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REGULATED FUNCTIONS ON TOPOLOGICAL SPACES

Abstract

A regulated function on the real line is a real valued function whose left-hand and right-hand limits exist at all points. In this paper we examine a generalization of regulated functions to functions defined on Davison Spaces, which are topological spaces with a little extra structure. Properties of such functions are discussed. Our main result concerns the set of discontinuities of these functions. We also prove that regulated functions defined on the natural numbers, with the cofinite topology, coincide with convergent sequences.

1 Introduction.

In a short paper in 1979, T. M. K. Davison introduced a generalization of regulated functions motivated by the question "what does a regulated function on $\mathbb{R}^n$ look like?". There seems to have been no other attempt to generalize the idea of a regulated function to higher dimensions and, to the best of our knowledge, these ideas of Davison have not been developed elsewhere. He develops a theory to describe regulated functions from a topological space to $\mathbb{R}$. In this paper, we resurrect Davison's beautiful generalization. We modify his definition. We examine examples of regulated functions on $\mathbb{R}^2$ and the natural numbers, whereby regulated functions coincide with convergent sequences. We prove a result regarding the set of discontinuities of a regulated function.

A function $f : [a, b] \to \mathbb{R}$ is called regulated if its left-hand and right-hand limits exist at all points. Examples of regulated functions on an interval $[a, b]$ include step functions, functions of bounded variation, càdlàg functions and

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continuous functions. Regulated functions were studied as far back as the early twentieth century by Hobson [9]. Many results are known regarding regulated functions defined on a closed subinterval of the real line. A regulated function has countably many discontinuities and is the uniform limit of a sequence of step functions. The class of bounded regulated functions is complete and forms a Banach algebra. The class of functions of bounded variation is dense in the class of regulated functions. See for example [2], [4], [5], [6], [8] and [12] for more on regulated functions. More recently, it's been shown that a function is regulated if and only if it is of bounded ϕ-variation [14].

Regulated functions give an alternative introduction to the Riemann integral [1]. They also are important in the study of Fourier series [7]. Key properties of the class provide an understanding of generalized differential and integral equations [15]. Recently, applications have been found in the area of mathematical hysteresis [10], and even more recently, applications have been found in stochastic processes, see for example [11] and [13] .

In this paper, we introduce Davison's generalization of a regulated function in Section 2. To allow us to do this we introduce the idea of a Davison space (X, τ, ω), which is a topological space, (X, τ), with an associated algebra of sets, ω. We extend Davison's definition so that the functions can map to a normed vector space, and for certain results to a metric space. In Section 3, we examine the Davison space on $\mathbb{R}$ that leads to regulated functions in the classical sense. In Section 4, we examine regulated functions defined on the natural numbers, $\mathbb{N}$. We prove that, in this setting, regulated functions coincide with convergent sequences. Section 5 briefly discusses regulated functions on a subset of $\mathbb{R}^2$ and examines how to create examples of Davison spaces and hence regulated functions.

In Section 6 we show that constant and continuous functions are regulated. We introduce the definition of an ω-atom, which is the characteristic function of an element of the algebra, ω, and an ω-molecule, which is a finite linear combination of ω-atoms. We give a global characterization of a regulated function on a compact set, and show that a function is regulated if and only if it is the uniform limit of ω-molecules. The final section contains our main theorem, which asserts that the set of discontinuities of a regulated function is at most a countable collection of boundaries of elements in the associated algebra.

2 Davison Spaces and Regulated Functions.

Davison [3], thought it reasonable to consider what a regulated function on $\mathbb{R}^n$ should look like. To do this, Davison introduced the term **appropriate**

family of sets as follows.

Definition 2.1. *[**Appropriate Family**] Let X be a topological space. A family $\mathcal{F}$ of subsets of X is appropriate if (i) $\mathcal{F}$ is a lattice with respect to union and intersection of sets, (ii) if $A, B \in \mathcal{F}$, and $A \supset B$, then $A \backslash B$ is in $\mathcal{F}$ and (iii) the open sets of X which are in $\mathcal{F}$ form a basis for the topology on X.*

In other words, Davison requires $\mathcal{F}$ to be a ring of subsets of X whose intersection with the topology forms a base for the topology. We now modify Davison's definition slightly. We use different terminology with the hope of simplifying the definition. We also require that $\mathcal{F}$ contains the set itself, X; i.e., that it is an algebra. We call the resulting triple, consisting of the set, the topology and the algebra, a Davison space.

