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POINTS OF CONTINUITY, QUASICONTINUITY, CLIQUISHNESS, AND UPPER AND LOWER QUASICONTINUITY

Abstract

The quadruplet $(C(f), Q(f), E(f), A(f))$ is characterized, where $C(f)$, $Q(f)$, $E(f)$ and $A(f)$ are the sets of all continuity, quasicontinuity, upper and lower quasicontinuity and cliquishness points of a real function f of real variable, respectively.

Let X be a topological space. For a subset A of X denote by $\mathrm{Cl}(A)$ the closure of A. The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ stand for the set of all real, rational and positive integer numbers, respectively. If A is a subset of $\mathbb{R}$ and $x \in \mathbb{R}$, then $\mathrm{dist}(x, A) = \inf\{|x-a| : a \in A\}$ is the distance of x from A.

A real function $f : X \to \mathbb{R}$ is said to be quasicontinuous (cliquish) at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and for each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $|f(x) - f(y)| < \varepsilon$ for each $y \in G$ ($|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$) [6].

A function $f : X \to \mathbb{R}$ is said to be upper (lower) quasicontinuous at $x \in X$ if for each $\varepsilon > 0$ and for each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $f(y) < f(x) + \varepsilon$ $(f(y) > f(x) - \varepsilon)$ for each $y \in G$ [3].

Denote by $C(f)$ the set of all continuity points of a function $f : X \to \mathbb{R}$, by $Q(f)$ the set of all quasicontinuity points of f, by $A(f)$ the set of all cliquishness points of f and by $E(f)$ the set of all points of both upper and lower quasicontinuity of f. It is well-known that $C(f) \subset Q(f) \subset A(f)$, $C(f)$ is G_δ, $A(f)$ is closed [5], $Q(f) \subset E(f)$ [3] and $A(f) \setminus C(f)$ is of first category [2].

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In the paper [2], the triplet $(C(f), Q(f), A(f))$ is characterized. In this paper, we will characterize the quadruplet $(C(f), Q(f), E(f), A(f))$.

In [4] it is shown that if a function $f : X \to \mathbb{R}$ is upper and lower quasicontinuous at each point $x \in X$, then f is cliquish. However, the inclusion $E(f) \subset A(f)$ does not hold. If $f(0) = 0$, $f(x) = 2$ for positive rational x, $f(x) = 1$ for positive irrational x, $f(x) = -2$ for negative rational x and $f(x) = -1$ for negative irrational x, then $0 \in E(f) \setminus A(f)$. However, the set $E(f) \setminus A(f)$ is small.

Theorem 1. *Let $f : X \to \mathbb{R}$ be a function. Then the set $E(f) \setminus A(f)$ is nowhere dense.*

Proof. Suppose that the set $E(f) \setminus A(f)$ is not nowhere dense. Then there is a nonempty open set K such that $E(f) \setminus A(f)$ is dense in K. Let $L = K \setminus A(f)$. Since $A(f)$ is closed the set L is nonempty open and $E(f)$ is dense in L.

Let $x_0 \in L$. Then there is an $\varepsilon > 0$ and a nonempty open set $M \subset L$ such that the following holds.

$$\text{If } \emptyset \neq G \subset M \text{ is open, there are } y, z \in G \text{ with } |f(y) - f(z)| \geq 8\varepsilon. \qquad (*)$$

Since $E(f)$ is dense in L there is $x_1 \in E(f) \cap M$. Hence, there is a nonempty open set $U_1 \subset M$ such that $f(y) < f(x_1) + \varepsilon$ for each $y \in U_1$. Further there is $x_2 \in E(f) \cap U_1$ and hence there is a nonempty open set $U_2 \subset U_1$ such that $f(y) > f(x_2) - \varepsilon$ for each $y \in U_2$. Thus for each $y \in U_2$ we have $f(x_2) - \varepsilon < f(y) < f(x_1) + \varepsilon$.

