ξ exists, for $\beta > 2$ and $t \geq 1/(\beta - 1)$. Further, Flatto [11] shows if $\beta > 0$ is

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rational (that is, $\beta = p/q, (p,q) = 1$, and $p, q \in N$), then $\xi > 0$ exists if $\beta > 2$ and $t > (q-1)/(q(\beta-1)) = (q-1)/(p-q)$.

In answer to a question of Erdős [10], Pollington [14] proved that if any positive real sequence $\{\beta_n\}_{n=0}^{\infty}$ satisfies $\beta_{n+1}/\beta_n \geq \alpha > 1$, for all $n \in W$, then there is a set of real ξ of Hausdorff dimension 1 so that the sequence $\{\xi\beta_n\}$ is not dense on $[0,1)$. Boshernitzan [2] and Ajtai, Havas, and Komlós [1] have shown that the fixed lower bound on the ratio of consecutive β_n is necessary. In lemma 1 of the latter, the authors show that for any sequence of real $\alpha_n > 1, \alpha_n \to 1$, there exists a sequence of positive integers $\{\beta_n\}_{n=0}^{\infty}$ such that $\beta_{n+1}/\beta_n \geq \alpha_n$, for all n, and for any irrational ξ, the sequence $\xi\beta_n$ is uniformly distributed (mod 1). For rational $\xi = p/q, (p,q) = 1$, then the sequence $\xi\beta_n$ is uniformly distributed (mod 1) over the set $\{0, 1/q, 2/q, \ldots, (q-1)/q\}$.

This motivates the question: "For what sequences $\{\beta_n\}_{n=0}^{\infty}$ will $\xi > 0$ exist such that $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$, where $0 \leq a < 1$ and $0 < t < 1$?" This paper identifies a class of sequences $\{\beta_n\}_{n=0}^{\infty}$ for which this is true.

2 Existence of a Set of ξ

Let $\{\beta_n\}_{n=0}^{\infty}$ be a positive, strictly increasing sequence of real numbers, and $\{\alpha_n\}_{n=0}^{\infty}$ be a positive sequence of real numbers such that $\alpha_n = \beta_{n+1}/\beta_n$, for all $n \in W$.

Let $A_n = \bigcup_{m=-\infty}^{\infty} Q(m,n)$, with $Q(m,n) = [(a+m)\beta_n^{-1}, (b+m)\beta_n^{-1}]$, for all $n \in W$ and $m \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, and a, b are real numbers such that $a < b$.

For $a, k \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, define

$$\delta(a,k) = \begin{cases} 1, \text{if } k \text{ is irrational,} \\ (q-1)/q + \{a(q-p)\}/q, \text{if } k = p/q, \text{ where } p, q \text{ are relatively} \\ \text{prime integers, and } q > 0. \end{cases}$$

It can be shown that $\delta(a,k) = \sup_{m\in\mathbb{Z}}\{a - (a+m)k\}$. This fact is used extensively in this paper. For example, $\delta(0, p/q) = (q-1)/q$, where p and q are relatively prime positive integers.

Theorem 1. *Let $\{\beta_n\}_{n=0}^{\infty}$ be a positive increasing sequence of real numbers such that $\beta_{n+1}/\beta_n = \alpha_n$, for all $n \in W$. If $0 \leq a < 1, 0 < t < 1$, $b = a + t$, and $\alpha_n \geq 1 + \delta(a, \alpha_n)/t$, for all $n \in W$, then for every $m \in \mathbb{Z}$, there exists a $\xi \in [a+m, b+m]$, such that $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$.*

The following lemmas are needed to prove this theorem.

In what follows, a, b, and t are real numbers such that $0 \leq a < 1$, $0 < t < 1$, and $b = a + t$.