Definition 2.2. *[**Davison Space**] A Davison space is a triple (X, τ, ω) where (X, τ) is a topological space and ω is an algebra of sets on X, such that $\tau \cap \omega$ is a base for τ.*

It can be shown that this definition is equivalent to the definition of an appropriate family provided the appropriate family contains the whole space. Next, we give a version of Davison's generalization of the concept of a regulated function. In [3], Davison's definition is for real valued functions. We allow functions have values in any normed vector space.

Definition 2.3. *[**Regulated Function**] Let (X, τ, ω) be a Davison space and $(Y, \|\cdot\|)$ a normed vector space. Then $f : X \to Y$ is said to be ω-regulated at $x \in X$ if given $\varepsilon > 0$, there exist $A_1, A_2, ..., A_n \in \omega$ such that $\cup_{i=1}^{n} A_i$ is a neighbourhood of x and $\|f(s) - f(t)\| < \varepsilon$ for $s, t \in A_i$, $i = 1, ..., n$. We say f is ω-regulated on X if it is ω-regulated at x for all $x \in X$.*

If it is clear with which algebra we are dealing we omit ω and just use the term **regulated**. This definition can easily be generalized when Y is a metric space. This definition of a regulated function coincides with the classical definition if we take the usual topology on $\mathbb{R}$ and the algebra generated by open intervals. We call such a function $\mathbb{R}$-regulated. In other words, to say a function is $\mathbb{R}$-regulated is to say that the left and right limits exist at all point in $\mathbb{R}$. In the next section we examine $\mathbb{R}$-regulated functions.

3 $\mathbb{R}$-Regulated Functions.

Let τ be the usual topology on $\mathbb{R}$. Let ω be the algebra generated by intervals of the form $(a, b), -\infty \leq a < b \leq +\infty$. Then $(\mathbb{R}, \tau, \omega)$ is a Davison space.

The algebra ω consists of finite unions of intervals of all types; open, closed, half-open, half-closed and singletons. The usual topology on $\mathbb{R}$ is the topology generated by open intervals. Thus, $\omega \cap \tau$ contains these open intervals and hence is a base for τ.

Example 3.1. *Consider the function*

$$H(x) = \begin{cases} -1, x < 0, \\ 0, x = 0, \\ 1, x > 0. \end{cases}$$

Then H is regulated at $x = 0$, which is seen by taking $A_1 = (-a, 0)$, $A_2 = \{0\}$, $A_3 = (0, a)$, for any $a > 0$. For $x > 0$, take $A = (0, \infty)$ and for $x < 0$ take $A = (-\infty, 0)$. Thus, H is regulated on $\mathbb{R}$.

Another interesting $\mathbb{R}$-regulated function is Thomae's function.

Example 3.2. *Let $g : [0, 1] \to \mathbb{R}$ be Thomae's function defined by*

$$g(x) = \begin{cases} \frac{1}{q}, x = \frac{p}{q}, \\ 0, x \notin \mathbb{Q}, \end{cases}$$

where p and q are coprime positive integers and $\mathbb{Q}$ denotes the rational numbers. Then g is $\mathbb{R}$-regulated.

PROOF. Let $\varepsilon > 0$ be given. By the Archimedean property there exists $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. Now, there are at most finitely many values of x such that $|g(x)| > \frac{1}{N}$, precisely $\sum_{i=1}^{N-1} \phi(i)$ values, where $\phi(n)$ is the Euler-phi function, defined as the number of natural numbers less than and coprime with n. Thus, there are at most finitely many values $\{x_i\}_{i=1}^{n}$ such that $|g(x_i)| > \varepsilon$. So, for all but finitely many values of $x \in [0, 1]$, $|g(x) - 0| < \varepsilon$, for $\varepsilon > 0$. Let $\delta = \min_i \frac{|c - x_i|}{2}$. For $c \in (0, 1)$ consider $A_1 = \{c\}$, $A_2 = (c - \delta, c)$ and $A_3 = (c, c + \delta)$. Then $A = A_1 \cup A_2 \cup A_3$ is a neighbourhood of c and for $i \in \{1, 2, 3\}$, $s, t \in A_i \Rightarrow |g(s) - g(t)| < \varepsilon$. For $c = 0$, choose $A = A_1 \cup A_3$ and for $c = 1$ take $A = A_1 \cup A_2$. $\square$

In fact, any function defined on $[0, 1]$ as follows is regulated. Let $\{a_n\}_{n=1}^{\infty}$ be a null sequence. Enumerate the rational numbers in any way, $r_1, r_2, r_3, ...$ and define

$$f(x) = \begin{cases} a_i, x = r_i, \\ 0, x \notin \mathbb{Q}. \end{cases}$$

Then, using a similar argument to that used in the last example it can be shown that f is $\mathbb{R}$-regulated.

