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A DARBOUX, BAIRE ONE FINED POINT PROBLEM

Abstract

K. Ciesielski asked whether the composition of two derivatives from the unit interval to itself always has a fixed point. The question is equivalent to asking if the composition of two Darboux, Baire one maps of $[0, 1]$ to $[0, 1]$ has a fixed point. The question is answered affirmatively for three subclasses of the Darboux, Baire one maps of $[0, 1]$ to $[0, 1]$

1 Introduction

At the Mini Conference in Analysis held at Auburn University in March, 1999 and in [5] K. Ciesielski asked if the composition of two derivatives from $I = [0, 1]$ to I has a fixed point. Using a result due to Maximoff (See [7].) the question is equivalent to asking if the composition of two Darboux, Baire class one functions, $\mathcal{DB}_1$, from I to I has a fixed point. The question is shown to have a positive solution for three subclasses of $\mathcal{DB}_1$. The first of these classes contains the so-called Croft function and the proof is quite constructive. The second class contains the Darboux, Baire* one functions, $(\mathcal{DB}_1^*)$. (The class $\mathcal{B}_1^*$ was given that name by R. J. O'Malley in [6] but is also called the class of generalized continuous functions.) The third class involves a new subclass of $\mathcal{B}_1$ which is introduced here. It is denoted by $\mathcal{B}_1^\omega$. It is shown that $\mathcal{B}_1^*$ and $\mathcal{B}_1^\omega$ are distinct classes. The authors hope this class will be useful in other settings.

A complete solution to Ciesielski's question by Marianna Csörnyei, Toby C. O'Neil and David Preiss appears in the Research Articles of this issue of the

Real Analysis Exchange. A second solution by Márton Elekes, Tamás Keleti and Vilmos Prokaj will appear in Vol. 27, No. 1. None the less the reader may find interesting the solutions to the special cases presented here.

2 Preliminaries

Throughout I denotes $[0, 1]$. Let $f : I \to I$. Recall that f is upper semicontinuous at x if $f(x) \geq \limsup_{y \to x} f(y)$ and is upper semicontinuous on I if it is upper semicontinuous at each point in I. Let $C_f = \{x \in I; f \text{ is continuous at } x\}$.

Definition 2.1. Let $f : I \to I$. Then f is a Croft-type function means f is upper semi-continuous on I and $f(C_f)$ is a singleton.

Definition 2.2. Let

$$\mathcal{B}_1^* = \{f : I \to I; \text{ for each nonempty perfect set } P \subset I \text{ there is a portion}$$
$$Q \subset P \text{ such that } f \upharpoonright P \text{ is continuous at each point of } Q\}.$$

It is well known that a function $f \in \mathcal{B}_1^*$ if and only if for each $n \in \mathbb{N}$ there is a closed set E_n and a continuous function $f_n : E_n \to I$ with $E_n \subset E_{n+1}$ such that $\cup_{n \in \mathbb{N}} E_n = I$ and $f(x) = f_n(x)$ for each $n \in \mathbb{N}$ and for each $x \in I$.

Definition 2.3. Let

$$\mathcal{B}_1^\omega = \{f : I \to I; \text{ for each nonempty perfect set } P \text{ there is } x \in P \text{ such that}$$
$$f \text{ is continuous at } x\}.$$

A function f is in Darboux, Baire$^\omega$ class one, $\mathcal{DB}_1^\omega$, if and only if f has the Darboux property and at most countably many discontinuities.

We use the following observation of Ciesielski's. Define a transformation, T, of the unit square, $I \times I$, by $TA = T(A) = \{(y, x); (x, y) \in A\}$, where A is an arbitrary subset of the unit square. Let f and g be functions. Then there exists $x \in I$ such that $g(f(x)) = x$ if and only if the sets $\{(x, f(x)); x \in I\}$ and $\{(g(y), y); y \in I\}$ have non-empty intersection; that is, $f \cap Tg \neq \emptyset$.