Let $v_1, v_2, \ldots, v_m \in \mathbb{R}$ be such that $(f(x_2) - \varepsilon, f(x_1) + \varepsilon) \subset \bigcup_{i=1}^{m} (v_i - \varepsilon, v_i + \varepsilon)$. Then $U_2 = \bigcup_{i=1}^{m} U_2 \cap f^{-1}((v_i - \varepsilon, v_i + \varepsilon))$ and hence there is $j \in \mathbb{N}$ such that $U_2 \cap f^{-1}((v_j - \varepsilon, v_j + \varepsilon))$ is not nowhere dense in U_2.

Therefore there is a nonempty open set $J \subset U_2$ and $v \in \mathbb{R}$ such that $f^{-1}((v - \varepsilon, v + \varepsilon))$ is dense in J.

Put $A = \{y \in J : |f(y) - v| < \varepsilon\}$, $B = \{y \in J : f(y) \geq v + 3\varepsilon\}$ and $C = \{y \in J : f(y) \leq v - 3\varepsilon\}$. Then A is dense in J and also $B \cup C$ is dense in J. If namely $B \cup C$ is not dense in J, then there is a nonempty open set $P \subset J$ such that $P \cap (B \cup C) = \emptyset$. Then $f(y) \in (v - 3\varepsilon, v + 3\varepsilon)$ for each $y \in P$ and thus $|f(y) - f(z)| < 6\varepsilon$ for each $y, z \in P$, which contradicts to (*).

This yields that B is not nowhere dense in J or C is not nowhere dense in J. Suppose that B is not nowhere dense in J; the case C is not nowhere dense in J is similar. Then there is a nonempty open set $T \subset J$ such that B is dense in T. There is a point $z_0 \in E(f) \cap T$. We have two possibilities:

a) If $f(z_0) \leq 2\varepsilon + v$, then every nonempty open set $U \subset T$ contains a point $z \in B$ and hence $f(z) \geq v + 3\varepsilon \geq f(z_0) + \varepsilon$. This yields $z_0 \notin E(f)$, a contradiction.

b) If $f(z_0) > 2\varepsilon + v$, then every nonempty open set $U \subset T$ contains a point $z \in A$ and hence $f(z) < v + \varepsilon < f(z_0) - \varepsilon$. Again, $z_0 \notin E(f)$, a contradiction.

Therefore the set $E(f) \setminus A(f)$ is nowhere dense. $\square$

Since $A(f)$ is closed, $E(f) = X$ implies $A(f) = X$ (see [4]).

Lemma 1. *([1],[8]) If $f_1 : X \to \mathbb{R}$ is quasicontinuous (cliquish) [upper and lower quasicontinuous] at x and $f_2 : X \to \mathbb{R}$ is continuous at x, then $f_1 + f_2$ is quasicontinuous (cliquish) [upper and lower quasicontinuous] at x.*

Theorem 2. *Let C, Q, E and A be subsets of $\mathbb{R}$. Then $C = C(f)$, $Q = Q(f)$, $E = E(f)$ and $A = A(f)$ for some $f : \mathbb{R} \to \mathbb{R}$ if and only if $C \subset Q \subset A \cap E$, C is G_δ, A is closed, $A \setminus C$ is of first category and $E \setminus A$ is nowhere dense.*

PROOF. The sufficiency for this proof follows from our previous remarks and Theorem 1. To prove the necessity, first note that the set $A \setminus C$ is a F_σ set of first category, hence by [7] we can write $A \setminus C = \bigcup_{n\in\mathbb{N}} D_n$, where the sets D_n are closed nowhere dense and pairwise disjoint. Since every nowhere dense set $S \subset \mathbb{R}$ can be written as $S = S_1 \cup S_2$, where S_1 is a nowhere dense perfect set, S_2 is countable and S_1 and S_2 are disjoint, we can write $A \setminus C = \bigcup_{i\in\mathbb{N}}(A_i \cup B_i)$, where sets A_i are nowhere dense perfect (maybe empty), B_i are singleton (or empty) and all A_i and B_j are mutually disjoint.

