RESEARCH

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 395–404

J. S. Cánovas, Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Campus Muralla del Mar, 30203 Cartagena, Spain. email: Jose.Canovas@upct.es

A. Linero Bas, Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain. email: lineroba@um.es

G. Soler López,* Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Paseo Alfonso XIII, 52, 30203 Cartagena, Spain. email: Gabriel.Soler@upct.es

ON CLOSED SUBGROUPS ASSOCIATED WITH INVOLUTIONS

Abstract

Given an involution f on $(0,\infty)$, we prove that the set $\mathcal{C}(f) := \{\lambda > 0 : \lambda f \text{ is an involution}\}$ is a closed multiplicative subgroup of $(0,\infty)$ and therefore $\mathcal{C}(f)$ is $\{1\}$, $(0,\infty)$ or $\lambda^{\mathbb{Z}} = \{\lambda^n : n \in \mathbb{Z}\}$ for some $\lambda > 0$, $\lambda \neq 1$. Moreover, we provide examples of involutions possessing each one of the above types as the set $\mathcal{C}(f)$ and prove that the unique involutions f such that $\mathcal{C}(f) = (0,\infty)$ are $f(x) = \frac{c}{x}$, $c > 0$.

1 Introduction.

Given a metric space X, by an *involution* we mean a continuous self–map $f : X \to X$ such that $(f \circ f)(x) = x$ for any $x \in X$ and f is not the identity on X. Involutions on $X = \mathbb{R}$, the set of real numbers, are usually called *strong involutions* and have been studied for a variety of reasons [11]. Among them, we can mention that the identity $f(f(x)) = x$ is one of the oldest functional

Key Words: involutions, closed subgroups, difference and functional equations
Mathematical Reviews subject classification: Primary: 39B22; Secondary: 26A18
Received by the editors January 2, 2008
Communicated by: Emma D'Aniello
*This paper has been partially supported by Grant FS 00684/PI/04 (Fundación Séneca, Comunidad Autónoma de la Región de Murcia, Spain), and by Grant MTM2005-03868 (Ministerio de Educación y Ciencia, Spain).

equations called the *Babbage equation* in honor of Charles Babbage ([1]) . It is also worth emphasizing that involutions have some applications, for instance to perimeter equals area problems [10], differential equations [8, 9], biomedical models [2], stability motions [3] and to the so called pantograph equation [4].

In this paper we focus our attention on the cases $X = (0,\infty)$ and $X = \mathbb{R}$. These involutions appear in the study of global periodicity of first order difference equations of the type $x_{n+1} = f(x_n)$, where f is a continuous self–map on $(0,\infty)$ (see [5] for more details). From the dynamical point of view involutions on $(0,\infty)$ and $\mathbb{R}$ are very simple: all the orbits of the system are periodic of period two or fixed points. The classification of these involutions is as follows: if $f : (0,\infty) \to (0,\infty)$ is an involution, then f is topologically conjugate to $\varphi(x) = \frac{1}{x}$, that is, there is an homeomorphism $h : (0,\infty) \to (0,\infty)$ so that $f \circ h = h \circ \varphi$ (see the last section of this paper). Then, up to conjugacy, there is only one involution on $(0,\infty)$. A similar result holds for involutions on $\mathbb{R}$ replacing $1/x$ by the map $f(x) = -x$.

Notice that for the map $1/x$ and for any $c > 0$, the map c/x is also an involution. This also happens with $-x$ and $c - x$ in the case of involutions on $\mathbb{R}$. Therefore, it makes some sense to wonder if something similar happens for any involution on $(0,\infty)$ or $\mathbb{R}$. To be precise, let f be an involution on $(0,\infty)$ and define $\mathcal{C}(f)$ to be the set of all the positive real numbers λ such that λf is also an involution. Similarly, for any involution on $\mathbb{R}$ define $\mathcal{A}(f)$ to be the set of all real numbers c such that $c + f$ is also an involution on the reals. Clearly $1 \in \mathcal{C}(f)$ and $0 \in \mathcal{A}(f)$ and therefore $\mathcal{C}(f)$ and $\mathcal{A}(f)$ are non–empty. Denote by $\mathbb{Z}$ the set of integers and for any $\lambda \in (0,\infty)$, let $\lambda^{\mathbb{Z}} = \{\lambda^m : m \in \mathbb{Z}\}$ and $\lambda\mathbb{Z} = \{\lambda n : n \in \mathbb{Z}\}$. Using this notation, we can state our main results.