3 Croft-Type Functions

Theorem 3.1. *The composition of any two Croft functions has a fixed point.*

Remark 3.2. Note that the composition of two Croft functions need not be of Baire class one but must be in Darboux Baire class two and such functions do not necessarily have a fixed point [1, page 6]. For example, the composition

of the Croft function [2, Example 2.2, page 11] with a shifted version itself, when the shifted version takes the value $\alpha > 0$ at the origin, is in Baire class two but not in Baire class one. This follows from the observation that on every open interval the composition takes every value between zero and α.

PROOF OF THEOREM 3.1. Let $f, g : I \to I$ be Croft functions; and, without loss of generality, suppose that $f(C_f) = g(C_g) = \{0\}$. Let f_n and g_n be non-increasing sequences of continuous functions from I to I that converge to f and g respectively. We proceed step-wise.

Step 1. Let $(x_1, y_1) \in \overline{f} \cap \overline{Tg}$. We consider the case where $y_1 < f(x_1)$ and $x_1 < g(y_1)$ first; later, we show that we can always reduce to this case. Observe that the points (x_1, y_1), $(x_1, f(x_1))$, $(g(y_1), f(x_1))$, and $(g(y_1), y_1)$ form a non-degenerate rectangle. Call this rectangle R_1. For convenience let R_0 denote the unit square, $I \times I$.

Assume that non-degenerate rectangles $R_1 \supset \cdots \supset R_n$ have been chosen, where the corners of R_k are (x_k, y_k), $(x_k, f(x_k))$, $(g(y_k), f(x_k))$, and $(g(y_k), y_k)$, $k = 1, \ldots, n$. Moreover, assume the length of the sides of R_k are less than one half the length of the sides of R_{k-1} and, when $n \geq 2$, that $f \restriction [x_k, g(y_k)]$ is bounded above by the upper side of R_{k-2} and $g \restriction [y_k, f(x_k)]$ is bounded on the right by the right side of R_{k-2}, $k = 2, \ldots, n$.

Step n, $n \geq 2$. Choose $\epsilon_n \in (0, (1/2) \min\{f(x_n) - y_n, g(y_n) - x_n\})$ so small that $f \restriction [x_n, x_n + \epsilon_n]$ is bounded above by $f(x_{n-1})$ and that $g \restriction [y_n, y_n + \epsilon_n]$ is bounded on the right by $g(y_{n-1})$. (Such a choice is possible since f and g are upper semicontinuous.) Hence, $f \restriction [x_n, x_n + \epsilon_n]$ remains below the upper side of R_{n-1} and $g \restriction [y_n, y_n + \epsilon_n]$ remains to the left of the right side of R_{n-1}. Choose $x'_n \in (x_n, x_n + \epsilon_n)$ and $y'_n \in (y_n, y_n + \epsilon_n)$ such that $f(x'_n) = g(y'_n) = 0$. Since f and g are Darboux, there exist points $x_{n+1} \in (x_n, x'_n)$ and $y_{n+1} \in (y_n, y'_n)$ such that $f(x_{n+1}) = y'_n$ and $g(y_{n+1}) = x'_n$. Observe that the points (x_{n+1}, y_{n+1}), $(x_{n+1}, f(x_{n+1}))$, $(g(y_{n+1}), f(x_{n+1}))$, and $(g(y_{n+1}), y_{n+1})$ form a non-degenerate rectangle R_{n+1} properly contained in R_n, with side length less than one half of that of R_n, and such that $f \restriction [x_{n+1}, g(y_{n+1})]$ is bounded above by the upper side of R_{n-1}, and $g \restriction [y_{n+1}, f(x_{n+1})]$ is bounded on the right by the right side of R_{n-1}.

By induction we have a nested sequence of closed rectangles, $\{R_n\}_{n=1}^{\infty}$, with side lengths tending to zero. Let $\{(x, y)\} = \bigcap_{n=1}^{\infty} R_n$. Observe that $(x, y) \in \overline{f} \cap \overline{Tg}$, which follows from the fact that the upper left and lower right corners of each rectangle are elements of f and Tg respectively.

