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ON THE STATISTICAL AND *I* VARIATION OF DOUBLE SEQUENCES

Abstract

We extend the notion of finite statistical variation of single sequences by Faisant et al (2005) to double sequences using the double natural density of the set $\mathbb{N}\times\mathbb{N}$. Certain consequences of this notion are investigated including its relation with statistical convergence introduced earlier by Mursaleen and Edely (2003). We also introduce the more general concept of I variation of double sequences and investigate it's relation with I and I^* convergence.

1 Introduction.

The usual notion of convergence does not always capture in fine detail the properties of the vast class of sequences that are not convergent. One way of including more sequences under preview is to consider those sequences that are convergent when restricted to some 'big' set of natural numbers. By a 'big' set one understands a set $K \subset \mathbb{N}$ having asymptotic density equal to 1. Investigation in this line was initiated by Fast [5] and independently by Schoenberg [17], who introduced the idea of statistical convergence. Since then a lot of work has been done in this area (in particular after the works of Fridy [6] and Šalát [16]). Recently a similar approach was taken by Faisant et al [4] to introduce the idea of finite statistical variation of sequences.

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Statistical convergence was further extended to I as also I^*-convergence by Kostyrko et al [7] (also independently by Nuray and Ruckle [13]) in 2001. Detailed investigations on these topics can be found in [7], [8], [9], and [10] where more references can be found.

For double sequences, statistical convergence was introduced by Mursaleen and Edely [12] in 2003 using the double natural density (also by Móricz [11] who studied it for multiple sequences). Double sequences were also studied in [2] and also in [1] and [3]. In particular a thorough investigation of I and I^*-convergence of double sequences was very recently done in [2]. It can be observed from [2] and [12] that the pattern of investigation for double sequences is not always analogous to that of single sequences. In this paper we continue in this line by defining finite statistical (also I) variations of double sequences and mainly investigate in the line of [4] where it again appears that the examples and methods of proofs are not always analogous to that for single sequences [4].

2 Basic Definitions and Notation.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{R}$ the set of all real numbers.

Recall that a subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if $d(A) = \lim_{n\to\infty} \frac{|A|_n}{n}$, where $|A|_n$ is the cardinality of the set $\{k \in A : k \leq n\}$. By the convergence of a double sequence we mean the convergence in Pringsheim's sense (see [15]). A double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ of real numbers is said to be convergent to $\xi \in \mathbb{R}$ if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{jk} - \xi| < \epsilon$ whenever $j, k \geq N_\epsilon$. In this case we write $\lim_{j\to\infty,k\to\infty} x_{jk} = \xi$.

If $A \subset \mathbb{N} \times \mathbb{N}$ has the property that for any $(m, n) \in \mathbb{N} \times \mathbb{N}$, there is a $(j, k) \in A$ such that $j > m, k > n$ then $\{x_{jk}\}_{(j,k)\in A}$ is called a subsequence of the double sequence $\{x_{jk}\}_{j,k\in\mathbb{N}}$. Pringsheim convergence of a subsequence is also similarly defined as above.

A double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{jk}| < M$ for all $j, k \in \mathbb{N}$. That is $\|x\|_{(\infty,2)} = \sup_{j,k} |x_{jk}| < \infty$.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let $K(n, m)$ be the numbers of $(j, k) \in K$ such that $j \leq n, k \leq m$. If the sequence $\{\frac{K(n,m)}{n.m}\}_{n,m\in\mathbb{N}}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by $d_2(K) = \lim_{m\to\infty,n\to\infty} \frac{K(n,m)}{nm}$.

Definition 1. [12] A double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ if for any $\epsilon > 0$, we have $d_2(A(\epsilon)) = 0$, where $A(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N}; |x_{jk} - \xi| \geq \epsilon\}$.

Next we recall the following, where X represents an arbitrary set.

Definition 2. Let $X \neq \phi$. A class I of subsets of X is said to be an ideal in X provided (i) $\phi \in I$, (ii) $A, B \in I$ implies $A \cup B \in I$, and (iii) $A \in I, B \subset A$ implies $B \in I$. I is called a nontrivial ideal if $X \notin I$.

Definition 3. Let $X \neq \phi$. A non empty class F of subsets of X is said to be a filter in X provided (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$, and (iii) $A \in F, A \subset B$ implies $B \in F$. If I is a nontrivial ideal in X, $X \neq \phi$, then the class $F(I) = \{M \subset X; M = X \setminus A \text{ for some } A \in I\}$ is a filter on X, called the filter associated with I.

Definition 4. A nontrivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$. Throughout the paper we take I as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 5. A nontrivial ideal I of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$. Clearly a strongly admissible ideal is also admissible. Let $I_0 = \{A \subset \mathbb{N} \times \mathbb{N} : \exists\ m(A) \in \mathbb{N} \text{ such that } (i,j) \notin A \text{ whenever } i, j \geq m(A)\}$. Then I_0 is a nontrivial strongly admissible ideal and clearly an ideal I is strongly admissible if and only if $I_0 \subset I$.