If A_i is nonempty nowhere dense perfect we can write $\mathbb{R} \setminus A_i = \bigcup_{j\in\mathbb{N}} I^i_j$, where $I^i_j = (a^i_j, b^i_j)$ are pairwise disjoint intervals. We can assume that $A_i \subset \mathrm{Cl}(\bigcup_{j\in\mathbb{N}} I^i_{2j}) \cap \mathrm{Cl}(\bigcup_{j\in\mathbb{N}} I^i_{2j-1})$.

If $A_i = \emptyset$ put $s_i(x) = 0$ for each $x \in \mathbb{R}$. If $A_i \neq \emptyset$ define $s_i : \mathbb{R} \to \mathbb{R}$ by

$$s_i(x) = \begin{cases} -4^{-i}, & \text{if } x \in I^i_{2j} \text{ for some } j \in \mathbb{N}, \\ 0, & \text{if } x \in A_i \cap (E \setminus Q), \\ 2^{-i}, & \text{if } x \in A_i \cap (A \setminus E), \\ 4^{-i}, & \text{otherwise} \end{cases}$$

and put $s = \sum_{i=1}^{\infty} s_i$.

If $x \notin A_i$, then $x \in C(s_i)$. Since the series $\sum_{i=1}^{\infty} s_i(x)$ converges uniformly we obtain

$$\mathbb{R} \setminus \bigcup_{i\in\mathbb{N}} A_i \subset C(s). \tag{1}$$

Now, let $x \in A_i$. Then $x \notin \bigcup_{j \neq i} A_j$ and hence $x \in C(s_j)$ for each $j \neq i$ and

$$A_i \subset C\Big(\sum_{j \neq i} s_j\Big). \tag{2}$$

Let $x \in A_i \cap (A \setminus E)$. Then $s_i(x) = 2^{-i}$. Let U be an open neighbourhood of x. Then there is $j \in \mathbb{N}$ such that $U \cap I_j^i \neq \emptyset$. The set $G = U \cap I_j^i$ is a nonempty open subset of U and $s_i(y) = s_i(z)$ for each $y, z \in G$, i.e.

$$A_i \cap (A \setminus E) \subset A(s_i). \tag{3}$$

Let H be an arbitrary open nonempty subset of U and let c be such that $4^{-i} < c < 2^{-i}$. Since A_i is nowhere dense there is $z \in H \setminus A_i$ and $s_i(z) \leq 4^{-i} < c < 2^{-i} = s_i(x)$. Therefore s_i is not lower quasicontinuous at x and

$$A_i \cap (A \setminus E) \subset \mathbb{R} \setminus E(s_i). \tag{4}$$

Let $x \in A_i \cap (E \setminus Q)$. Then $s_i(x) = 0$. Let U be an open neighbourhood of x. Then there is $j \in \mathbb{N}$ such that $H = U \cap I_{2j}^i \neq \emptyset$. For each $y, z \in H$ we have $s_i(y) = s_i(z)$ and hence,

$$A_i \cap (E \setminus Q) \subset A_i. \tag{5}$$

Moreover, for each $y \in H$ we have $s_i(y) = -4^{-i} < 0 = s_i(x)$, thus s_i is upper quasicontinuous at x. Further, there is $k \in \mathbb{N}$ such that $U \cap I_{2k-1}^i \neq \emptyset$ and for each $y \in U \cap I_{2k-1}^i$ we have $s_i(y) = 4^{-i} > 0 = s_i(x)$, thus s_i is lower quasicontinuous at x. Therefore we have

$$A_i \cap (E \setminus Q) \subset E(s_i). \tag{6}$$

Now, let G be an arbitrary open set. There is $z \in G \setminus A_i$ and $|s_i(z)| = 4^{-i}$, hence we have $|s_i(z) - s_i(x)| = 4^{-i}$. This yields

$$A_i \cap (E \setminus Q) \subset \mathbb{R} \setminus Q(s_i). \tag{7}$$

Now, let $x \in A_i \cap (Q \setminus C)$. Then $s_i(x) = 4^{-i}$. Let U be an open neighbourhood of x. Then there is $j \in \mathbb{N}$ such that $H = U \cap I_{2j}^i \neq \emptyset$. For each $y \in H$ we have $s_i(y) = 4^{-i} = s_i(x)$ and