Theorem 1.1. *Let $f : (0,\infty) \to (0,\infty)$ be an involution. Then the set $\mathcal{C}(f)$ is a closed multiplicative subgroup of $(0,\infty)$ and one of the following cases happens:*

1. $\mathcal{C}(f) = \{1\}$.
2. $\mathcal{C}(f) = \lambda^{\mathbb{Z}}$ *for some* $\lambda > 0$, $\lambda \neq 1$.
3. $\mathcal{C}(f) = (0,\infty)$.

In addition, for any set described above there is an involution f having this set as $\mathcal{C}(f)$.

As a consequence, we easily prove our second main result.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an involution. Then the set $\mathcal{A}(f)$ is a closed additive subgroup of $\mathbb{R}$ and one of the following cases happens:*

1. *$\mathcal{A}(f) = \{0\}$.*

2. *$\mathcal{A}(f) = \lambda\mathbb{Z}$ for some $\lambda > 0$, $\lambda \neq 0$.*

3. *$\mathcal{A}(f) = \mathbb{R}$.*

In addition, for any set described above there is an involution f having this set as $\mathcal{A}(f)$.

The next section will be devoted to proving our main results and their consequences. To this end, we need three technical lemmas stated below. We use $\mathcal{I}$ to denote the set of involutions on $(0, \infty)$.

Lemma 1.3. *Let $f : (0, \infty) \to (0, \infty)$ be an involution and let $\lambda \in (0, \infty)$. Then the following conditions are equivalent:*

1. *$\lambda f \in \mathcal{I}$.*

2. *$\lambda f(u) = f\left(\frac{u}{\lambda}\right)$, for all $u > 0$.*

PROOF. We first prove (1) $\Rightarrow$ (2). Take $u = \lambda f(x)$, or equivalently $x = f\left(\frac{u}{\lambda}\right)$. Applying (1), we obtain

$$\lambda f(u) = f\left(\frac{u}{\lambda}\right).$$

Since any $u \in (0, \infty)$ can be written as $u = \lambda f(x)$ because of the bijectivity of λf, (1) $\Rightarrow$ (2) follows. Finally (2) $\Rightarrow$ (1) is obtained by computing for any $u > 0$,

$$(\lambda f)^2(u) = (\lambda f)(\lambda f(u)) = (\lambda f)\left(f\left(\frac{u}{\lambda}\right)\right) = \lambda f^2\left(\frac{u}{\lambda}\right) = \lambda\frac{u}{\lambda} = u.$$

□

Lemma 1.4. *Let $f \in \mathcal{I}$, $\lambda \in \mathcal{C}(f)$ and let $y > 0$, then*

$$f(\lambda^m y) = \frac{1}{\lambda^m} f(y), \text{ for all } m \in \mathbb{Z}.$$

Proof. From Lemma 1.3 $f(y) = \frac{1}{\lambda} f(\frac{1}{\lambda} y)$, so

$$f(\lambda^{-1} y) = \lambda f(y).$$

We apply the previous equality and also Lemma 1.3, with $u = \lambda^{-1} y$, to deduce

$$f(\lambda^{-2} y) = \lambda f(\lambda^{-1} y) = \lambda^2 f(y).$$

Then, by recurrence, it is easily seen that $f(\lambda^{-m} y) = \lambda^m f(y)$, for all $m = 0, 1, 2, ...$ For negative integers m it is enough to take $u = \lambda^{-m} y > 0$ and then applying the above result we obtain $f(\lambda^m u) = \lambda^{-m} f(u)$, $f(y) = \lambda^{-m} f(\lambda^{-m} y)$, and finally $f(\lambda^{-m} y) = \lambda^m f(y)$. □

Lemma 1.5. *Let G be a closed multiplicative subgroup of $(0, \infty)$, then either $G = \{1\}$ or $G = \lambda^{\mathbb{Z}}$ for some $\lambda \in (0, \infty) \setminus \{1\}$ or $G = (0, \infty)$.*