Definition 6. [3] (see also [2]). A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is said to converge to $\xi \in \mathbb{R}$ with respect to the ideal I, if for every $\epsilon > 0$ the set $A(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N}; |x_{jk} - \xi| \geq \epsilon\} \in I$. In this case we say that x is I-convergent and we write $I - \lim_{j \to \infty, k \to \infty} x_{jk} = \xi$.

Remark 1. Note that If I is the ideal I_0 then I-convergence coincides with the usual convergence and if we take $I_d = \{A \subset \mathbb{N} \times \mathbb{N}; d_2(A) = 0\}$ then I_d-convergence becomes statistical convergence. I-convergent double sequences may be unbounded, for example, let I be the ideal I_0 of $\mathbb{N} \times \mathbb{N}$. If we define $\{x_{jk}\}_{j,k \in \mathbb{N}}$ by

$$x_{jk} = \begin{cases} k & \text{if } j = 1, \\ 2 & \text{if } j \neq 1, \end{cases}$$

Then $\{x_{jk}\}_{j,k \in \mathbb{N}}$ is unbounded but I-convergent.

Definition 7. [2] A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is said to be I^*-convergent to $\xi \in \mathbb{R}$ if and only if there exists a set $M \in F(I)$, i.e. $\mathbb{N} \times \mathbb{N} \setminus M \in I$, such that $\lim_{j \to \infty, k \to \infty, (j,k) \in M} x_{jk} = \xi$, where $\{x_{jk}\}_{j,k \in M}$ is a subsequence of $\{x_{jk}\}_{j,k \in \mathbb{N}}$, and we write $I^* - \lim_{j \to \infty, k \to \infty} x_{jk} = \xi$.

We shall denote by $C_2(I)$ ($C_2^*(I)$) the set of all I-convergent (I^*-convergent) double sequences of real numbers.

3 The set $W_2(I)$.

From this stage onwards we assume that the set $\mathbb{N} \times \mathbb{N}$ (or any subset of $\mathbb{N} \times \mathbb{N}$) is ordered with respect to the relation

$$(i,j) \begin{cases} < (i_1, j_1) & \text{if } i + j < i_1 + j_1 \text{ or } i < i_1 \text{ when } i + j = i_1 + j_1, \\ = (i_1, j_1) & \text{if } i = i_1, j = j_1. \end{cases} \tag{1}$$

We now introduce the following definition.

Definition 8. cf. [4]. A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is said to be of finite I-variation if there exists a set $K = \{(j_1, k_1) < (j_2, k_2) < ...\} \in F(I)$ such that $Var\ x|_K = \sum_{i=1}^{\infty} |x_{j_{i+1} k_{i+1}} - x_{j_i k_i}| < +\infty$, where K is ordered by the relation (1).

Note 1. The definition of double sequences of finite statistical variation immediately follows from Definition 8 taking $I = I_d$. The set of all double sequences of finite I-variation will be denoted by $W_2(I)$. It is easy to verify that if $K \supset L$, then $Var\ x|_K \geq Var\ x|_L$. It should be noted that a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers having finite I-variation on a set $K \in F(I)$ can have infinite I-variation on a superset of K. Consider the following example.

Example 1. Let us consider a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ defined as follows:

$$x_{jk} = \begin{cases} 1 & \text{if } j = m^2, k = n^2 \text{ for some } m, n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the set $K = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j = m^2, k = n^2 \text{ for some } m, n \in \mathbb{N}\} \in I_d$, so $\mathbb{N} \times \mathbb{N} \setminus K = \{(j_1, k_1) < (j_2, k_2) < ...\} \in F(I_d)$ and $Var\ x|_{\mathbb{N} \times \mathbb{N} \setminus K} =$

$\sum_{i=1}^{\infty} |x_{j_{i+1}k_{i+1}} - x_{j_i k_i}| = 0 < +\infty$. This shows that $x \in W_2(I_d)$. Now if we consider the superset $M = (\mathbb{N} \times \mathbb{N} \setminus K) \cup (N \times \{1\})$ of $\mathbb{N} \times \mathbb{N} \setminus K$, then

$$Var\ x|_M \geq \sum_{k=2}^{\infty} |x_{k^2 1} - x_{(k^2-1)2}| = \sum_{k=2}^{\infty} 1 = +\infty.$$

The following results show that the idea of I-variation is closely related to the concepts of I and I^*-convergence.

Theorem 1. *(i) For ideals $I \subset J$, we have $W_2(I) \subset W_2(J)$, $C_2^*(I) \subset C_2^*(J)$, and $C_2(I) \subset C_2(J)$. (ii) For every strongly admissible ideal I, we have $W_2(I) \subset C_2^*(I) \subset C_2(I)$.*

Proof. (i) The proof is straightforward and so is omitted.

