RESEARCH

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 365–374

Zbigniew Grande, Institute of Mathematics, Kazimierz Wielki University, Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland.
email: grande@ukw.edu.pl

ON THE GRAPH CONVERGENCE OF SEQUENCES OF FUNCTIONS

Abstract

Let (X, T_X) and (Y, T_Y) be topological spaces. A sequence (f_n) of functions $f_n : X \to Y$ is graph convergent to $f : X \to Y$ if for each set $U \in T_X \times T_Y$ containing the graph $Gr(f)$ of f there is an index k such that $Gr(f_n) \subset U$ for nk. It is proved that if (X, T_X) is a T_1 space, then the graph convergence implies the pointwise convergence. Moreover the uniform and graph convergences are compared, and the graph limits of sequences of continuous (quasicontinuous, cliquish, almost continuous or Darboux) functions are investigated.

Let (X, T_X) and (Y, T_Y) be topological spaces and let $\mathbb{R}$ be the set of all reals considered with the natural topology T_e. Denote by $T_X \times T_Y$ the product topology in $X \times Y$.

We will say that a sequence of functions $f_n : X \to Y$ graph converges to a function $f : X \to Y$ if for each set $U \in T_X \times T_Y$ containing the graph $Gr(f)$ of the function f there is a positive integer k such that $Gr(f_n) \subset U$ for all nk.

Theorem 1. *If (X, T_X) is a T_1 topological space and a sequence of functions $f_n : X \to Y$ graph converges to a function f, then the sequence (f_n) pointwise converges to f.*

Proof. Of course, assume to a contrary, that there is a point $x \in X$ such that the sequence $(f_n(x))$ does not converge to $f(x)$. Then there is a set $V \in T_Y$ containing $f(x)$ and a sequence (n_k) of positive integers such that

$$n_{k+1} n_k \text{ and } f_{n_k}(x) \in Y \setminus V \text{ for all } k \geq 1.$$

Key Words: graph convergence, uniform and pointwise convergence, continuity, quasicontinuity, cliquishness, almost continuity, Darboux property
Mathematical Reviews subject classification: 26A15
Received by the editors July 9, 2007
Communicated by: Brian S. Thomson

Let

$$U = ((X \setminus \{x\}) \times Y) \cup (X \times V).$$

Then

$$U \in T_X \times T_Y \text{ and } Gr(f) \subset U.$$

However for each positive integer m there is $n_m m$ such that the graph $Gr(f_{n_m})$ is not contained in U, a contrary to the graph convergence of (f_n). This completes the proof. $\square$

As an immediate corollary we obtain:

Corollary 1. *If (X, T_X) is a T_1 topological space and (Y, T_Y) is a Hausdorff topological space, then the limit f of a graph convergent sequence of functions $f_n : X \to Y$ is unique.*

Remark 1. *If a topological space (Y, T_Y) is such that there are points a, b, $a_n \in Y$, $n \geq 1$, such that $a \neq b$ and a sequence (a_n) converges to a and b, then for each singleton topological space $X = \{x\}$ with the discrete topology T_X the sequence of functions $f_n : X \to Y$ defined by $f_n(x) = a_n$, $n \geq 1$, graph converges to different functions $f(x) = a$ and $g(x) = b$. Evidently such (Y, T_Y) is not any Hausdorff space.*

Remark 2. *If a topological space (X, T_X) does not satisfy separation axiom T_1, then there are functions $f, g, f_n : X \to \mathbb{R}$, $n \geq 1$, ($\mathbb{R}$ is considered with the topology T_e), such that $f \neq g$ and the sequence f_n graph converges to f and g.*

PROOF. Since (X, T_X) does not satisfy axiom T_1, there are two different points $a, b \in X$ such that every set $U \in T_X$ containing a contains also b. For $n \geq 1$ put

$$f_n(x) = f(x) = 0 \text{ for } x \in X,$$

and let

$$g(x) = 0 \text{ for } x \neq b \text{ and } g(b) = 1.$$

Evidently, $g \neq f$ and the sequence (f_n) graph converges to f. We will prove that (f_n) graph converges to g. For this fix a set $W \in T_X \times T_e$ with $Gr(g) \subset W$. Let

$$W_1 = W \cap (X \times (-\frac{1}{3}, \frac{1}{3})) \text{ and } W_2 = W \cap (X \times (\frac{2}{3}, \frac{4}{3})).$$

