Real Analysis Exchange
Summer Symposium 2002 pp. 487–488

E. Talvila, Department of Mathematics and Statistics, University College of the Fraser Valley, Abbotsford, BC, Canada V2S 7M8.
email: `Erik.Talvila@ucfv.ca`

CHARACTERIZING INTEGRALS OF RIEMANN INTEGRABLE FUNCTIONS

If f is Riemann integrable on $\mathbb{R}$ ($f \in \mathcal{R}([0,1])$), define $F(x) = \int_0^x f(t)\,dt$. Since f is bounded, there exists $M \in \mathbb{R}$ with $|f(x)| \leq M$ for every $x \in [0,1]$. Hence, for $0 \leq x < y \leq 1$ we have

$$|F(x) - F(y)| \leq \int_x^y |f(t)|\,dt \leq M \cdot |x - y|,$$

and so F is necessarily Lipschitz on $[0,1]$. However, it is relatively straightforward to see that $\mathcal{R}([0,1])$ is not simply the class of derivatives of Lipschitz functions on $[0,1]$. To see this, let C be a Cantor set in $[0,1]$ with positive measure, a so-called *fat Cantor set.* If $I \subset [0,1]$ is an interval whose endpoints are in C but whose interior is disjoint from C then let F be a tent function on I with $F = 0$ at the endpoints of I and F has slope 1 on one half of I and slope -1 on the other half of I. Then F is Lipschitz on $[0,1]$ and $|F'| = 1$ where it is defined, but $F' \notin \mathcal{R}([0,1])$. This construction was pointed out by Z. Buczolich, L. Larson, T. Trainor and L. Moonens.

Question 1. What is a (geometric) characterization of the class of Riemann integrable functions? The corresponding result for Lebesgue integrals is that $g \in L^1([0,1])$ if and only if there is a function G that is absolutely continuous on $[0,1]$ such that $G' = g$ almost everywhere on $(0,1)$. And then $\int_0^1 g = G(1) - G(0)$.

Key Words: Riemann integrable
Mathematical Reviews subject classification: 26A42
Received by the editors June 12, 2008