4 Regulated Functions on $\mathbb{N}$.

Example 4.1. *Let $\mathbb{N}$ denote the natural numbers and let τ be the cofinite topology on $\mathbb{N}$ (that is $A \in \tau$ if the complement of A, A^c, is finite or $A = \emptyset$, where A is a subset of $\mathbb{N}$). Let ω be the cofinite algebra (that is $A \in \omega$ if A or A^c is finite). Then $(\mathbb{N}, \tau, \omega)$ is a Davison space.*

Proof. First, we show τ is a topology on $\mathbb{N}$. Clearly, $\mathbb{N}$ and $\emptyset$ are contained in τ. Let $A, B \in \tau$. Then A^c and B^c are finite. Hence, $A^c \cup B^c$ is finite. Thus, $(A \cap B)^c$ is finite and $A \cap B \in \tau$. Also, if $A_i \in \tau$, $i \in I$, then A_i^c is finite for $i \in I$. So, $\cap_{i \in I} A_i^c$, being an arbitrary intersection of finite sets is finite. Whence, $(\cup_{i \in I} A_i)^c$ is finite and $\cup_{i \in I} A_i \in \tau$. Hence, τ is a topology. Next, we'll show ω is an algebra. Again, clearly $\emptyset, \mathbb{N} \in \omega$. Let $A \in \omega$. Then A or its complement is finite. Thus, A^c is finite or its complement is finite. Thus, $A^c \in \omega$. Finally, let $A, B \in \omega$. Then if A and B are both finite, $A \cup B$ is finite and hence contained in ω. If at least one of A or B has finite complement, then $A^c \cap B^c = (A \cup B)^c$ is finite and $A \cup B \in \omega$. We have shown ω is an algebra. Recall that τ consists of all subsets of $\mathbb{N}$ with finite complement and $\emptyset$. All these sets are contained in ω. So $\omega \cap \tau$ is a base for τ. In fact, $\omega \cap \tau = \tau$. $\square$

A function which is regulated with respect to this Davison space is called $\mathbb{N}$-regulated. We note that this Davison space is in fact compact. We include a proof of this result as a reference could not be found.

Proposition 4.2. *Let τ be the cofinite topology on $\mathbb{N}$. Then $\mathbb{N}$ is compact.*

Proof. Let A_i, $i \in I$, be an open cover for $\mathbb{N}$. Thus, each A_i is of the form $\{m_1, m_2, ..., m_{l_i}, n_i, n_i + 1, n_i + 2, ...\} = \{m_1, m_2, ..., m_{l_i}\} \cup \mathbb{N} \backslash \{1, 2, ..., n_i - 1\}$ where $m_j \in \mathbb{N}$ for all j and $m_{l_i}, n_i \in \mathbb{N}$. Let B be an element of the open cover. There are only finitely many natural numbers not in B, say $\{r_1, r_2, ..., r_{j_1}\}$. But, A_i, $i \in I$, is an open cover for $\mathbb{N}$ so there exists $i_k \in I$ such that $r_k \in A_{i_k}$ for all $k = 1, ..., j_i$. Then $B, A_{i_1}, ..., A_{i_{j_1}}$ is a finite open cover for $\mathbb{N}$. Thus, $\mathbb{N}$ is compact. $\square$

Theorem 4.3. *Let Y be a normed vector space. A function $f : \mathbb{N} \to Y$ is $\mathbb{N}$-regulated if and only if $\{f(n)\}_{n=1}^{\infty}$ is a convergent sequence.*

Proof. Suppose $\{f(n)\}_{n=1}^{\infty}$ is a convergent sequence. Then given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $m, n \geq N => \|f(n) - f(m)\| < \varepsilon$. Let $p \in \mathbb{N}$. Let $\varepsilon > 0$ be given. We want to show f is regulated at p. Let $A_1 = \{p\}$ and $A_2 = \{N, N + 1, ...\}$. Then $A = A_1 \cup A_2$ is an open set containing p, and $s, t \in A_i, i = 1, 2 => \|f(s) - f(t)\| < \varepsilon$. So, f is regulated at p. Since $p \in \mathbb{N}$ is arbitrary, f is regulated on $\mathbb{N}$.