Observe that $Gr(g) \subset W_1 \cup W_2 \subset W$. Since $(a, g(a)) = (a, 0) \in W_1$ and $W_1 \cap W_2 = \emptyset$, there are a set $V \in T_X$ and an open interval $I \subset (-\frac{1}{3}, \frac{1}{3})$ with

$(a,0) \in V \times I \subset W_1$. From the choice of points a and b it follows that $b \in V$. So the point $(b,0) \in W_1 \subset W$ and $Gr(f_n) \subset W$ for $n \geq 1$. This completes the proof. □

Theorem 2. *Let (X, T_X) be a T_1 topological space. If there is an isolated infinite closed set $A \subset X$, then there is a sequence of functions $f_n : X \to \mathbb{R}$ which uniformly converges to a function $f : X \to \mathbb{R}$ and which is not graph convergent to f.*

PROOF. Let $A = \{a_n; n \geq 1\}$ be an isolated infinite subset of X and for $n = 1, 2, \ldots$ let $W_n \in T_X$ be such that $\{a_n\} = A \cap W_n$. Moreover for $n \geq 1$ put

$$f_n(x) = \frac{1}{n} \text{ for } x \in A, \text{ and } f_n(x) = 0 \text{ otherwise on } X.$$

Then the set

$$U = ((X \setminus A) \times \mathbb{R}) \cup \bigcup_n (W_n \times (-\frac{1}{2n}, \frac{1}{2n})) \in T_X \times T_e,$$

and the graph of the function $f = 0$ on X is contained in U. Evidently, the sequence (f_n) uniformly converges to f.

We will prove that (f_n) is not graph convergent to f. For this we observe that for $n \geq 1$ the points

$$(a_n, f_n(a_n)) = (a_n, \frac{1}{n}) \in (X \times \mathbb{R}) \setminus U,$$

so the graphs $Gr(f_n)$ are not contained in U. This finishes the proof. □

However in some special cases the uniform convergence implies the graph convergence.

Theorem 3. *If (X, ρ_X) and (Y, ρ_Y) are compact metric spaces, a function $f : X \to Y$ is continuous and a set $U \in T_X \times T_Y$ contains the graph $Gr(f)$ of f, then there is a positive real r such that each function $g : X \to Y$ with* $\sup\{\rho_Y(f(x), g(x)); x \in X\}r$ *has the graph $Gr(g) \subset U$*

PROOF. Since the function $d : (X \times Y) \to \mathbb{R}$ defined as $d(x,y) = \rho_Y(f(x), y)$ is continuous and the set $(X \times Y) \setminus U$ is compact, the real $r = \min\{d(x,y); (x,y) \in (X \times Y) \setminus U\}$ is positive and the graph $Gr(g)$ of each function $g : X \to Y$ with $\sup\{\rho_y(f(x), g(x)); (x \in X\}r$ is contained in U. □

As an immediate consequence of this theorem we obtain:

Corollary 2. *If (X, ρ_X) and (Y, ρ_Y) are compact metric spaces and a function $f : X \to Y$ is continuous, then the uniform convergence of a sequence (f_n) to f implies the graph convergence of (f_n) to f.*

Theorem 4. *Let (X, T_X) be a topological space and let (Y, ρ_Y) be a metric space. Suppose that a sequence (f_n) of functions $f_n : X \to Y$ graph converges to a continuous function $f : X \to Y$. Then the sequence (f_n) uniformly converges to f.*

PROOF. Observe that for every positive real r the set

$$A_r(f) = \bigcup_{x \in X} (\{x\} \times K(f(x), r))$$

belongs to $T_X \times T_Y$, where T_Y is the topology determined by the metric ρ_Y. Of course, if a point $(x, y) \in A_r(f)$, then $\rho_Y(y, f(x))r$. Let

$$s = \frac{r - \rho_Y(y, f(x))}{3}.$$

Then $s0$ and from the continuity of f it follows that there is a set $U \in T_X$ such that $x \in U$ and $f(U) \subset K(f(x), s)$. Fix a point $(u, z) \in U \times K(y, s) \in T_X \times T_Y$. Then

$$\rho_Y(z, f(u)) \leq \rho_Y(z, y) + \rho_Y(y, f(x)) + \rho_Y(f(x), f(u))s + r - 3s + s = r - sr,$$