(ii) Let $x \in W_2(I)$. Then there exists a set $K = \{(j_1, k_1) < (j_2, k_2) < \ldots\} \in F(I)$ such that $Var\ x|_K = \sum_{i=1}^{\infty} |x_{j_{i+1}k_{i+1}} - x_{j_i k_i}| < +\infty$. Consequently $\lim_{n\to\infty} \sum_{i=1}^{n} (x_{j_{i+1}k_{i+1}} - x_{j_i k_i}) = \lim_{n\to\infty} (x_{j_{n+1}k_{n+1}} - x_{j_1 k_1}) = l$, i.e. $\lim_{n\to\infty} x_{j_{n+1}k_{n+1}} = l - x_{j_1 k_1} = l_0 \in R$.

Let $\epsilon > 0$ be given. Then there exists a $n_0 \in \mathbb{N}$ such that $|x_{j_{n+1}k_{n+1}} - l_0| < \epsilon$, $\forall n \geq n_0$. Choose $p = max\{j_{n_0+1}, k_{n_0+1}\} + 1$. Then evidently for any $(m, n) \in K$ with $m, n \geq p$ (Since I is strongly admissible, there are infinitely many indices like this in K) $|x_{mn} - l_0| < \epsilon$, i.e. $\lim_{m\to\infty, n\to\infty, (m,n)\in K} x_{mn} = l_0$. This shows that $x \in C_2^*(I)$. Hence $W_2(I) \subset C_2^*(I)$.

The proof of $C_2^*(I) \subset C_2(I)$ is given in [2, Th. 1] and so is omitted. □

Strong admissibility is essential for Theorem 1 (ii) as shown by the following examples.

Example 2. Let $\Delta = \{(m, n) \in \mathbb{N}\times\mathbb{N} : m = n\}$, the diagonal of $\mathbb{N}\times\mathbb{N}$. Let $I = \{A \cup B : A \subset \mathbb{N} \times \mathbb{N} \setminus (\Delta \cup (\{1\} \times \mathbb{N}))$ and B is a finite suset of $\Delta \cup\ (\{1\} \times \mathbb{N})\}$. Then I is not strongly admissible. Consider the sequence $\{x_{jk}\}_{j,k\in\mathbb{N}}$ defined by $x_{jk} = \begin{cases} 1 & \text{if } (j,k) \in \Delta, \\ max\{j,k\} & \text{for } (j,k) \in \mathbb{N}\times\mathbb{N}\setminus\Delta \end{cases}$. Then for $K = \Delta \cup (\{1\} \times \mathbb{N}) \in F(I)$, $\lim_{j\to\infty, j\to\infty, (j,k)\in K} x_{jk} = 1$ and so $\{x_{jk}\}_{j,k\in\mathbb{N}}$ is I^*-convergent to 1 but it is not I-convergent.

Example 3. Let $I = \{A \cup B : A \subset \mathbb{N} \times \mathbb{N} \setminus (\{1\} \times \mathbb{N})$ and B is at most a finite subset of $\{1\} \times \mathbb{N}\}$. Then again I is not strongly admissible. Consider the sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ defined by $x_{jk} = \begin{cases} 1 & \text{if } j = 1, \\ max\{j,k\} & \text{otherwise} \end{cases}$. Then

$K = \{1\} \times \mathbb{N} \in F(I)$ and $Var\ x|_K = 0 < \infty$ but there is no $K \in F(I)$ for which $\{x_{jk}\}_{j,k \in K}$ is a subsequence of $\{x_{jk}\}_{j,k \in \mathbb{N}}$ and $\lim_{j\to\infty, j\to\infty, (j,k)\in K} x_{jk}$ finitely exists.

Before we prove the next result we introduce the following as in [4]. Let us denote by $l_2^\infty(I)$ the set of all double sequences $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ such that there exists $K \in F(I)$ satisfying $x|_K$ is bounded. Then $l_2^\infty(I)$ is a real vector subspace of $R^{\mathbb{N}\times\mathbb{N}}$ and $C_2(I) \subset l_2^\infty(I)$.

Now for $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$, $y = \{y_{jk}\}_{j,k\in\mathbb{N}} \in l_2^\infty(I)$, we define $\|x\|_\infty = inf\{\lambda \in R^+ : \exists K \in F(I) \text{ such that } \forall (j,k) \in K, |x_{jk}| \leq \lambda\}$, and $\rho(x,y) = \sup_{j,k} |x_{jk} - y_{jk}|$. The reason behind introducing this topology is same as in [4].