$$A_i \cap (Q \setminus C) \subset Q(s_i). \tag{8}$$

Since $\liminf\limits_{y \to x} s_i(y) = -4^{-i}$ and $\limsup\limits_{y \to x} s_i(y) = 4^i$ we have

$$A_i \cap (Q \setminus C) \subset \mathbb{R} \setminus C(s_i). \tag{9}$$

If $B_i = \emptyset$ put $t_i(x) = 0$. If $B_i = \{b_i\}$ define a function $t_i : \mathbb{R} \to \mathbb{R}$ by

$$t_i(x) = \begin{cases} 4^{-i}, & \text{if } x > b_i \text{ or } x = b_i \text{ and } b_i \in Q \setminus C, \\ 0, & \text{if } x = b_i \text{ and } b_i \in E \setminus Q, \\ 2^{-i}, & \text{if } x = b_i \text{ and } b_i \in A \setminus E, \\ -4^{-i}, & \text{if } x < b_i. \end{cases}$$

and put $t = \sum\limits_{i=1}^{\infty} t_i$.

If $x \neq b_i$, then $x \in C(t_i)$ and since the series $\sum\limits_{i=1}^{\infty} t_i$ converges uniformly we obtain

$$\mathbb{R} \setminus \bigcup_{i \in \mathbb{N}} B_i \subset C(t). \tag{10}$$

Now, let $x = b_i$. Since B_j are pairwise disjoint we have $x \in C(t_j)$ for each $j \neq i$ and

$$B_i \subset C\Big(\sum_{j \neq i} t_j\Big). \tag{11}$$

It is easy to see that

$$B_i \cap (A \setminus E) \subset A(t_i) \setminus E(t_i), \tag{12}$$

$$B_i \cap (E \setminus Q) \subset E(t_i) \cap A(t_i) \setminus Q(t_i), \tag{13}$$

$$B_i \cap (Q \setminus C) \subset Q(t_i) \setminus C(t_i). \tag{14}$$

If $A = \mathbb{R}$ we put $u(x) = 0$. Now, let $A \neq \mathbb{R}$. Then $A \cup \mathrm{Cl}(E)$ is a closed set and hence $\mathbb{R} \setminus (A \cup \mathrm{Cl}(E)) = \bigcup_{i \in M} (a_i, b_i)$, where $M \subset \mathbb{N}$ and all intervals (a_i, b_i) are pairwise disjoint. Since $A \neq \mathbb{R}$ and the set $E \setminus A$ is nowhere dense the set M is nonempty.

For each $i \in M$ let c_j^i, d_j^i be such that for each $j \in \mathbb{N}$ $a_i < c_{j+1}^i < d_j^i < c_j^i < c_1^i = b_i$ and $\lim\limits_{j \to \infty} c_j^i = a_i$. Define a function $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x) = \begin{cases} \min\{1, \operatorname{dist}(x, A)\}, & \text{if } x \in (d^i_{2j}, c^i_{2j}) \setminus \mathbb{Q} \\ & \text{for some } i \in M \text{ and } j \in \mathbb{N}, \\ \min\{2, 2\operatorname{dist}(x, A)\}, & \text{if } x \in (d^i_{2j}, c^i_{2j}) \cap \mathbb{Q} \\ & \text{for some } i \in M \text{ and } j \in \mathbb{N}, \\ \min\{3, 3\operatorname{dist}(x, A)\}, & \text{if } x \in ([c^i_{j+1}, d^i_j] \cup (\operatorname{Cl}(E \setminus A) \setminus E)) \cap \mathbb{Q} \\ & \text{for some } i \in M \text{ and } j \in \mathbb{N}, \\ 0, & \text{if } x \in A \cup E, \\ \max\{-1, -\operatorname{dist}(x, A)\}, & \text{if } x \in (d^i_{2j-1}, c^i_{2j-1}) \setminus \mathbb{Q} \\ & \text{for some } i \in M \text{ and } j \in \mathbb{N}, \\ \max\{-2, -2\operatorname{dist}(x, A)\}, & \text{if } x \in (d^i_{2j-1}, c^i_{2j-1}) \cap \mathbb{Q} \\ & \text{for some } i \in M \text{ and } j \in \mathbb{N}, \\ \max\{-3, -3\operatorname{dist}(x, A)\}, & \text{if } x \in ([c^i_{j+1}, d^i_j] \cup (\operatorname{Cl}(E \setminus A) \setminus E)) \setminus \mathbb{Q} \\ & \text{for some } i \in M \text{ and } j \in \mathbb{N}. \end{cases}$$