and consequently $(x, y) \in U \times K(y, s) \subset A_r(f)$. So $A_r(f)$ is an open set belonging to $T_X \times T_Y$ containing $Gr(f)$. Consequently, there is a positive integer k such that for nk we obtain $Gr(f_n) \subset A_r(f)$, where from

$$\rho_X(f_n(x), f(x))r \text{ for } x \in X \text{ and } nk.$$

This implies that the sequence (f_n) uniformly converges to f. □

Remark 3. *Let $X = \{0\} \cup \{\frac{1}{n}; n \geq 1\}$ and let $T_X = T_e/X$ be the natural topology generated by T_e. Let $g(0) = 0$ and $g(\frac{1}{n}) = 1$ for $n \geq 1$. Moreover for $k \geq 1$ let $g_k(\frac{1}{n}) = 1$ for $n \leq k$ and $g_k(x) = 0$ otherwise on X. Then the functions g_k, $k \geq 1$, are T_X-continuous (***i.e.*** they are continuous as applications from (X, T_X) to $(\mathbb{R}, T_e)$), the sequence (g_k) graph converges to g, and g is not continuous at 0.*

By a direct modification in next example we show that there is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}, \ \ n \geq 1$, which graph converges to a discontinuous function $f : \mathbb{R} \to \mathbb{R}$.

Example 1. For $n \geq 1$ let $I_n = [a_n, b_n] = [\frac{1}{n} - \frac{1}{8^n}, \frac{1}{n} + \frac{1}{8^n}]$ and let $c_n = \frac{1}{n}$ denote the center of the interval I_n. Moreover for $n = 1, 2, \ldots$ let

$$f_n(x) = \begin{cases} 1 & for & x = c_k, \ \ k \leq n \\ 0 & for & x \in \mathbb{R} \setminus \bigcup_{k \leq n}(a_k, b_k) \\ linear & \text{on the intervals} & [a_k, c_k] \ \text{and} \ [c_k, b_k], \ \ k \leq n, \end{cases}$$

and let

$$f(x) = \begin{cases} 1 & for & x = c_n, \ \ n \geq 1 \\ 0 & for & x \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n) \\ linear & \text{on the intervals} & [a_n, c_n] \ \text{and} \ [c_n, b_n], \ \ n \geq 1. \end{cases}$$

Observe that the sequence (f_n) is graph convergent to f. Of course, if an open (**i.e.** belonging to $T_e \times T_e$) set $U \subset \mathbb{R}^2$ contains the graph $G(f)$ of the function f, then there is a positive real r such that $K((0, 0), r) \subset U$. Let a positive integer k be such that $\frac{1}{k}r$. So for nk we have

$$Gr(f_n) \subset Gr(f) \cup ([0, \frac{1}{n}] \times \{0\}) \subset U \cup K((0, 0), r) = U,$$

and consequently the sequence (f_n) graph converges to f.

All functions f_n, $n = 1, 2, \ldots$, are continuous (as applications from $(\mathbb{R}, T_e)$ to $(\mathbb{R}, T_e)$), but the function f is not continuous at 0.

So the graph limit of a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ may be discontinuous. Such a limit must be of the first Baire class, because it is the limit of a pointwise convergent sequence of continuous functions. However the following theorem is true.

Theorem 5. *Let (X, ρ) be a separable complete metric space which is dense in itself. If a sequence (f_n) of the first Baire class functions $f_n : X \to \mathbb{R}$ is graph convergent to a function f, then f is also of the first Baire class.*

Proof. Assume, to a contradiction, that f is not of Baire 1 class. Then, by Baire's theorem, there is a nonempty closed set $A \subset X$ such that the restricted function f/A is discontinuous at each point $x \in A$. Observe that A must be a perfect subset of X.

For each point $x \in A$ there are intervals $[a(x), b(x)]$ and $[c(x), d(x)]$ with rational endpoints such that $c(x), a(x), f(x), b(x), d(x)$ and $x \in cl(A \cap f^{-1}(\mathbb{R} \setminus [c(x), d(x)]))$, where $cl(P)$ denotes the closure of the set P.