We claim $y = f(x)$ and $x = g(y)$. Proceeding toward a contradiction, suppose that $y \neq f(x)$. Then $y < f(x)$ since $(x, y) \in \overline{f}$. This implies that at some step, say k, $f(x)$ is above the top side of R_{k-2}; a contradiction. The argument for $x = g(y)$ is similar.

Finally, we consider the other possibilities for (x_1, y_1).

Case $y_1 = f(x_1)$ and $x_1 = g(y_1)$. We are done, since $y_1 = f(g(y_1))$.

Case $y_1 < f(x_1)$ and $0 < x_1 = g(y_1)$. Select $\alpha < x_1$ such that $f(\alpha) = 0$. Then there exists $\alpha < \xi_1 < x_1$ such that $f(\xi_1) = (f(x_1) + y_1)/2$. Then we have $y_1 < f(\xi_1)$ and $\xi_1 < g(y_1)$ which is the first case considered.

Case $y_1 < f(x_1)$ and $0 = x_1 = g(y_1)$. Without loss of generality assume $g(f(x_1)) \neq 0$. Since g is Darboux, there exists a $f(x_1) > \eta_1 > y_1$ such that $g(\eta_1) = g(f(x_1))/2$. Then, we have that $\eta_1 < f(x_1)$ and that $x_1 < g(\eta_1)$, which is the first case considered.

The other two cases are similar.   □

4   The Class $\mathcal{DB}_1^*$

Let P be some property of the class of Darboux, Baire one functions. Define the maximal outer P-compositional class of $\mathcal{DB}_1$ as

$$\mathcal{M}_o^P = \{f \in \mathcal{DB}_1; f \circ g \text{ has property } P \text{ for all } g \in \mathcal{DB}_1\}$$

and the maximal inner P-compositional class of $\mathcal{DB}_1$ as

$$\mathcal{M}_i^P = \{f \in \mathcal{DB}_1; g \circ f \text{ has property } P \text{ for all } g \in \mathcal{DB}_1\}.$$

First we study the case where P is the property of being Darboux, Baire one, and then the case where P is the property of having a fixed point.

The following basic result seems to be new (as far as the authors know) and is of interest since it determines a class of dynamical systems larger than the continuous functions.

Theorem 4.1. *The composition of Darboux, Baire* one functions is again a Darboux, Baire* one function.*

Proof. Let $f, g \in \mathcal{DB}_1^*$ and $P \subset I$ be perfect. By definition there is an interval $(a, b) \subset I$ such that $(a, b) \cap P \neq \emptyset$ and $f \restriction ((a, b) \cap P)$ is continuous. By considering a slightly smaller interval, we may assume that $[a, b] \cap P = P'$ is perfect and $f \restriction P'$ is continuous. Then $f(P')$ is closed as the continuous image of a compact set.

If there is an isolated point $c \in f(P')$, then $(f \restriction P')^{-1}(c)$ is open in P'. Hence, there is an interval $(a', b') \subset (a, b)$ such that $g \circ f$ is constant on $(a', b') \cap P$.

Otherwise $f(P')$ is perfect and there is an interval $(c, d) \subset I$ such that $(c, d) \cap f(P') \neq \emptyset$ and $g \restriction ((c, d) \cap f(P'))$ is continuous. Then $g \circ f$ is continuous on the relatively open set $(f \restriction P')^{-1}((c, d))$ and hence on some non-empty set of the form $(a', b') \cap P$. Since P was arbitrary, $g \circ f$ is Darboux, Baire* one.

Theorem 4.2. *Let $f, g : I \to I$ be Darboux, Baire* one and Darboux, Baire one respectively. Then $g \circ f$ is Darboux, Baire one.*

PROOF. Let $P \subset I$ be perfect. By definition there is an interval $(a, b) \subset I$ such that $(a, b) \cap P \neq \emptyset$ and $f \upharpoonright ((a, b) \cap P)$ is continuous. Assume, as we may, that $[a, b] \cap P = P'$ is perfect and $f \upharpoonright P'$ is continuous. Then $f(P')$ is closed as the continuous image of a compact set.