Let $x \in A$ and let $\varepsilon > 0$. Then $u(x) = 0$. Since $\operatorname{dist}(x, A)$ is continuous there is a neighbourhood U of x such that $|\operatorname{dist}(y, A)| = |\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| < \varepsilon/3$ for each $y \in U$. Hence for each $y \in U$ we have $|u(x) - u(y)| = |u(y)| \le 3\operatorname{dist}(x, A) < \varepsilon$. Therefore we get

$$A \subset C(u). \tag{15}$$

Now, let $x \notin A$. Let $a = \operatorname{dist}(x, A)$ if $A \ne \emptyset$ and $a = 2$ if $A = \emptyset$. Then $U = (x - a/4, x + a/4)$ is a neighbourhood of x. Let $G \subset U$ be an arbitrary nonempty open set and let $b = \min\{1, \frac{1}{8}a\} > 0$.

Let $z \in G$ and $A \ne \emptyset$. Then $|x - z| < \frac{a}{4}$. Let $w \in A$. Then $a = \operatorname{dist}(x, A) \le |x - w| \le |x - z| + |z - w| < \frac{a}{4} + |z - w|$. Therefore for each $w \in A$ we have $|z - w| > \frac{3}{4}a$ and hence $\operatorname{dist}(z, A) \ge \frac{3}{4}a$. On the other hand, there is $v \in A$ such that $|v - x| < \frac{9}{8}a$ and hence $\operatorname{dist}(z, A) \le |v - z| \le |z - x| + |x - v| < \frac{a}{4} + \frac{9}{8}a = \frac{11}{8}a$. Therefore, if $G \subset U$ is a nonempty open set and $A \ne \emptyset$ we have

$$\frac{3}{4}a < \operatorname{dist}(z, A) < \frac{11}{8}a. \tag{16}$$

There are three possibilities:

a) $P = (G \setminus \operatorname{Cl}(E \setminus A)) \cap (d^i_{2j}, c^i_{2j}) \ne \emptyset$ for some $i \in M$ and $j \in \mathbb{N}$. Then there are points $z_1 \in P \cap \mathbb{Q}$ and $z_2 \in P \setminus \mathbb{Q}$. According to

(16) we have $u(z_1) = \min\{2, 2\operatorname{dist}(z_1, A)\} \geq \min\{2, \frac{3}{2}a\}$ and $u(z_2) = \min\{1, \operatorname{dist}(z_2, A) \leq \min\{1, \frac{11}{8}a\}$. We obtain

$$|u(z_1) - u(z_2)| \geq \min\{2, \frac{3}{2}a\} - \min\{1, \frac{11}{8}a\} \geq \min\{1, \frac{1}{8}a\} = b.$$

b) $P = (G \setminus \mathrm{Cl}(E \setminus A)) \cap (d^i_{2j-1}, c^i_{2j-1}) \neq \emptyset$ for some $i \in M$ and $j \in \mathbb{N}$. Then for $z_1 \in P \cap \mathbb{Q}$ and $z_2 \in P \setminus \mathbb{Q}$ we have

$$|u(z_1) - u(z_2)| \geq \max\{-1, -\frac{11}{8}a\} - \max\{-2, -\frac{3}{2}a\} \geq \min\{1, \frac{1}{8}a\} = b.$$