Since the set of all pairs of intervals with rational endpoints is countable, there are closed intervals $I = [a, b]$ and $J = [c, d]$ with rational endpoints such that the set

$$B = \{x \in A; [a(x), b(x)] = I \text{ and } [c(x), d(x)] = J\}$$

is of the second category in A. The functions f_n are of the first Baire class, so the sets $C(f_n/A)$ of all continuity points of the restrictions f_n/A are residual in A, and consequently the intersection

$$E = B \cap \bigcap_{n=1}^{\infty} C(f_n/A)$$

is of the second category in A. From the graph convergence of the sequence (f_n) it follows its pointwise convergence. Thus for each point $x \in E$ there is a positive integer $n(x)$ such that for all $n \geq n(x)$ the inequality $a(x) f_n(x) b(x)$ holds. Let $E_k = \{x \in E; n(x) = k\}$ for $k \geq 1$. Observe that

$$E = \bigcup_{k=1}^{\infty} E_k.$$

Since E is of the second category in A, there is a positive integer j such that E_j is of the second category in A.

Let $U \subset X$ be a nonempty open set in X such that $U \cap A \neq \emptyset$ and $U \cap E_j$ is dense in $U \cap A$. Let $u \in U \cap B$ be a point. Since $u \in U \cap A \cap cl(f^{-1}(\mathbb{R} \setminus J))$, there are points $u_n \in A \cap U$ such that

$$\lim_{n \to \infty} u_n = u \text{ and } f(u_n) \in \mathbb{R} \setminus [c, d].$$

So the set $G = U \cap A \cap f^{-1}(\mathbb{R} \setminus [c, d])$ is dense in $U \cap A$. Fix two reals $c_1 \in (c, a)$ and $d_1 \in (b, d)$. Since the restricted functions f_n/E, $n \geq 1$, are continuous and since for kj the sets $U \cap A \cap f_k^{-1}((a, b))$ are dense in $U \cap A$, the sets $H_k = U \cap A \cap (f_k)^{-1}(\mathbb{R} \setminus [c_1, d_1])$, kj, are nowhere dense in $U \cap A$. But $f = \lim_{n \to \infty} f_n$, so for every point $x \in U \cap A$ with $f(x) \in \mathbb{R} \setminus [c, d]$ there is an index m such that $x \in H_m$. Thus the set $G \subset K = \bigcup_n H_n$. Let n_1 be the first positive integer such that $G \cap H_{n_1} \neq \emptyset$. Fix a point $a_1 \in G \cap H_{n_1}$. Since $G \subset K$ is dense in $U \cap A$, there are an index $n_2 n_1$ and a point $a_2 \in G \cap (H_{n_2} \setminus H_{n_1})$ with $\rho(a_2, a_1) \frac{1}{2}$. Similarly, by induction, if we have indices $n_i n_{i-1}$ and points $a_i \in G \cap (H_{n_i} \setminus \bigcup_{li} H_{n_l})$, $1 i \leq k$, such that $\rho(a_i, a_1) \frac{1}{i}$, then we find an index $n_{k+1} n_k$ and a point $a_{k+1} \in G \cap (H_{n_{k+1}} \setminus \bigcup_{l \leq k} H_{n_l})$ such that $\rho(a_{k+1}, a_1) \frac{1}{k+1}$. Since $\lim_{n \to \infty} a_n = a_1$, the set $L = \{a_n; n \geq 1\}$ is closed. Let

$$W = ((X \setminus L) \times \mathbb{R}) \cup (X \times (\mathbb{R} \setminus [c, d])).$$

Evidently W is an open set in $X \times \mathbb{R}$ containing the graph $Gr(f)$. Moreover for each index k there is an index $n_j k$ and a point $a_{j+1} \notin H_{n_j}$, thus $f_{n_j}(a_j) \in [c_1, d_1]$, and consequently the graph $Gr(f_{n_j})$ is not contained in W. This contradicts the graph convergence of (f_n) to f, and the obtained contradiction completes the proof. □

Recall that a function $f : X \to \mathbb{R}$ is quasicontinuous (resp. cliquish) at a point $x \in X$ if for each set $U \in T_X$ containing x and for each real $\eta 0$ there is a nonempty set $V \in T_X$ contained in U and such that $f(V) \subset (f(x)-\eta, f(x)+\eta)$ (resp. the diameter $diam(f(V))\eta$) ([4, 6]).