Suppose $f : \mathbb{N} \to Y$ is regulated. Then given $x \in \mathbb{N}$ and $\varepsilon > 0$ there exists $A_i \in \omega, i = 1, ..., k$ such that $A_1 \cup A_2 \cup ... \cup A_k$ is a neighbourhood of x and $s, t \in A_i => \|f(s) - f(t)\| < \varepsilon$. Suppose $\{f(n)\}_{n=1}^{\infty}$ is not convergent. Then for any N there exists N_1 and $N_2 \geq N$ such that $\|f(N_1) - f(N_2)\| > \varepsilon$. Thus, there does not exist $j \in \{1, ..., k\}$ such that $\{N, N+1, N+2, ...\} \subset A_j$ for some fixed N. Thus, each A_i is finite. Hence, $A_1 \cup A_2 \cup ... \cup A_k$ is finite. But, any neighbourhood of x must contain a set of the form $\{N_0, N_0 + 1, ...\}$ for some fixed N_0 since any non trivial open set has finite complement. Thus, there does not exist a finite collection of $A_i's$ such that $A_1 \cup A_2 ... \cup A_k$ is a neighbourhood of x and $s, t \in A_i => \|f(s) - f(t)\| < \varepsilon$. Thus, f cannot be regulated at x. We get a contradiction. Thus, $\{f(n)\}_{n=1}^{\infty}$ has to be a convergent sequence. □

Example 4.4. *Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = 1/n$ for $n \in \mathbb{N}$. Then f is $\mathbb{N}$-regulated on $\mathbb{N}$, but since a function is continuous with respect to the cofinite topology if and only if it is constant, f is nowhere continuous on $\mathbb{N}$.*

Monotone functions $f : [a, b] \to \mathbb{R}$ are $\mathbb{R}$-regulated, but, in general monotone functions need not be regulated as the following example illustrates.

Example 4.5. *Consider $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$ where $(\mathbb{N}, \tau, \omega)$ is the Davison space described in Example 4.1. Clearly, f is monotonic increasing, but by Theorem 4.3, since $\{f(n)\}_{n=1}^{\infty}$ is not a convergent sequence, f is not $\mathbb{N}$-regulated.*

5 More Examples of Regulated Functions.

Example 5.1 (Regulated on $\mathbb{R}^2$)**.** *Let Ω be a compact subset of $\mathbb{R}^2$. Let τ be the usual topology on $\mathbb{R}^2$ and ω the algebra of sets with finite perimeter (See [16], Chapter 4, Section 2). Then this is a Davison space. We call a function regulated with respect to this Davison space an $\mathbb{R}^2$-regulated function. For example the characteristic function of a set with finite perimeter is $\mathbb{R}^2$-regulated (See Lemma 6.5).*

We can also find examples of regulated functions on $\mathbb{Z}^d$, with the cofinite topology, where $\mathbb{Z}$ denotes the integers and $\mathbb{R}^n$ for $n \geq 3$. So how do we manufacture a Davison space? Here are some possible ways.

- Start with a topological space (X, τ). Generate the smallest algebra, ω, containing τ. Then (X, τ, ω) is a Davison space.
- Take any collection of sets in X. Generate the smallest topology τ containing the collection. Find the smallest algebra, ω, containing τ. Then (X, τ, ω) is a Davison space.

- We could also start with a class of functions with certain properties and pick a suitable algebra and topology so that the functions are regulated.

6 Vector-Valued Regulated Functions.

In Section 4 we saw that monotone functions need not be regulated and it is possible to have a regulated function which is discontinuous everywhere. Next we look at some results which are true for all regulated functions. We state some results from Davison's paper [3] in a more generalized version than Davison. First we introduce some notation.

Notation. *Let X denote a Davison space, (X, τ, ω), and Y denote a normed vector space, $(Y, ||.||)$. Let $S = S(X, Y)$ denote the class of regulated functions from X to Y.*

The following result shows that the class of regulated functions is non-empty.