Lemma 1. *If $\alpha_n > 0$, for all $n \in W, \xi \in \mathbb{R}$, then $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$, if and only if $\xi \in \bigcap_{n=0}^{\infty} A_n$.*

Proof. Given $\alpha_n > 0$, for all $n \in W$, $\xi \in \mathbb{R}$, $0 \leq a < 1$, and $b = a + t$, if $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$, and j $= \lfloor \xi\beta_n - a \rfloor$, then $a + j \leq \xi\beta_n < b + j$, that is, $\xi \in Q(j, n)$, for all $n \in W$. Hence, $\xi \in \bigcap_{n=0}^{\infty} A_n$. On the other hand, if $\xi \in \bigcap_{n=0}^{\infty} A_n$, then for all $n \in W$, there exists $m \in \mathbb{Z}$ so that $\xi \in Q(m, n)$, that is, $a \leq \{\xi\beta_n - m\} \leq b$. Thus, $0 \leq \{\xi\beta_n - a - m\} \leq t$, implying $m = \lfloor \xi\beta_n - a \rfloor$. Hence, $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$. Notice that this lemma proves that the set of all ξ that satisfy $0 \leq \{\xi\beta_n - a\} \leq t$ is a closed set, since it is equal to the intersection of the closed sets A_n. □

Define $m' = \lceil (a + m)\alpha_n - a \rceil \in \mathbb{Z}$, where $n \in W$. Here, $\lceil x \rceil$ is the least integer greater than x.

Lemma 2. *If $m \in \mathbb{Z}$, $n \in W$, $\alpha_n > 1 + \frac{\delta(a,\alpha_n)}{t}$, and $m' = \lceil (a+m)\alpha_n - a \rceil \in \mathbb{Z}$, then $Q(m', n+1) \subseteq Q(m, n)$.*

Proof. By definition of m', $(a + m)\alpha_n \leq a + m'$. Also,
$a + m' - \delta(a, \alpha_n) \leq (a + m)\alpha_n$. Hence,

$$(a + m' - \delta(a, \alpha_n))\beta_{n+1}^{-1} \leq (a + m)\beta_n^{-1} \leq (a + m')\beta_{n+1}^{-1}.$$

Let $x \in Q(m', n + 1)$. Then $x \geq (a + m)\beta_n^{-1}$, and

$$x \leq (b + m')\beta_{n+1}^{-1} \leq (\delta(a, \alpha_n) + b - a)\beta_{n+1}^{-1} + (a + m' - \delta(a, \alpha_n))\beta_{n+1}^{-1}.$$

So

$$x \leq (\delta(a, \alpha_n) + b - a)\beta_{n+1}^{-1} + (a + m)\beta_n^{-1} \leq (b + m)\beta_n^{-1}.$$

Hence, $x \in Q(m, n)$. □

The theorem can now be proven.

Proof. Let $\alpha_n \geq 1 + \delta(a, \alpha_n)/t$, for all $n \in W$. Define the sequence $\{m_n\}_{n=0}^{\infty}$ where m_0 is an arbitrary integer, and $m_{n+1} = \lceil (a - m_n)\alpha_n - a \rceil$, for all $n \in W$. Also define the sequence $\{I_n\}_{n=0}^{\infty}$ of closed bounded intervals $I_n = Q(m_n, n)$. From Lemma 2, $I_{n+1} \subseteq I_n$, for all $n \in W$. Further, $I_n \subseteq A_n$, for all $n \in W$.

Suppose $I_j \subseteq \bigcap_{n=0}^{j} A_n$. Then $I_{j+1} \subseteq I_j \subseteq \bigcap_{n=0}^{j} A_n$ and also $I_{j+1} \subseteq A_{j+1}$. Thus $I_{j+1} \subseteq \bigcap_{n=0}^{j+1} A_n$. Since $I_0 \subseteq A_0 = \bigcap_{n=0}^{0} A_n$, by the principle of induction, $I_j \subseteq \bigcap_{n=0}^{j} A_n$, for all $j \in W$.