Theorem 2. *(i) $x \longmapsto \|x\|_\infty$ is a seminorm on $l_2^\infty(I)$ and $\|x-y\|_\infty \leq \rho(x,y)$. (ii) $\overline{W_2(I)} = C_2(I)$, where $\overline{W_2(I)}$ is the closure of $W_2(I)$ in $l_2^\infty(I)$.*

The proof is parallel to Proposition 2 in [4] and so is omitted.

Remark 2. The mapping $x \longmapsto \|x\|_\infty$ is not a norm. For example let us take $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ defined as $x_{jk} = \begin{cases} 1 & \text{if } (j,k) \in \{1\}\times\mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$ then $\|x\|_\infty = 0$ but $x \neq 0$ (Taking $I = I_0$ or I_d).

4 Basic Inclusions with Respect to the Ideal I_d.

In this section we shall precisely establish the following:

$$W_2(I_d) \subsetneqq C_2^*(I_d) = C_2(I_d) \subsetneqq l_2^\infty(I_d).$$

We start with the last inclusion $C_2(I_d) \subsetneqq l_2^\infty(I_d)$. For this we first recall that an admissible ideal I of $\mathbb{N}\times\mathbb{N}$ is a maximal admissible ideal if and only if for any $A \subset \mathbb{N}\times\mathbb{N}$, either $A \in I$ or $A^c \in I$, where c stands for complement (see [2]). We now prove the following result.

Theorem 3. *For a strongly admissible ideal I, $C_2(I) = l_2^\infty(I)$ if and only if I is a maximal ideal.*

PROOF. Let I be a maximal strongly admissible ideal and $x = \{x_{jk}\}_{j,k\in\mathbb{N}} \in l_2^\infty(I)$. Then there exists a set $K \in F(I)$ such that for all $(j,k) \in K$, $|x_{jk}| \leq \lambda$, for some $\lambda \in R^+$. Therefore we can find $a, b \in \mathbb{R}$ such that $a \leq x_{jk} \leq b$ for all $(j,k) \in K$. Define $K_1 = \{(j,k) \in K : a \leq x_{jk} \leq \frac{a+b}{2}\}$ and $L_1 = \{(j,k) \in K :$

$\frac{a+b}{2} \leq x_{jk} \leq b\}$. Then $K = K_1 \cup L_1$. Since I is nontrivial so $K \notin I$. Hence both K_1 and L_1 can not belong to I. Without any loss of generality let $K_1 \notin I$ and we rewrite this set as $A_1 = \{(j,k) \in K : a = a_1 \leq x_{jk} \leq b_1 = \frac{a+b}{2}\} \notin I$.

Let $K_2 = \{(j,k) \in K : a_1 \leq x_{jk} \leq \frac{a_1+b_1}{2}\}$ and $L_2 = \{(j,k) \in K : \frac{a_1+b_1}{2} \leq x_{jk} \leq b_1\}$. Then again $A_1 = K_2 \cup L_2 \notin I$ and so by similar arguments we obtain a set $A_2 = \{(j,k) \in K : a_2 \leq x_{jk} \leq b_2\}$ such that $A_2 \subset A_1$, $A_2 \notin I$ and $b_2 - a_2 = \frac{b-a}{4}$. Proceeding in this way we obtain $A_1 \supset A_2 \supset ... \supset A_n \supset ...$ where $A_n = \{(j,k) \in K : a_n \leq x_{jk} \leq b_n\} \notin I$ and $b_n - a_n = \frac{b-a}{2^n}$. So there exists $l \in \cap_{n\geq 1}[a_n, b_n]$.

Now let $\epsilon > 0$ be given. We choose $p \in \mathbb{N}$, such that $[a_n, b_n] \subset (l-\epsilon, l+\epsilon)$ for all $n \geq p$. Let $A(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - l| \geq \epsilon\}$ and $A'(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \geq p \wedge k \geq p \wedge |x_{jk} - l| \geq \epsilon\}$. If $(j,k) \in A'(\epsilon)$ then $x_{jk} \notin [a_p, b_p]$ and so $(j,k) \notin A_p$. Therefore $A'(\epsilon) \subseteq A_p^c$. Since I is maximal and $A_p \notin I$, so $A_p^c \in I$ and so $A'(\epsilon) \in I$. Hence $A(\epsilon) \subset A'(\epsilon) \cup (\{1,2,...,p-1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,...,p-1\}) \cup K^c \in I$, since I is strongly admissible. Therefore $I - \lim_{j\to\infty, k\to\infty} x_{jk} = l$. This implies $l_2^\infty(I) \subset C_2(I)$ and so $l_2^\infty(I) = C_2(I)$.