Proposition 6.1. *Constant functions are regulated.*

Proof. Let $f(x) = a$ for all $x \in X$ and some $a \in Y$. Then independent of $\varepsilon > 0$ given, choose X itself as a neighbourhood of a point $x \in X$. Then for $s, t \in X$, $\|f(s) - f(t)\| = \|a - a\| = 0 < \varepsilon$. Since X is contained in any algebra on X, f is regulated. □

One would hope that continuous functions are regulated and indeed they are. In [3], Davison comments that continuous functions on a locally compact space are regulated. We generalize and prove this result.

Proposition 6.2. *Let (X, τ, ω) be a Davison space. If f is continuous on X, then f is ω-regulated on X.*

Proof. Let $f : X \to Y$ be continuous. Thus, given $x \in X$ and $\varepsilon > 0$ there exists a neighbourhood of x, N_x, such that $z \in N_x$ implies $\|f(x) - f(z)\| < \frac{\varepsilon}{2}$. Since (X, τ, ω) is a Davison space, $\omega \cap \tau$ is a base for τ. So, ω contains a base for τ. Hence, there exists an open set $B_x \in \omega \cap \tau \subset \omega$ such that $x \in B_x \subset N_x$. Thus, $B_x \in \omega$ and $x \in B_x$. Since $B_x \subset N_x,$ $s, t \in B_x \Rightarrow \|f(s) - f(t)\| \leq \|f(s) - f(x)\| + \|f(x) - f(t)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, f is regulated at x. Since x is arbitrary f is regulated at all $x \in X$. □

We note that if the algebra in a Davison space is contained in the topology, then all regulated functions are continuous.

Definition 6.3. *[ω-**atom**] We call χ_A the characteristic function of A, an ω-atom, where $A \in \omega$.*

Definition 6.4. *[ω-**molecule**] We call a finite linear combination of ω-atoms an ω-molecule.*

Lemma 6.5. *Let $A \in \omega$. Then an ω-atom, the characteristic function of A, is regulated.*

This follows directly from the definition of a regulated function taking $A_1 = A$ and $A_2 = X \backslash A$.

Lemma 6.6. *$S(X, Y)$ is a linear subspace of the vector space of all functions from X to Y.*

Thus, ω-molecules, being finite linear combinations of ω-atoms, are regulated. Here, ω-molecules play the same role as step functions play for classical regulated functions $f : [a, b] \to \mathbb{R}$. The next result gives a global characterization of a regulated function on a compact set. We note that Davison, [3] proved the following for $Y = \mathbb{R}$ and that our proof is similar to Davison's but is included for completeness.

Theorem 6.7. *Let X be compact. Then $f : X \to Y$ is ω-regulated if and only if given $\varepsilon > 0$, there exist $A_1, A_2, ..., A_n \in \omega$, $n \in \mathbb{N}$ such that (i) $X = A_1 \cup A_2 \cup ... \cup A_n$ and (ii) if $s, t \in A_i$, then $\|f(s) - f(t)\| < \varepsilon$, $i = 1, 2, ..., n$.*

PROOF. Suppose f is ω-regulated. Let $\varepsilon > 0$ be given. For each $x \in X$, there exist $B_1, ..., B_k \in \omega$, for some $k \in \mathbb{N}$, such that $s, t \in B_i \Rightarrow \|f(s) - f(t)\| < \varepsilon$ and $\mathcal{B}_x = \cup_{i=1}^{k} B_i$ is a neighbourhood of x. Consider the family of sets $\{\mathcal{B}_x : x \in X\}$. This family forms an open cover of X. Since X is compact there exists a finite subcollection of $\{\mathcal{B}_x : x \in X\}$ that covers X, say $\mathcal{B}_{x_1}, \mathcal{B}_{x_2}, ..., \mathcal{B}_{x_m}$, $m \in N$. Each $\mathcal{B}_{x_i}$ is a finite collection of elements of ω, $A_{i_1}, ..., A_{i_p}$, some $p \in \mathbb{N}$, where $s, t \in A_{i_j} \Rightarrow \|f(s) - f(t)\| < \varepsilon$. Thus $\cup_{i=1}^{m} \mathcal{B}_{x_i}$ is a finite collection of elements of ω, A_{i_j}, such that their union is X and $s, t \in A_{i_j} \Rightarrow \|f(s) - f(t)\| < \varepsilon$.