In Example 2 we show that there exists a sequence of quasicontinuous functions graph converges to a function which is not quasicontinuous.

Example 2. For $n \geq 1$ let $I_n = [-\frac{1}{n}, \frac{1}{n}]$ and

$$f_n(x) = 1 \text{ for } x \in I_n \text{ and } f_n(x) = 0 \text{ otherwise on } \mathbb{R}.$$

Then the functions $f_n, \ n \geq 1$, are quasicontinuous and the sequence (f_n) graph converges to the function

$$f(0) = 1 \text{ and } f(x) = 0 \text{ otherwise on } \mathbb{R},$$

which is not quasicontinuous at 0.

However the following theorem is true.

Theorem 6. *Let (X, ρ) be a separable complete metric space which is dense in itself. If a sequence of cliquish functions $f_n : X \to \mathbb{R}, \ n \geq 1$, graph converges to a function f, then f is also cliquish.*

Proof. The proof is completely similar to that of Theorem 5 in which the set A is an open set $V \subset X$. Moreover we use the known result which says that a function $f : X \to \mathbb{R}$ is cliquish iff its set of continuity points is dense ([6]). □

Recall that a function $f : X \to Y$ is almost continuous (in the sense of Stallings) if for each set $U \in T_X \times T_Y$ containing the graph $Gr(f)$ there is a continuous function $g : X \to Y$ such that $Gr(g) \subset U$ ([7]).

Remark 4. *If a sequence of almost continuous functions $f_n : X \to Y$ graph converges to a function $f : X \to Y$, then f is also almost continuous.*

Proof. If $U \in T_X \times T_Y$ contains the graph $Gr(f)$, then there is a positive integer k with $Gr(f_k) \subset U$. Since the function f_k is almost continuous, there is a continuous function $g : X \to Y$ such that $Gr(g) \subset U$. So f is almost continuous and the proof is finished. □

Corollary 3. *Since each continuous function is almost continuous, the graph limit of a sequence of continuous functions is an almost continuous function.*

Each almost continuous function $f : \mathbb{R} \to \mathbb{R}$ (with respect to the topologies $T_X = T_Y = T_e$) has the Darboux property and each Baire 1 Darboux function $f : \mathbb{R} \to \mathbb{R}$ is almost continuous ([1]), so the graph limit of a sequence of Darboux Baire 1 functions is almost continuous and consequently it has the Darboux property.

However there are uniformly convergent sequences of Darboux functions whose limits do not have the Darboux property ([2]).

Theorem 7. *Let I be an open interval and let (f_n) be a sequence of Darboux functions $f_n : I \to \mathbb{R}$ which graph converges to a function $f : I \to \mathbb{R}$ with respect to the topology $T_e \times T_e$. Then f has also the Darboux property.*

PROOF. Assume, to a contradiction, that f does not have the Darboux property. Then there are points $a, b \in I$ with ab and $f(a) \neq f(b)$ and a real number

$$c \in (\min(f(a), f(b)), \max(f(a), f(b)))$$

such that $f(x) \neq c$ for all points $x \in (a, b)$. Let

$$r = \frac{\min(|c - f(a)|, |c - f(b)|}{2},$$

and let

$$U = ((I \times \mathbb{R}) \setminus (([a, b] \times \{c\}) \cup (\{a, b\} \times \mathbb{R}))) \cup$$
$$\cup (\{a\} \times (f(a) - r, f(a) + r)) \cup (\{b\} \times (f(b) - r, f(b) + r)).$$

Then $U \in T_e \times T_e$ and $U \supset Gr(f)$. Consequently, there is a positive integer k such that $Gr(f_k) \subset U$. Observe that

$$\min(f_k(a), f_k(b)) \min(f(a), f(b)) + rc$$
$$\max(f(a), f(b)) - r \max(f_k(a), f_k(b))$$

and $f_k(x) \neq c$ for $x \in (a, b)$. This is contradictory with the Darboux property of the function f_k, and the proof is completed. □

In the same manner we can prove an analogous theorem for functionally connected functions.

Recall that a function $f : I \to \mathbb{R}$ is functionally connected if for each continuous function $g : [a, b] \to \mathbb{R}$ with $a, b \in I$ and ab and $(f(a) - g(a))(f(b) - g(b))0$ there is a point $c \in (a, b)$ such that $f(c) = g(c)$ ([3]).