By Cantor's nested interval theorem, there exists a $\xi \in \mathbb{R}$ such that $\xi \in I_j$, for all $j \in W$. Hence, $\xi \in \bigcap_{n=0}^{\infty} A_n$, and by Lemma 1, $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$. In addition, $\xi \in I_0 = [a + m_0, b + m_0]$. □

Theorems 1 and 2 of Tijdeman [15] are special cases of this theorem when α_n is constant for all $n \in W$.

3 Uncountability of the Set of ξ

Flatto [11] gives a condition for uncountability of the set of ξ that satisfies $0 \leq \{\xi\beta^n\} \leq t$, for all $n \in W$. If $\beta > 3$ and $2/(\beta-1) < t < 1$, then for any integer m, there exists such a set of ξ with cardinality of the continuum and where $\xi \in [m, m+1)$.

In what follows, an improvement of this theorem is given for certain sequences of positive real numbers $\{\beta_n\}_{n=0}^{\infty}$, where $\beta_{n+1}/\beta_n \geq \alpha > 1$, for all $n \in W$.

Theorem 2. *Let $\{\beta_n\}_{n=0}^{\infty}$ be a positive increasing sequence of real numbers such that $\alpha_n = \beta_{n+1}/\beta_n$, for all $n \in W$. Given $0 \leq a < 1, 0 < t < 1$, $b = a+t$, and $\alpha_n \geq 1+\delta(a,\alpha_n)/t$, for all $n \in W$, if there is a strictly increasing sequence of whole numbers $\{k_i\}_{i=0}^{\infty}$ such that*

$$t\alpha_{k_i}\alpha_{k_i+1} - (1+\delta(a,\alpha_{k_i}))\alpha_{k_i+1} - t - \delta(a,\alpha_{k_i+1}) \geq 0,$$

then for any $m \in \mathbb{Z}$, there exists a set of ξ with the cardinality of the continuum so that $\xi \in [a+m, b+m]$ and ξ satisfies $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$.

To prove this theorem we need Lemma 3.

Lemma 3. *Suppose $m \in \mathbb{Z}$, $n \in W$,*

$$t\alpha_n\alpha_{n+1} - (1+\delta(a,\alpha_n)\alpha_{n+1} - t - \delta(a,\alpha_{n+1}) \geq 0, \tag{1}$$

$\alpha_n \geq 1+\delta(a,\alpha_n)/t$, and $\alpha_{n+1} \geq 1+\delta(a,\alpha_{n+1})/t$. If

$$\begin{aligned} m_1 =& \lceil (a+m)\alpha_n - a \rceil, \\ m_2 =& \lceil (a+m_1)\alpha_{n+1} - a \rceil, \\ \textit{and } m_2^{'} =& \lceil (a+m_1+1)\alpha_{n+1} - a \rceil, \end{aligned}$$

then

$$\begin{aligned} Q(m_2, n+2) \subseteq& Q(m_1, n+1), \\ Q(m_2^{'}, n+2) \subseteq& Q(m_1+1, n+1), \\ Q(m_2, n+2) \subseteq& Q(m, n), \\ \textit{and } Q(m_2^{'}, n+2) \subseteq& Q(m, n). \end{aligned}$$

Proof. By Lemma 2, $Q(m_1, n+1) \subseteq Q(m,n)$, $Q(m_2, n+2) \subseteq Q(m_1, n+1)$, and $Q(m_2^{'}, n+2) \subseteq Q(m_1, n+1)$. Thus, $Q(m_2, n+2) \subseteq Q(m,n)$.

If $x \in Q(m_2', n+2)$, then $x \geq (a + m_2')\beta_{n+2}^{-1}$. This implies that $x > (a + m_2)\beta_{n+2}^{-1} \geq (a + m)\beta_n^{-1}$.