Conversely, let I be not maximal. Then there exists $M \subset \mathbb{N} \times \mathbb{N}$ such that $M \notin I$ and $M^c \notin I$. We define a double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ as $x_{jk} = \begin{cases} 2 & \text{if } (j,k) \in M, \\ 0 & \text{otherwise.} \end{cases}$ Clearly $x \in l_2^\infty(I)$ but $x \notin C_2(I)$. Indeed for any $l \in R$ there exists an $\epsilon > 0$ such that $A(\epsilon) = \{(j,k) : |x_{jk} - l| \geq \epsilon\}$ is equal to either M or M^c or $\mathbb{N} \times \mathbb{N}$ and neither of these sets belongs to I. Therefore $x \notin C_2(I)$. This completes the proof. □

Remark 3. Since the ideal I_d is not maximal so in view of above we can conclude that $C_2(I_d) \subsetneqq l_2^\infty(I_d)$.

Theorem 4. $C_2^*(I_d) = C_2(I_d)$.

Proof. In view of Theorem 1 (ii) it is sufficient to prove that $C_2(I_d) \subset C_2^*(I_d)$. Let $\{x_{jk}\}_{j,k\in\mathbb{N}}$ be a double sequence, I_d-convergent to $\xi \in \mathbb{R}$. Put $A_1 = \{(j,k) : |x_{jk} - \xi| \geq 1\}$ and $A_n = \{(j,k) : \frac{1}{n} \leq |x_{jk} - \xi| < \frac{1}{n-1}\}$. From the assumption it follows that $d_2(A_n) = 0$ for each $n \in \mathbb{N}$.

Observe that also $d_2(\cup_{n=1}^p A_n) = 0$ for $p \in \mathbb{N}$. For $p \in \mathbb{N}$, let T_p be a natural number such that $\frac{1}{jk}\text{card}\{(m,n) : m \leq j \wedge n \leq k \wedge (m,n) \in \cup_{i=1}^p A_i\} < \frac{1}{p}$ for $j \geq T_p$ and $k \geq T_p$. We can obviously assume that the sequence $\{T_p\}_{p\in\mathbb{N}}$ is increasing. Let $C_p = \{(m,n) : T_p \leq \min\{m,n\} < T_{p+1}\}$, $D_p = C_p \cap \cup_{i=1}^p A_i$ for $p \in \mathbb{N}$ and $D = \cup_{p=1}^\infty D_p$. We shall show that $d_2(D) = 0$. Indeed, if $\eta > 0$ and $p \in \mathbb{N}$ is such that $\frac{1}{p} < \eta$ then for $(j,k) \in C_p$ we have $(\{1,2,...,j\} \times$

$\{1,2,...,k\}) \cap D \subset (\{1,2,...,j\} \times \{1,2,...,k\}) \cap \cup_{i=1}^{p} A_i$, so $\frac{1}{jk}\text{card}\{(m,n) : m \leq j \wedge n \leq k \wedge (m,n) \in D\} < \frac{1}{p}$ for such k and j. Hence $d_2(D) = 0$.

Simultaneously for $k \geq T_p$, $j \geq T_p$, $(j,k) \notin D$ we have $|x_{jk} - \xi| < \frac{1}{p}$, so $\{x_{jk}\}_{j,k\in\mathbb{N}}$ I_d^*-converges to ξ. Hence the proof is completed. □

Remark 4. In [7] it was proved that I and I^*-convergence of ordinary sequences of real numbers are equivalent if and only if the ideal $I \subset 2^{\mathbb{N}}$ satisfies the following condition (AP) (see also [4]):

Definition 9. (AP). An admissible ideal $I \subset 2^{\mathbb{N}}$ satisfies the condition (AP) if for every countable family of mutually disjoint sets $\{A_n\}_{n\in\mathbb{N}}$ belonging to I, there exists a countable family of sets $\{B_n\}_{n\in\mathbb{N}}$ such that $A_n \Delta B_n$ is a finite set for $n \in \mathbb{N}$ and $B = \cup_{n=1}^{\infty} B_n \in I$.

If $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ is an admissible ideal fulfilling the condition (AP) (the definition of (AP) for ideals of subsets of $\mathbb{N} \times \mathbb{N}$ is in practice the same as above) then as in Theorem 3.2 in [7] one can easily prove that for any double sequence $\{x_{jk}\}_{j,k\in\mathbb{N}}$ in R, $I - \lim_{j\to\infty,k\to\infty} x_{jk} = \xi$ implies $I^* - \lim_{j\to\infty,k\to\infty} x_{jk} = \xi$. However unlike single sequences, the condition (AP) is not necessary for the equivalence of I and I^*-convergence of double sequences. For example consider the ideal I_0 (which corresponds to the Pringsheim's convergence). Obviously for the ideal I_0, I_0 and I_0^*-convergence are equivalent. But note that the sets $B_i = \{i\} \times \mathbb{N}$ belong to I_0 and they form a decomposition of $\mathbb{N} \times \mathbb{N}$. If we omit from $\mathbb{N} \times \mathbb{N}$ only finitely many elements of each B_i (or some B_i 's), the resulting set does not belong to I_0. This shows that the ideal I_0 does not have the property (AP).