Theorem 8. *If a sequence of functionally connected functions $f_n : I \to \mathbb{R}$ graph converges to a function $f : I \to \mathbb{R}$, then the function f is also functionally connected.*

Proof. Assume, to a contradiction, that there are points $a, b \in I$ with ab and a continuous function $g : [a, b] \to \mathbb{R}$ such that

$$(g(a) - f(a))(g(b) - f(b))0 \ \text{ and } \ g(x) \neq f(x) \ \text{ for all } \ x \in [a, b].$$

Let

$$r = \frac{\min(|g(a) - f(a)|, |g(b) - f(b)|)}{2}$$

and let

$$U = ((I \times \mathbb{R}) \setminus (Gr(g) \cup (\{a, b\} \times \mathbb{R})) \cup (\{a\} \times (f(a) - r, f(a) + r)) \cup \\ \cup (\{b\} \times (f(b) - r, f(b) + r)).$$

Then $U \in T_e \times T_e$ and $U \supset Gr(f)$. Consequently, there is a positive integer k such that $Gr(f_k) \subset U$. If $g(a)f(a)$ and $g(b)f(b)$, then

$$g(a)f(a) - rf_k(a) \ \text{ and } \ g(b)f(b) + rf_k(b) \\ \text{and } \ f_k(x) \neq g(x) \ \text{ for } \ x \in (a, b).$$

This is contradictory with the functional connectivity of the function f_k. If $g(a)f(a)$ and $g(b)f(b)$, then

$$g(a)f(a) + rf_k(a) \ \text{ and } \ g(b)f(b) + rf_k(b) \\ \text{and } \ f_k(x) \neq g(x) \ \text{ for } \ x \in (a, b).$$

This is also contradictory with the functional connectivity of the function f_k. We have considered all cases, so the proof is completed. □

Finishing observe that the graph topology T_{gr} in the set Y^X of all functions from X to Y may be generated by the family of all sets

$$W(f, U) = \{g \in Y^X; Gr(g) \subset U\}, \ \text{ where} \\ U \in T_X \times T_Y, \ \ f \in Y^X \ \text{ and } \ Gr(f) \subset U.$$

Then a sequence of functions $f_n : X \to Y$ is graph convergent to a function $f : X \to Y$ if and only if it is convergent to f with respect to the graph topology T_{gr}.

From Theorems 5 and 7 (cf. also Remark 4) it follows immediately

Corollary 4. *Let $I \subset \mathbb{R}$ be a nondegenerate interval. If a sequence of continuous functions $f_n : I \to \mathbb{R}$ graph converges to a function f, then f is of the first class of Baire and has the Darboux property.*

On the other hand the following theorem is true.

Theorem 9. *Let $I \subset \mathbb{R}$ be a nondegenerate interval and let DB_1 be the space of all Darboux Baire 1 functions $f : I \to \mathbb{R}$ considered with the topology T_{gr}. Then the set C of all continuous functions $f : I \to \mathbb{R}$ is dense in DB_1.*

PROOF. Fix a set $W(f, U)$, where $f \in DB_1$ and $U \subset \mathbb{R}^2$ is an open set such that $Gr(f) \subset U$. Since f is an almost continuous (in the sense of Stallings) function, there is a continuous function $g : I \to \mathbb{R}$ with $Gr(g) \subset U$. So $g \in W(f, U)$ and the proof is completed. □

Acknowledgement. The author is grateful to the referee for his suggestion concerning the present form of Theorem 3.

References

[1] J.B. Brown, *Almost continuous Darboux functions and Reed's pointwise convergence criteria*, Fund. Math., **86** (1974), 1–7.

[2] A.M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math. 659, Springer-Verlag, Berlin, 1978.

[3] R.G. Gibson and T. Natkaniec, *Darboux like functions*, Real Anal. Exchange, **22(2)** (1996-97), 492–533.

[4] S. Kempisty, *Sur les fonctions quasicontiues*, Fund. Math., **19** (1932), 184–197.

[5] E. Klein and A.C. Thompson, *Theory of Correspondences*, A Wiley-Interscience Publication, John Wiley and Sons, 1984.

[6] T. Neubrunn, *Quasi-continuity*, Real Anal. Exchange, **14(2)** (1988/89), 259–306.

[7] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math., **47** (1959), 249–263.