Conversely, suppose that, given $\varepsilon > 0$ there exists a finite collection of elements $A_i \in \omega$, such that $\cup_{i=1}^{n} A_i = X$ and $s, t \in A_i$ implies $\|f(s) - f(t)\| < \varepsilon$. We want to show that f is ω-regulated. Let $x \in X$. Then $x \in A_i$ for some $i \in \{1, 2, ..., n\}$ and X is a neighbourhood of x. Also $s, t \in A_i$ implies that $\|f(s) - f(t)\| < \varepsilon$. Hence, f is ω-regulated at x. Since x is arbitrary f is ω-regulated at x for all $x \in X$. $\square$

Lemma 6.8. *If X is compact, then ω-regulated functions are bounded.*

Lemma 6.9. *If X is compact, then $S(X,Y)$ with the supremum norm, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, is closed in $B(X,Y)$, the set of bounded functions from X to Y.*

The proofs of these two lemmas are similar to those given by Davison in [3].

Theorem 6.10. *Let (X,τ,ω) be a compact Davison space. Then a function from X to Y is ω-regulated if and only if it is the uniform limit of ω-molecules.*

We include the proof of this theorem because we use different terminology to Davison and generalize the setting, but we emphasize that the ideas stem from Davison's in [3].

Proof. Let f be regulated and $\varepsilon > 0$ be given. By Theorem [6.7], there exists $A_1, A_2, ..., A_n \in \omega$ such that (i) $X = A_1 \cup ... \cup A_n$ and (ii) if $s,t \in A_i$, then $\|f(s) - f(t)\| < \frac{\varepsilon}{2}$. Let $B_1 = A_1$ and $B_i = A_i \backslash \cup_{j=1}^{i-1} B_j$. Then $B_i \in \omega$, since an algebra is closed under finite unions and differences. Also, $B_i \cap B_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{n} B_i = X$ and $\|f(s) - f(t)\| < \varepsilon$ for $s,t \in B_i$. Choose any $x_i \in B_i$. Define the ω-molecule, s, as follows:

$$s(x) = f(x_i), x \in B_i.$$

Recall that an ω-molecule is a finite linear combination of ω-atoms and an ω-atom is the characteristic function of A, where $A \in \omega$. So, $s = \sum_{i=1}^{m} f(x_i)\chi_{B_i}$. We have shown that given $\varepsilon > 0$ there exists an ω-molecule, s, such that $\|f - s\|_\infty < \varepsilon$. Thus, f is the uniform limit of ω-molecules.

By Lemma 6.9, the limit of a uniformly convergent sequence of regulated functions is regulated. Since an ω-molecule is regulated, a uniformly convergent sequence of ω-molecules is regulated. □

7 Discontinuities of a Regulated Function.

The set of discontinuities of an ω-atom is the boundary of an element in ω. Since an ω-molecule is a finite linear combination of ω-atoms, the set of discontinuities of an ω-molecule is at most a finite union of boundaries of elements in ω. This leads us to our main result.

Theorem 7.1. *Let (X,τ,ω) be a compact Davison space and $f : X \to Y$ an ω-regulated function. The set of discontinuities of f is at most a countable union of boundaries of elements of the algebra, ω.*

PROOF. Since f is regulated and X is compact, f is the uniform limit of a sequence, f_n, of ω-molecules. Let A_n denote the set of points of continuity of f_n. Let $A = \cap_{n=1}^{\infty} A_n$. Then A is a subset of the set of points of continuity of f. Let B denote the set of points of discontinuity of f. Thus, $B \subseteq A^c = (\cap_{n=1}^{\infty} A_n)^c = \cup_{n=1}^{\infty} A_n^c$. Note that each A_n^c is a finite union of boundaries of elements of ω. Hence, B is at most a countable union of boundaries of elements of ω. $\square$

This provides a simple proof of the following classical result.

Corollary 7.2. *Let $f : [a, b] \to \mathbb{R}$ be an $\mathbb{R}$-regulated function. Then f has at most countably many discontinuities.*

Also, as a consequence of Theorem 7.1 the set of discontinuities of an $\mathbb{R}^2$-regulated function, described in Example 5.1, is at most a countable collection of rectifiable curves.

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