The equality of the sets $C_2^*(I)$ and $C_2(I)$ (for double sequences) is governed by the following condition (AP2).

Definition 10. (AP2). We say that an admissible ideal $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the condition (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to I, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in I_0$, i.e. $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \cup_{j=1}^{\infty} B_j \in I$ (hence $B_j \in I$ for each $j \in \mathbb{N}$). The details of the equivalence of I and I^*-convergence of double sequences and condition (AP2) can be seen in [2].

Finally we prove the following result.

Theorem 5. *Let I be a strongly admissible ideal satisfying $\overline{d_2}(K) > \frac{1}{2}$ for every $K \in F(I)$, then $W_2(I) \subsetneqq C_2^*(I)$.*

Proof. It is known that for any strongly admissible ideal I, $W_2(I) \subset C_2^*(I)$ from Theorem 1, (ii). Now let us write $\mathbb{N} \times \mathbb{N} = \{(a_1, b_1) < (a_2, b_2) < ...\}$, ordered by the relation (1) and define $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ by $x_{jk} = \frac{(-1)^i}{a_i} + \frac{(-1)^i}{b_i}$ if $(j,k) = (a_i, b_i)$. Then x is I^*-convergent to 0 with $K = \mathbb{N} \times \mathbb{N} \in F(I)$. Thus $x \in C_2^*(I)$.

Now let $K \in F(I)$. Consider $E = \{(a_i, b_i) \in K : (a_{i+1}, b_{i+1}) \in K\}$. Then $(a_i, b_i) \in K \setminus E$ implies $(a_{i+1}, b_{i+1}) \notin K \setminus E$, so $\overline{d_2}(K \setminus E) \leq \frac{1}{2}$. thus $\overline{d_2}(K \setminus E) \leq \frac{1}{2} < \overline{d_2}(K)$, which implies $\overline{d_2}(E) > 0$.

Let $E_1 = \{i \in \mathbb{N} : (i,j) \in E \text{ for some } j\}$ and $E_2 = \{j \in \mathbb{N} : (i,j) \in E \text{ for some } i\}$. Then clearly $E \subset E_1 \times E_2$. Furthermore it is easy to check that for any $(m,n) \in \mathbb{N} \times \mathbb{N}$, $E(m,n) \leq |E_1|_m |E_2|_n$. We now claim that $\overline{d}(E_1)$ and $\overline{d}(E_2)$ must be positive. For otherwise let $\overline{d}(E_1) = 0$. Let $\epsilon > 0$ be given. Then there is a $m_0 \in \mathbb{N}$ such that $\frac{|E_1|_m}{m} < \epsilon \ \forall \ m \geq m_0$. Then for any $(m,n) \in \mathbb{N} \times \mathbb{N}$ with $m, n \geq m_0$

$$\frac{E(m,n)}{mn} \leq \frac{|E_1|_m |E_2|_n}{mn} \leq \frac{|E_1|_m}{m} < \epsilon$$

which shows that $\overline{d_2}(E) = 0$, a contradiction to the fact that $\overline{d_2}(E) > 0$. Hence

$$\begin{aligned} Var\ x|_K \geq Var\ x|_E &\geq \sum_{(a_i,b_i) \in E} \left(\frac{1}{a_i} + \frac{1}{b_i} + \frac{1}{a_{i+1}} + \frac{1}{b_{i+1}}\right) \\ &\geq \sum_{a_i \in E_1} \frac{1}{a_i} + \sum_{b_i \in E_2} \frac{1}{b_i} = \infty. \end{aligned}$$

The last equality is a consequence of a theorem of Powel-Šalát (see [14]). □

Note 2. Since for the ideal I_d, $\overline{d_2}(K) = 1$ for all $K \in F(I_d)$ so from the above theorem it follows that $W_2(I_d) \subsetneqq C_2^*(I_d)$.

Remark 5. As $C_2^*(I)$ is a semi-normed space, so if $W_2(I) \subsetneqq C_2^*(I)$ then $W_2(I)$ is a proper linear subspace of $C_2^*(I)$ and consequently $C_2^*(I) \setminus W_2(I)$ is dense in $C_2^*(I)$. In particular taking $I = I_d$ we can conclude that the set of all double sequences of finite statistical variation as also its complement, both are proper dense subsets of the space of all statistically convergent sequences.

5 When $W_2(I) \subsetneqq C_2^*(I)$?

In this section we continue to examine the inclusion $W_2(I) \subset C_2^*(I)$ and show that this is a strict inclusion for certain other ideals I also in addition to I_d as also the class of ideals given in Theorem 5.