c) $(G \setminus \mathrm{Cl}(E \setminus A)) \cap (\bigcup_{i\in M} \bigcup_{j\in\mathbb{N}} (d^i_j, c^i_j)) = \emptyset$. Since $E \setminus A$ is nowhere dense there are $z_1 \in (G \setminus \mathrm{Cl}(E \setminus A)) \cap \mathbb{Q}$ and $z_2 \in (G \setminus \mathrm{Cl}(E \setminus A)) \setminus \mathbb{Q}$. We have $u(z_1) = \min\{3, 3\operatorname{dist}(z_1, A)\} \geq \min\{3, \frac{9}{4}a\}$ and $u(z_2) = \max\{-3, -3\operatorname{dist}(z_2, A)\} \leq -\min\{3, \frac{9}{4}a\}$ and hence

$$|u(z_1) - u(z_2)| \geq 2\min\{3, \frac{9}{4}a\} > b.$$

Therefore u is not cliquish in x and

$$\mathbb{R} \setminus A \subset \mathbb{R} \setminus A(u). \tag{17}$$

Now, let $x \in E \setminus A$ and let $U = (x - \delta, x + \delta)$, $\delta > 0$, be a neighbourhood of x. Then $u(x) = 0$ and there is $0 < \delta_1 < \delta$ such that $(x - \delta_1, x + \delta_1) \cap A = \emptyset$. Since $E \setminus A$ is nowhere dense there is an interval $(c, d) \subset (x, x + \delta_1)$ such that $(c, d) \cap \mathrm{Cl}(E \setminus A) = \emptyset$. This yields $(c, d) \subset \bigcup_{i\in M}(a_i, b_i)$ and since (a_i, b_i) are disjoint there is $i \in M$ such that $(c, d) \subset (a_i, b_i)$. Since $x \notin (a_i, b_i)$ we have $x \leq a_i \leq c < x + \delta_1$. Since $\lim\limits_{j\to\infty} c^i_j = a_i$ there is $j \in \mathbb{N}$ such that $a_i < c^i_j < x + \delta_1$.

For each $y \in (d^i_{2j}, c^i_{2j}) \subset U$ we have $y(y) > 0 = u(x)$, i.e. u is lower quasicontinuous at x. Similarly, for each $y \in (d^i_{2j-1}, c^i_{2j-1}) \subset U$ we have $u(y) < 0 = u(x)$, i.e. u is upper quasicontinuous at x. Thus

$$E \setminus A \subset E(u). \tag{18}$$

Finally, let $x \notin (E \cup A)$. Let $a = \operatorname{dist}(x, A)$ if $A \neq \emptyset$ and $a = 3$ if $A = \emptyset$. We have three possibilities:

a) $x \in (d^i_{2j}, c^i_{2j})$ for some $i \in M$ and $j \in \mathbb{N}$. Then $U = (x - a/4, x + a/4) \cap (d^i_{2j}, c^i_{2j})$ is a neighbourhood of x. Let G be an open nonempty open subset of U.

   If $x \in \mathbb{Q}$, then $u(x) = \min\{2, 2\,\mathrm{dist}(x, A)\}$ and $b = \min\{1, \frac{11}{8}a\} < u(x)$. There is a point $z \in G \setminus \mathbb{Q}$ and according to (16) we have $\frac{3}{4}a < \mathrm{dist}(z, A) < \frac{11}{8}a$ for $A \neq \emptyset$. Therefore $u(z) = \min\{1, \mathrm{dist}(z, A)\} \leq \min\{1, \frac{11}{8}a\} = b < u(x)$, i.e. u is not lower quasicontinuous at x.