Proof. It is well–known (see [6, page 233]) that any closed additive subgroup H of $\mathbb{R}$ is either $H = \{0\}$ or $H = \{ma : m \in \mathbb{Z}\}$ for some $a \in \mathbb{R} \setminus \{0\}$ or $H = \mathbb{R}$. We apply this result to the closed additive subgroup $H = \log G := \{\log g : g \in G\}$ to obtain that either $H = \{0\}$ or $H = \{na : n \in \mathbb{Z}\}$ for some $a \in \mathbb{R}$ or $H = \mathbb{R}$. Since $G = \{e^h : h \in H\}$, we finally have one of the following alternatives: (1) $G = \{1\}$, (2) $G = \lambda^{\mathbb{Z}}$ for $\lambda = e^a$ or (3) $G = (0, \infty)$. □

2 Main Theorems.

Proof of Theorem 1.1. First, we fix an involution f and prove that $\mathcal{C}(f)$ is a closed multiplicative subgroup of $(0, \infty)$. Note that $1 \in \mathcal{C}(f)$. Let $\lambda \in \mathcal{C}(f)$ and prove that $\lambda^{-1} \in \mathcal{C}(f)$. To this end, use Lemma 1.4 to obtain

$$\frac{1}{\lambda} f\left(\frac{1}{\lambda} f(x)\right) = \frac{1}{\lambda} f(f(\lambda x)) = \frac{1}{\lambda} \lambda x = x,$$

and so $\frac{1}{\lambda} f$ is an involution by Lemma 1.3. Finally, given $\lambda, \mu \in \mathcal{C}(f)$, we prove that $\lambda\mu \in \mathcal{C}(f)$. Since f is a bijective map, given an arbitrary $x > 0$ there exists a unique δ_x such that $\mu f(x) = f(\delta_x)$. Then, using Lemma 1.3,

$$\begin{aligned} \lambda\mu f(\lambda\mu f(x)) &= \lambda\mu f(\lambda f(\delta_x)) = \mu[\lambda f(\lambda f(\delta_x))] \\ &= \mu\delta_x = \mu f^{-1}(\mu f(x)) = \mu f^{-1}(f(\frac{x}{\mu})) \\ &= \mu \frac{x}{\mu} = x, \end{aligned}$$

and so $\lambda\mu f$ is an involution and therefore $\lambda\mu \in \mathcal{C}(f)$. So, we deduce that $\mathcal{C}(f)$ is a multiplicative subgroup of $(0, \infty)$.

Now, assume that $\{\lambda_n\}_n \subset \mathcal{C}(f)$ is a convergent sequence, and let λ be its limit. By the continuity of f and Lemma 1.3 we obtain

$$f(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} f\left(\frac{x}{\lambda_n}\right) = \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right).$$

By Lemma 1.3 we deduce that λf is an involution. Then $\mathcal{C}(f)$ is a closed set. By Lemma 1.5, we conclude that $\mathcal{C}(f)$ has to have the form described in the statement of the result.

To finish, we construct involutions with a prescribed set $\mathcal{C}(f)$. Firstly, note that if $f(x) = 1/x$, then $\mathcal{C}(f) = (0, \infty)$.

Now, for any $\lambda > 0$, $\lambda \neq 1$, we construct an involution f such that $\mathcal{C}(f) = \lambda^{\mathbb{Z}}$. First, note that since $(\lambda^{-1})^{\mathbb{Z}} = \lambda^{\mathbb{Z}}$, we may assume that $\lambda > 1$ (if λ was smaller than 1, then we will work with $\lambda^{-1} > 1$). Let $\widetilde{f} : [1, \lambda] \to [1, \lambda]$ be a continuous bijective (strictly decreasing) map so that (1) $\widetilde{f} \circ \widetilde{f} = \mathrm{Id}_{[1,\lambda]}$ and (2) if μ is the only fixed point from $\widetilde{f}$ then $\mu \neq \sqrt{\lambda}$. Let $x \in (0, \infty)$ so that $\lambda^k \leq x < \lambda^{k+1}$ for some $k \in \mathbb{Z}$, then $f(x) := \lambda^{-k}\widetilde{f}(\lambda^{-k}x)$. Now it is easy to check that f is continuous for any $x \in (\lambda^k, \lambda^{k+1})$. Moreover, if $x = \lambda^k$, then:

$$\begin{aligned}
\lim_{x\to(\lambda^k)^+} f(x) &= \lim_{x\to(\lambda^k)^+} \lambda^{-k}\widetilde{f}(\lambda^{-k}x) \\
&= \lambda^{-k} \lim_{x\to 1^+} \widetilde{f}(x) = \lambda^{-k}\lambda = \lambda^{-k+1},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to(\lambda^k)^-} f(x) &= \lim_{x\to(\lambda^k)^-} \lambda^{-k+1}\widetilde{f}(\lambda^{-k+1}x) \\
&= \lambda^{-k+1} \lim_{x\to\lambda^-} \widetilde{f}(x) = \lambda^{-k+1}1 = \lambda^{-k+1}.
\end{aligned}$$

Therefore f is continuous in any point. Next we show that f is an involution. Let $x \in (0, \infty)$ such that $\lambda^k \leq x < \lambda^{k+1}$ for some $k \in \mathbb{Z}$. Since $\lambda^{-k}\widetilde{f}(\lambda^{-k}x) \in (\lambda^{-k}, \lambda^{-k+1})$ it holds

$$\begin{aligned}
f(f(x)) &= f(\lambda^{-k}\widetilde{f}(\lambda^{-k}x)) = \lambda^k\widetilde{f}(\lambda^k\lambda^{-k}\widetilde{f}(\lambda^{-k}x)) \\
&= \lambda^k\widetilde{f}(\widetilde{f}(\lambda^{-k}x)) = \lambda^k\lambda^{-k}x = x.
\end{aligned}$$

Let us now prove that λf is an involution. Take $x \in (0, \infty)$ such that $\lambda^k \leq$

$x < \lambda^{k+1}$ ($k \in \mathbb{Z}$). Observe that $\lambda \cdot \lambda^{-k} \widetilde{f}(\lambda^{-k}x) \in [\lambda^{-k+1}, \lambda^{-k+2}]$, then

$$\begin{aligned} \lambda f(\lambda f(x)) &= \lambda f(\lambda \cdot \lambda^{-k} \widetilde{f}(\lambda^{-k}x)) = \lambda \cdot \lambda^{k-1} \widetilde{f}(\lambda^{k-1} \cdot \lambda \cdot \lambda^{-k} \widetilde{f}(\lambda^{-k}x)) \\ &= \lambda \cdot \lambda^{k-1} \widetilde{f}(\widetilde{f}(\lambda^{-k}x)) = \lambda \cdot \lambda^{k-1} \lambda^{-k} x = x. \end{aligned}$$

Moreover μf is not an involution because $\mu f(\mu f(\lambda)) = \mu f(\mu) = \mu\mu \neq \lambda$ (use Lemma 1.3 and recall that $f(\lambda) = 1$). So f is an involution and applying the first part of our Main Theorem $\mathcal{C}(f) = \lambda^{\mathbb{Z}}$.

Finally, we construct an involution f such that $\mathcal{C}(f) = \{1\}$. Let φ be the involution defined by $\varphi(x) = \frac{1}{x}$ and let $\alpha(x) = \frac{x^2}{1+x}$ for any $x \in (0,\infty)$. Then the map $\alpha : (0,\infty) \to (0,\infty)$ is bijective and its inverse map is $\alpha^{-1}(x) = \frac{x+\sqrt{x^2+4x}}{2}$. Now it is a simple task to check that $f := \alpha^{-1} \circ \varphi \circ \alpha$ is an involution. Let $k \in (0,\infty)\backslash\{1\}$, then we claim that the map

$$kf(x) = k \left(\frac{\frac{1+x}{x^2} + \sqrt{\left(\frac{1+x}{x^2}\right)^2 + 4\left(\frac{1+x}{x^2}\right)}}{2} \right)$$

is not an involution. Assume the opposite, then according to Lemma 1.3 we have

$$kf(x) = f(\frac{x}{k}), \text{ for all } x > 0.$$

Hence

$$k \left(\frac{\frac{1+x}{x^2} + \sqrt{\left(\frac{1+x}{x^2}\right)^2 + 4\left(\frac{1+x}{x^2}\right)}}{2} \right) = \frac{\frac{1+\frac{x}{k}}{\left(\frac{x}{k}\right)^2} + \sqrt{\left(\frac{1+\frac{x}{k}}{\left(\frac{x}{k}\right)^2}\right)^2 + 4\left(\frac{1+\frac{x}{k}}{\left(\frac{x}{k}\right)^2}\right)}}{2}.$$