On the other hand, $x \leq (b+m_2')\beta_{n+2}^{-1}$. If α_n and α_{n+1} satisfy inequality (1), then

$$(\delta(a, \alpha_n) + 1)\alpha_{n+1} + t + \delta(a, \alpha_{n+1}) \leq (b + m)\alpha_n\alpha_{n+1} - (a + m)\alpha_n\alpha_{n+1}.$$

This inequality yields

$$\begin{aligned}(b + m)\alpha_n\alpha_{n+1} \geq &[(a + m)\alpha_n + \delta(a, \alpha_n) + 1]\alpha_{n+1} + t + \delta(a, \alpha_{n+1})\\ =&[a + \lceil (a + m)\alpha_n - a\rceil + 1]\alpha_{n+1} + t + \delta(a, \alpha_{n+1})\\ =&(a + m_1 + 1)\alpha_{n+1} + t + \delta(a, \alpha_{n+1})\\ \geq&(a + m_1 + 1)\alpha_{n+1} + (b - a) + \delta(a, \alpha_{n+1})\\ \geq&a + \lceil (a + m_1 + 1)\alpha_{n+1} - a\rceil + (b - a) = b + m_2'.\end{aligned}$$

Thus, $b + m_2' \leq (b + m)\alpha_n\alpha_{n+1}$, which implies that $x \leq (b + m)\beta_n^{-1}$. Hence, $x \in Q(m, n)$ and thus $Q(m_2', n + 2) \subseteq Q(m, n)$, which completes the proof of this lemma. □

To prove Theorem 2, a ξ will be constructed satisfying $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$, and that is related to some binary sequence. It will then be shown that there is a one-to-one correspondence between these ξ and the set of all binary sequences which have the cardinality of the continuum.

Proof. Let a, b, t, and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ have the properties described in the statement of the theorem. Let the sequence $\{k_i\}_{n=0}^{\infty}$ be further restricted by the condition $k_{i+1} - k_i \geq 2$. Note that this restriction is justifiable since any increasing sequence of integers has a subsequence where consecutive terms differ by at least two. Finally, the inequality (1) holds for all k_i, upon substituting k_i for n.

Let $R = \{r_i\}_{i=0}^{\infty}$ be a binary sequence; that is, $r_i \in \{0, 1\}$, for all $i \in W$.

Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of integers, with m_0 an arbitrary integer, and

$$m_{n+1} = \begin{cases} \lceil (a + m_n + r_j)\alpha_n - a\rceil, & \text{where } j \in W, n = k_j + 1,\\ \lceil (a + m_n)\alpha_n - a\rceil, & \text{otherwise.}\end{cases}$$

A sequence of intervals of I_n can now be defined by $I_n = Q(m_n, n)$. Note that given m_0, a, and $\{\alpha_n\}_{n=0}^{\infty}$, I_n is dependent only on the binary sequence R. So I_n will be a function from the binary sequences to a set of compact intervals in $\mathbb{R}$. This is denoted by $I_n(R) = Q(m_n, n)$.

If $n \neq k_j + 1$, for all $j \in W$, then $m_{n+1} = \lceil (a + m_n)\alpha_n - a\rceil$. Hence, $\alpha_n \geq 1 + \delta(a, \alpha_n)/t$ implies that $Q(m_{n+1}, n + 1) \subseteq Q(m, n)$, by Lemma 2, and $I_{n+1}(R) \subseteq I_n(R)$.