   If $x \notin \mathbb{Q}$, then $u(x) = \min\{1, \mathrm{dist}(x, A)\}$ and $b = \{2, \frac{3}{2}a\} > u(x)$. There is a point $z \in G \cap \mathbb{Q}$ and $u(z) = \min\{2, 2\,\mathrm{dist}(x, A)\} \geq \min\{2, \frac{3}{2}a\} = b > u(x)$, i.e. u is not upper quasicontinuous at x.

b) $x \in (d^i_{2j-1}, c^i_{2j-1})$ for some $i \in M$ and $j \in \mathbb{N}$. Then similarly as in a) we can show that $x \notin E(u)$.

c) $x \notin \bigcup_{i \in M} \bigcup_{j \in \mathbb{N}} (d^i_j, c^i_j)$. Let G be an open nonempty subset of $(x - a/4, x + a/4)$.

   If $x \in \mathbb{Q}$, then $u(x) = \min\{3, 3\,\mathrm{dist}(x, A)\}$. For each $y \in G \setminus \mathbb{Q}$ we have $u(y) \leq \min\{2, 2\,\mathrm{dist}(y, A)\} \leq \min\{2, \frac{11}{4}a\} = b < u(x)$ and u is not lower quasicontinuous at x.

   If $x \notin \mathbb{Q}$, then $u(x) = \max\{-3, -3\,\mathrm{dist}(x, A)\}$ and for each $y \in G \cap \mathbb{Q}$ we have $u(y) \leq \max\{-2, -2\,\mathrm{dist}(y, A)\} \geq \max\{-2, -\frac{11}{4}a\} > u(x)$, i.e. u is not upper quasicontinuous at x.

Therefore we have

$$\mathbb{R} \setminus (E \cup A) \subset \mathbb{R} \setminus E(u). \tag{19}$$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f = s + t + u$. We will show that f is the desired function.

1. Let $x \in C$. Then according to (1), (10) and (15) we have $x \in C(s) \cap C(t) \cap C(u)$ and hence $x \in C(f)$,

$$C \subset C(f). \tag{20}$$

2. Let $x \in Q \setminus C$. If $x \in A_i$ for some $i \in M$, then according to (10), (15) and (2) we obtain $x \in C(t) \cap C(u) \cap C(\sum_{j \neq i} s_j)$, according to (8) we have $x \in Q(s_i)$ and by (9) $x \notin C(s_i)$. Therefore by Lemma 1 we have $x \in Q(f) \setminus C(f)$.

If $x = b_i$ for some $i \in M$, then by (1), (15) and (11) we have $x \in C(s) \cap C(u) \cap C(\sum_{j \neq i} t_j)$ and by (14) $x \in Q(t_i) \setminus C(t_i)$. Hence, by Lemma 1 again $x \in Q(f) \setminus C(f)$ and

$$Q \setminus C \subset Q(f) \setminus C(f). \tag{21}$$

3. Let $x \in A \cap E \setminus Q$. If $x \in A_i$, then by (10), (15) and (2) we have $x \in C(t) \cap C(u) \cap C(\sum_{j \neq i} s_j)$. Further, $x \in A(s_i)$ by (5), $x \in E(s_i)$ by (6) and $x \notin Q(s_i)$ by (7), therefore $x \in A(f) \cap E(f) \setminus Q(f)$.

   If $x = b_i$, then (1), (15) and (11) imply $x \in C(s) \cap C(u) \cap C(\sum_{j \neq i} t_j)$ and (13) yields $x \in A(t_i) \cap E(t_i) \setminus Q(t_i)$. Hence $x \in A(f) \cap E(f) \setminus Q(f)$. Therefore

$$A \cap E \setminus Q \subset A(f) \cap E(f) \setminus Q(f). \tag{22}$$

4. Let $x \in A \setminus E$. If $x \in A_i$, then by (10), (15) and (2) we have $x \in C(t) \cap C(u) \cap C(\sum_{j \neq i} s_j)$, by (3) $x \in A(s_i)$ and by (4) $x \notin E(s_i)$, therefore $x \in A(f) \setminus E(f)$.