We first start with the ideal I_0. Consider the double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ defined by

$$x_{jk} = \frac{(-1)^i}{a_i} + \frac{(-1)^i}{b_i} \quad \text{if } (j,k) = (a_i, b_i),$$

where $\mathbb{N} \times \mathbb{N} = \{(a_1, b_1) < (a_2, b_2) < ... < (a_i, b_i) < (a_{i+1}, b_{i+1}) < ...\}$ ordered by the relation (1) as before. Since $\mathbb{N} \times \mathbb{N} \in F(I_0)$ and $\lim_{j \to \infty, k \to \infty} x_{jk} = 0$, so $x \in C_2^*(I)$. However $x \notin W_2(I_0)$ because for any $K \in F(I_0)$ we can choose a positive integer $m \in \mathbb{N}$ such that $K \supset \mathbb{N} \times \mathbb{N} \setminus \cup_{i=1}^{m} [(\{i\} \times \mathbb{N}) \cup (\mathbb{N} \times \{i\})]$. Then $Var\ x|_K \geq Var\ x|_M$ where $M = \{(a_i, b_i) : a_i > m \wedge b_i > m\}$. clearly $Var\ x|_M = \infty$ and so $Var\ x|_K = \infty$.

Another example could be the sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ where $x_{jk} = \frac{(-1)^{j+k}}{j+k}$. It is easy to check that $x \in C_2^*(I_0)$ but $x \notin W_2(I_0)$.

Now let I be an ideal and $A \subset \mathbb{N} \times \mathbb{N}$. As in [4] we define $\mathcal{I} =< I, A >$, the ideal generated by I and A as $\mathcal{I} = \{M \cup B : M \in I \text{ and } B \subset A\}$. Clearly $\mathcal{I} \neq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ if and only if $A^c \notin I$.

Theorem 6. *Let $A \subset \mathbb{N} \times \mathbb{N}$ be such that (i) A^c is not contained in finite union of rows and columns of $\mathbb{N} \times \mathbb{N}$, and (ii) A^c contains at most finite numbers of elements from each row and column of $\mathbb{N} \times \mathbb{N}$, then for the strongly admissible ideal $\mathcal{I} =< I_0, A >$, we have $W_2(\mathcal{I}) \subsetneqq C_2^*(\mathcal{I})$.*

PROOF. Let $K = A^c = \{(p_1, q_1) < (p_2, q_2) < ...\} \in F(\mathcal{I})$. We define $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ as :

$$x_{jk} = \begin{cases} 0 & \text{if } (j,k) \notin K, \\ \frac{(-1)^i}{i} & \text{if } (j,k) = (p_i, q_i) \in K. \end{cases}$$

Then $\lim_{j \to \infty, k \to \infty, (j,k) \in K} x_{jk} = 0$ and so $x \in C_2^*(\mathcal{I})$.

Now if $L \in F(\mathcal{I})$, then $L^c = J \cup C$ where $J \in I_0$ and $C \subset A$. So $L = J^c \cap C^c \supset J^c \cap A^c = K \setminus J$. Since $J \in I_0$, so there exists $m(J) \in \mathbb{N}$ such that $(p_j, q_j) \notin J$ whenever both $p_j, q_j \geq m(J)$. Therefore

$$\begin{aligned} L \supset K \setminus J \supset K \setminus \{(p_j, q_j) : \text{ either } p_j \text{ or } q_j < m(J)\} \\ \supset \{(p_j, q_j) \in K : p_j \geq m(J) \wedge q_j \geq m(J)\} = M. \end{aligned}$$

Then $Var\ x|_L \geq Var\ x|_M$. Since by the condition (ii), K contains only finite number of terms (p_j, q_j) where either p_j or $q_j < m(J)$ so clearly $Var\ x|_M = \infty$, which gives $Var\ x|_L = \infty$. So $x \notin W_2(\mathcal{I})$. □

Remark 6. It is not clear whether the result remains true when I_0 is replaced by any strongly admissible ideal I and it remains open.

Let $\sigma : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be an injective map such that $\cup_{n=1}^{\infty}\sigma(n)$ is a partition of $\mathbb{N}$. Let $\Delta_n = \{(j,k) : \min\{j,k\} \in \sigma(n)\}$. Then $\{\Delta_n\}_{n\in N}$ is a decomposition of $\mathbb{N}\times\mathbb{N}$. Note that for each $n \in \mathbb{N}$, both Δ_n and Δ_n^c are infinite. Now we define $I_\sigma = \{A \subset \mathbb{N}\times\mathbb{N} :$ there exists a finite set F such that $A \subset \cup_{n\in F}\Delta_n\}$. Then I_σ is a strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$.

Theorem 7. $W_2(I_\sigma) \subsetneqq C_2^*(I_\sigma)$.