If $n = k_j + 1$ for some $j \in W$, then $m_{n+1} = \lceil (a + m_n + r_j)\alpha_n - a \rceil$ and $t\alpha_{n-1}\alpha_n - (1 + \delta(a, \alpha_{n-1}))\alpha_n - t - \delta(a, \alpha_n) \geq 0$. Therefore, by Lemma 3, $Q(m_{n+1}, n + 1) \subseteq Q(m, n)$ and $Q(m_{n+1}, n + 1) \subseteq Q(m_{n-1}, n - 1)$, regardless of the value of r_j. Thus, $I_{n+1}(R) \subseteq I_n(R)$ and $I_{n+1}(R) \subseteq I_{n-1}(R)$. If $l \neq k_j$, for all $j \in W$, and $I_l(R) \subseteq \bigcap_{n=0}^{l} A_n$, then $I_{l+1}(R) \subseteq I_l(R)$ and $I_{l+1}(R) \subseteq \bigcap_{n=0}^{l} A_n$. Also $I_{l+1}(R) \subseteq A_{l+1}$, and hence, $I_{l+1}(R) \subseteq \bigcap_{n=0}^{l} A_n$. If $l = k_j$, for some $j \in W$, and $I_{l-1}(R) \subseteq \bigcap_{n=0}^{l-1} A_n$, then $I_{l+1}(R) \subseteq \bigcap_{n=0}^{l-1} A_n$. Further, $I_{l+1}(R) \subseteq I_l(R)$, which implies that $I_{l+1}(R) \subseteq A_l$. Since $I_{l+1}(R) \subseteq A_{l+1}$, it follows that $I_{l+1}(R) \subseteq \bigcap_{n=0}^{l+1} A_n$. So by an argument similar to the one used in Theorem 1, $I_l(R) \subseteq \bigcap_{n=0}^{l} A_n$, for all $l \neq k_j, j \in W$. By Cantor's nested interval theorem, there exists a real $\xi \in [m_0+a, m_0+b]$ such that $\xi \in \bigcap_{n=0}^{\infty} A_n$. Thus by Lemma 1, $0 \leq \{\xi\beta_n - a\} \leq t$ is true, for all $n \in W$.

Let $S = \{s_i\}_{i=0}^{\infty}, s_i \in \{0, 1\}$ be a second binary sequence distinct from R, that is, there is some whole number n such that $s_n \neq r_n$. A new sequence of integers $\{l_n\}_{n=0}^{\infty}$ can be constructed with l_0 an arbitrary integer, and

$$l_{n+1} = \begin{cases} \lceil (a + l_n + s_j)\alpha_n - a \rceil, & \text{where } j \in W, \\ \lceil (a + l_n)\alpha_n - a \rceil, & \text{otherwise.} \end{cases}$$

There is a real ξ' that is contained in all the sets $I_n(S)$ and such that $0 \leq \{\xi'\beta_n - a\} \leq t$ holds, for all $n \in W$.

Since $S \neq R$, there exists a whole number p such that $s_p \neq r_p$ and $s_i = r_i$, for $0 \leq i \leq p-1$. This means that $l_{k_p} = m_{k_p}$ and thus $Q(l_{k_p}, k_p) = Q(m_{k_p}, k_p)$; that is, $I_{k_p}(R) = I_{k_p}(S)$.

However, $s_p \neq r_p$ yields $l_{k_p+1} \neq m_{k_p+1}$, and hence,

$$Q(l_{k_p+1}, k_p + 1) \bigcap Q(m_{k_p+1}, k_p + 1) = \emptyset.$$

This implies that $\xi \neq \xi'$. Thus, for every binary sequence, there is a unique $\xi \in [a+m_0, b+m_0]$ so that $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$. The set of infinite binary sequences has the cardinality of the continuum and so must the set of ξ that satisfies $0 \leq \{\xi\beta_n - a\} \leq t$, for all $n \in W$, and $\xi \in [a + m_0, b + m_0]$. This proves the theorem. ☐

A useful corollary follows from this theorem when $\beta_n = k^n$, for all $n \in W$. This corollary improves a result of Flatto [11].

Corollary 1. *Given* $0 \leq a < 1, 0 < t < 1, b = a + t$, *and if*

$$k \geq \frac{1 + \delta(a, k) + \sqrt{(1 + \delta(a, k))^2 + 4t(t + \delta(a, k))}}{2t}, \tag{2}$$

then for every integer m there is a set of ξ with the cardinality of the continuum so that $\xi \in [a+m, b+m]$, and ξ satisfies $0 \le \{\xi k^n - a\} \le t$, for all $n \in W$.

Proof. Note that k satisfies $k \ge 1 + \delta(a,k)/t$, and

$$tk^2 - (1+\delta(a,k))k - t - \delta(a,k) \ge 0.$$

Let $\alpha_n = k$ and $\beta_n = k^n$, for all $n \in W$. The hypotheses of Theorem 2 are then met, and the corollary follows.

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