Multiplying last equation by $2x^2$ and simplifying we have

$$k^2 - k \quad = \quad k\sqrt{(1+x)^2 + 4(1+x)x^2} - \sqrt{k^2(k+x)^2 + 4k(k+x)x^2}.$$

Since

$$\begin{aligned} &\lim_{x\to 0} \left(k\sqrt{(1+x)^2 + 4(1+x)x^2} - \sqrt{k^2(k+x)^2 + 4k(k+x)x^2} \right) \\ &= \quad k - k^2, \end{aligned}$$

we obtain a contradiction. Then $\mathcal{C}(f) = \{1\}$. □

Theorem 2.1. *If f is an involution on $(0,\infty)$, and $\mathcal{C}(f) = (0,\infty)$, then*

$$f(x) = \frac{c}{x},$$

for some positive constant c.

Proof. If λf is an involution, according to Lemma 1.4 we have

$$f(\lambda) = \frac{1}{\lambda} f(1).$$

Since $\mathcal{C}(f) = (0,\infty)$, the above equality holds for all $\lambda > 0$, and consequently, $f(x) = \frac{c}{x}$, with $c = f(1)$. $\square$

Corollary 2.2. *The unique maps with the property that*

$$\lambda f(\lambda f(x)) = x$$

for any $x > 0$ and $\lambda > 0$ are $f(x) = c/x$, where $c > 0$ is constant.

Proof of Theorem 1.2. The result follows from the fact that $f : \mathbb{R} \to \mathbb{R}$ is an involution if and only if $g(x) = e^{f(\log x)}$ is an involution on $(0,\infty)$. Moreover, for any $c \in \mathbb{R}$ it follows that $c + f$ is an involution on $\mathbb{R}$ if and only if $e^c g$ is an involution on $(0,\infty)$. Use this fact and the proof of Theorem 1.1 to conclude the proof. $\square$

We have the following immediate consequences which follow from the proof of Theorem 1.2 and the corresponding results for involutions on $(0,\infty)$.

Theorem 2.3. *If f is an involution on $\mathbb{R}$, and $\mathcal{A}(f) = \mathbb{R}$, then*

$$f(x) = c - x,$$

for some $c \in \mathbb{R}$.

Corollary 2.4. *The unique maps with the property that*

$$\lambda + f(\lambda + f(x)) = x$$

for any x and λ are $f(x) = c - x$, where $c \in \mathbb{R}$ is constant.

Remark 2.5. *It is worth mentioning that the class of involutions on* $(0,\infty)$ *satisfying* $\mathcal{C}(f) = \{\lambda^n : n \in \mathbb{Z}\}$ *contains infinite pairwise linearly independent elements. For instance, if* $f, g \in \mathcal{I}$, $\mathcal{C}(f) = \lambda^{\mathbb{Z}}$ *and* $\mathcal{C}(g) = \mu^{\mathbb{Z}}$, $\lambda \neq \mu^n$, *for any* $n \in \mathbb{Z}$, *then* $g \neq \alpha f$. *Otherwise we would have that* $\alpha \in \mathcal{C}(f)$ *and so* $\alpha = \lambda^n$ *for* $n \in \mathbb{Z}$. *Then* $\mu g = \mu\lambda^n f$ *would be an involution and therefore* $\mu\lambda^n \in \mathcal{C}(f)$, *which is a contradiction. Similarly, the class of involutions on* $\mathbb{R}$ *such that* $\mathcal{A}(f) = \{\lambda n : n \in \mathbb{Z}\}$ *contains infinite elements* f *and* g *such that* $f - g$ *is not a constant.*

Remark 2.6. *In our main results, the structure of group of the domain of the involutions is necessary to state them. It is interesting to study similar questions for continuous involutions defined on topological groups.*

3 Dynamical classification of involutions.

For the sake of completeness, we give a proof of the fact that if f is an involution on $(0,\infty)$, then it is conjugate to the map $\varphi(x) = \frac{1}{x}$.