   If $x = b_i$, then by (1), (15) and (11) we have $x \in C(s) \cap C(u) \cap C(\sum_{j \neq i} t_j)$ and by (12) $x \in A(t_i) \setminus E(t_i)$ and hence $x \in A(f) \setminus E(f)$ and

$$A \setminus E \subset A(f) \setminus E(f). \tag{23}$$

5. Let $x \in E \setminus A$. Then according to (1) and (10) we have $x \in C(s) \cap C(t)$, by (18) we have $x \in E(u)$ and by (16) $x \notin A(u)$. Lemma 1 implies $x \in E(f) \setminus A(f)$ and

$$E \setminus A \subset E(f) \setminus A(f). \tag{24}$$

6. Let $x \in R \setminus (A \cup E)$. Then (1) and (10) imply $x \in C(s) \cap C(t)$, (19) yields $x \notin E(u)$ and (17) implies $x \notin A(u)$. From Lemma 1 we deduce

$$\mathbb{R} \setminus (A \cup E) \subset \mathbb{R} \setminus (A(f) \cup E(f)). \tag{25}$$

Finally, from (20), (21), (22), (23), (24) and (25) we conclude that $C = C(f)$, $Q = Q(f)$, $E = E(f)$ and $A = A(f)$.

$\square$

Remark 1. *Theorem 2 is not true for functions $f : \mathbb{R}^2 \to \mathbb{R}$. If $C = Q = \mathbb{R}^2 \setminus \{(0,0)\}$ and $A = E = \mathbb{R}^2$, then all the assumptions of Theorem 1 are satisfied, however there is no function $f : \mathbb{R}^2 \to \mathbb{R}$ with $C = C(f)$, $Q = Q(f)$, $E = E(f)$ and $A = A(f)$.*

PROOF. Assume that there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $C = C(f), \mathbb{R}^2 = A(f) = E(f)$. We will show that under this assumption, $(0,0) \in Q(f)$, which is a contradiction.

Let U be a neighbourhood of $(0,0)$ and let $\varepsilon > 0$. Let $\delta > 0$ be such that $T = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < \delta\} \subset U$ and let $W = T \setminus \{(0,0)\}$ and $a = f(0,0)$. Since $(0,0) \in E(f)$ there are nonempty open sets $G_1, G_2 \subset T$ such that $f(G_1) \subset (a - \varepsilon/2, \infty)$ and $f(G_2) \subset (-\infty, a + \varepsilon/2)$. Therefore there are $(y_1, z_1), (y_2, z_2) \in W$ such that $f(y_1, z_1) > a - \varepsilon/2$ and $f(y_2, z_2) < a + \varepsilon/2$.

If $f(y_1, z_1) \le a$, then $|f(y_1, z_1) - a| < \varepsilon/2$ and there is an open neighbourhood $G \subset W \subset U$ of (y_1, z_1) such that $|f(y,z) - f(y_1, z_1)| < \varepsilon/2$ for each $(y,z) \in G$. Therefore for each $(y,z) \in G$ we obtain $|f(y,z) - f(0,0)| \le |f(y,z) - f(y_1,z_1)| + |f(y_1,z_1) - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, i.e $(0,0) \in Q(f)$. If $f(y_2, z_2) \ge a$, then similarly we can show $(0,0) \in Q(f)$.

Finally, let $f(y_2, z_2) < a < f(y_1, z_1)$. The set W is connected and the function $f \restriction W$ is continuous, hence the set $f \restriction W(W) = f(W)$ is connected. Since $f(y_2, z_2), f(y_1, z_1) \in f(W)$ there is $(y_3, z_3) \in W$ such that $f(y_3, z_3) = a$. Since $(y_3, z_3) \in C(f)$ there is an open neighbourhood $G \subset U$ of (y_3, z_3) such that $|f(y_3, z_3) - f(y,z)| < \varepsilon$ for each $(y,z) \in G$. Therefore for each $(y,z) \in G$ we have $|f(y,z) - f(0,0)| \le |f(y,z) - f(y_3,z_3)| + |f(y_3,z_3) - a| < \varepsilon$ and $(0,0) \in Q(f)$. □

Problem 1. *Characterize the quadruplet $(C(f), Q(f), E(f), A(f))$ for real functions defined on a metric space, or at least for $\mathbb{R}^2$.*

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