PROOF. Let us write $\Delta_n = \{(a_1^n, b_1^n) < (a_2^n, b_2^n) < ...\}$ which is ordered by the relation (1). We define $x = \{x_{mn}\}_{m,n\in N}$ as $x_{mn} = \frac{(-1)^i}{k+i}$ if $(m,n) = (a_i^k, b_i^k)$.

Now let us choose $K = \mathbb{N}\times\mathbb{N} \in F(I_\sigma)$. Then for any $\epsilon > 0$, choose $M \in \mathbb{N}$ so that $M > \frac{1}{\epsilon}$. Now let $m_0 = max\ \{a_i^k : k \le M \wedge i \le M\}$, $n_0 = \max\{b_i^k : i \le M \wedge k \le M\}$ and $k_0 = \max\{m_0, n_0\}$. Now let $m \ge k_0, n \ge k_0$. Then writing $(m,n) = (a_i^k, b_i^k)$ we must have either $k > M$ or $i > M$. But in both the cases $|x_{mn}| = \frac{1}{k+i} \le \frac{1}{M} < \epsilon$. Thus we have $|x_{mn}| < \epsilon$ whenever $m \ge k_0, n \ge k_0$. This gives $I^* - \lim_{m\to\infty,n\to\infty} x_{mn} = 0$, and so $x \in C_2^*(I_\sigma)$.

Now let $K \in F(I_\sigma)$. Then $K \supset \cup_{n\ge P}\Delta_n$ for some $P \ge 1$. Hence

$$Var\ x|_K \ge Var\ x|_{\Delta_P} = \sum_{i=1}^{\infty}\left|\frac{(-1)^{i+1}}{P+(i+1)} - \frac{(-1)^i}{P+i}\right| \ge \sum_{j=P+1}^{\infty}\frac{1}{j} = \infty.$$

Therefore $x \notin W_2(I_\sigma)$. □

Remark 7. In general for any strongly admissible ideal $C_2^*(I) \subset C_2(I)$. But we shall show that for the strongly admissible ideal I_σ, $C_2^*(I_\sigma) \subsetneqq C_2(I_\sigma)$ if in adition $\sigma(n)$ is infinite for each n. We consider the double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ defined as $x_{mn} = \frac{1}{j}$ if and only if $(m,n) \in \Delta_j$. Put $\epsilon_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Let $\eta > 0$ be given. Choose $p \in \mathbb{N}$ such that $\frac{1}{p} < \eta$. Then $A(\eta) = \{(m,n) : |x_{mn} - 0| \ge \eta\} \subset \Delta_1 \cup \Delta_2 \cup ... \cup \Delta_p$. Hence $A(\eta) \in I_\sigma$ and $I_\sigma - \lim_{m\to\infty,n\to\infty} x_{mn} = 0$.

Now suppose that $I_\sigma^* - \lim_{m\to\infty,n\to\infty} x_{mn} = 0$. Then there exists $H \in I_\sigma$ such that for $M = \mathbb{N}\times\mathbb{N} \setminus H$ we have $\lim_{m\to\infty,n\to\infty,(m,n)\in M} x_{mn} = 0$. Then from the construction of the ideal I_σ, there exists $q \in \mathbb{N}$, such that $H \subset \Delta_1 \cup \Delta_2 \cup ... \cup \Delta_q$. But then $\Delta_{q+1} \subset \mathbb{N}\times\mathbb{N} \setminus H = M$. Since $\sigma(q+1)$ is infinite so it follows that for any $n_0 \in \mathbb{N}$, $|x_{mn} - 0| = \epsilon_{q+1} > 0$ hold for infinitely many (m,n)'s with $(m,n) \in \Delta_{q+1} \subset M$ and $m, n \ge n_0$. This contradicts the fact that $\lim_{m\to\infty, n\to\infty, (m,n)\in M} x_{mn} = 0$. We can also conclude from above that the ideal I_σ does not satisfy the condition (AP2).

In another direction, Theorem 5 can be further generalized as follows

Theorem 8. *Let I be a strongly admissible ideal such that every $K = \{(a_1, b_1) < (a_2, b_2) < ...\} \in F(I)$ contains a set $E(K) = \{(a_{i_1}, b_{i_1}) < (a_{i_2}, b_{i_2}) < ...\}$ with $i_k - i_{k-1}$ odd for all $k \in \mathbb{N}$ and such that either $\overline{d}(E_1(K)) > 0$ or $\overline{d}(E_2(K)) > 0$ where $E_1(K) = \{i : (i, j) \in E(K)$ for some $j\}$ and $E_2(K) = \{j : (i, j) \in E(K)$ for some $i\}$ (note that $\overline{d_2}(E(K))$ may be zero). Then for this I, $W_2(I) \subsetneqq C_2^*(I)$.*

6 Open Problem.

Like ordinary sequences [4], here also it remains open whether there exists a strongly admissible ideal I for which $W_2(I) = C_2^*(I)$.

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