By [7, Lemma 15.2, page 290], we have that if $f \in \mathcal{I}$, then

$$f(x) = \begin{cases} f_0(x) \text{ if } x \in (0, x_0), \\ x_0 \text{ if } x = x_0, \\ f_0^{-1}(x) \text{ if } x \in (x_0, \infty), \end{cases}$$

where $x_0 > 0$ and $f_0 : (0, x_0) \to (x_0, \infty)$ is a continuous strictly decreasing map such that $\lim_{x\to x_0} f_0(x) = x_0$. We are going to define a homeomorphism $h : (0,\infty) \to (0,\infty)$ such that

$$h \circ f = \varphi \circ h. \tag{1}$$

Notice that equality (1) can be rewritten for any $x \in (0,\infty)$ as

$$h(f(x)) = 1/h(x). \tag{2}$$

Then (2) gives us $h(x_0) = 1/h(x_0)$, and hence $h(x_0) = 1$. Now, define $h : (0, x_0] \to (0, 1]$ to be strictly increasing, continuous and such that $\lim_{x\to 0} h(x) = 0$. We extend this map to (x_0, ∞) as follows. Let $x \in (x_0, \infty)$ and let $y = f(x) \in (0, x_0)$. Define $h(x) := 1/h(y) = 1/h(f(x))$. Note that since $h(x) \neq 0$ for all $x \in (0, x_0]$ and

$$\lim_{\substack{x\to x_0 \\ x > x_0}} h(x) = \lim_{\substack{x\to x_0 \\ x < x_0}} 1/h(x) = 1,$$

the map h is continuous. Now, we prove that $h : (0, \infty) \to (0, \infty)$ is a homeomorphism. Since

$$\lim_{x\to\infty} h(x) = \lim_{x\to 0} 1/h(x) = \infty,$$

we conclude that h is surjective. Now, we finish by proving that h is strictly increasing. To this end, note that $h(x) < h(y)$ for any $x \in (0, x_0]$, $y \in (x_0, \infty)$ by definition. If $x > y > x_0$, then since f is strictly decreasing $f(x) < f(y)$. Then $h(f(x)) < h(f(y))$ and therefore $h(x) = 1/h(f(x)) > 1/h(f(y)) = h(y)$, which finishes the proof.

Acknowledgement. We wish to thank the referee for his/her suggestions and remarks which helped us to improve the paper.

References

[1] C. Babbage. *An essay towards the calculus of functions.* Philosophical Transaction of the Royal Society London, **105** (1815), 389–424.

[2] S. Busenberg and C. Travis. *On the use of reducible–functional differential equations in biological models.* J. Math. Anal. Appl., **89** (1982), 46–66.

[3] W. G. Castelan and E.F. Infante. *On a functional equation arising in the stability theory of difference–differential equations.* Quart. Appl. Math., **35** (1977), 311–319.

[4] G. Derfel and A. Iserles. *The pantograph equation in the complex plane.* J. Math. Anal. Appl., **213** (1997), 117–132.

[5] A. Linero, J. S. Cánovas, and G. Soler López. *On global periodicity of difference equations.* Taiwanese J. Math. (to appear).

[6] V. Jiménez López. *Ecuaciones Diferenciales: cómo enseñarlas, cómo aprenderlas.* Servicio de Publicaciones de la Universidad de Murcia, Murcia, 2000.

[7] M. Kuczma, *Functional equations in a single variable.* Polish Scientific Publishers, 1968.

[8] J. Wiener. *Differential equations with involutions.* Differ. Equations, **5** (1969), 1131–1137.

[9] J. Wiener. *Differential equations in partial derivatives with involutions.* Differ. Equations, **6** (1970), 1320–1322.

[10] J. Wiener, H. Chi, and H. Poorkarimi. *Involutions and problems involving perimeter and area.* College Mathematics Journal, **19** (1988), 250–252.

[11] J. Wiener and W. Watkins. *A glimpse into the wonderland of involutions.* Missouri J. Math. Sci., **14** (2002),175–182.