If there is an isolated point in $f(P')$, then the argument is the same as in Theorem 4.1. Otherwise $f(P')$ is perfect and $g \upharpoonright f(P')$ is continuous at some point $c \in f(P')$. Then $(g \circ f)$ is continuous at any point in $(f \upharpoonright P')^{-1}(c)$. Since P was arbitrary, $g \circ f$ is Darboux, Baire one.

When P is the property of being Darboux, Baire one, Theorem 4.2 shows that $\mathcal{DB}_1^* \subset \mathcal{M}_i^P$. On the other hand, we have the following for $\mathcal{M}_o^P$.

Theorem 4.3. *If P is the property of being Darboux, Baire one, then the maximal outer P-compositional class of $\mathcal{DB}_1$ is $\mathcal{C}$, the class of continuous functions.*

PROOF. Bruckner [1, Theorem 3.5] has shown that $C \subset \mathcal{M}_o^P$. It remains to show that if $f \in \mathcal{DB}_1 \setminus \mathcal{C}$, then there exists $g \in \mathcal{DB}_1$ such that $f \circ g \notin \mathcal{DB}_1$.

Let $f \in \mathcal{DB}_1 \setminus \mathcal{C}$ and suppose that f is discontinuous on the right at $x_0 \in [0, 1)$. (The alternative is similar.) Let

$$[a, b] = \left[\liminf_{x \to x_0^+} f(x), \limsup_{x \to x_0^+} f(x) \right].$$

Put $g(x) = x_0 + (1 - x_0)C(x)$, for $x \in I$ and where $C(x)$ is Croft's function [2, Example 2.2, page 11]. Since $C : I \to I$ is in $\mathcal{DB}_1$, g has these properties as well. (Shortly, we will use the additional facts that $C(C_C) = \{0\}$ and that C is non-zero on a dense set.)

We will be done when we show that $f \circ g$ takes every value in (a, b) in every interval. Hence, cannot be in $\mathcal{DB}_1$. To see this, let $U \subset I$ be a non-degenerate interval and observe that both $\{x \in I; C(x) = 0\}$ and $\{x \in I | C(x) > 0\}$ are dense in U. Since C is Darboux, it takes all small enough values in U. Therefore g takes all values in a small enough right neighborhood of x_0. Thus $f \circ g$ takes every value in (a, b) in U.

When P is the property of having a fixed point, then $\mathcal{M}_o^P = \mathcal{M}_i^P \overset{\text{def}}{=} \mathcal{M}^P$ since if $f \circ g$ has a fixed point, then $g \circ f$ does as well. Since each $f \in \mathcal{DB}_1$, has a fixed point, we obtain the following corollary which shows that $\mathcal{DB}_1^* \subset \mathcal{M}^P$.

Corollary 4.4. *Let $f \in \mathcal{DB}_1$ and $g \in \mathcal{DB}_1^*$. Then $f \circ g$ has a fixed point.*

5   The Class $\mathcal{DB}_1^\omega$

The following two examples show that the classes Darboux, Baire* one and Darboux, Baire$^\omega$ one are distinct.

Example 5.1. There exists $f \in \mathcal{DB}_1^* \setminus \mathcal{DB}_1^\omega$.

CONSTRUCTION. Let C be the middle thirds Cantor set and $\{(a_n, b_n)\}_{n=1}^\infty$ be an enumeration of the components of $I \setminus C$. Define $f : I \to I$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in C, \\ 1 - \left| \dfrac{2x - a_n - b_n}{a_n - b_n} \right| & \text{if } x \in (a_n, b_n). \end{cases}$$

Then $f \in \mathcal{DB}_1^*$ since $C_f = I \setminus C$ and $f \upharpoonright C \equiv 0$. Moreover $f \notin \mathcal{DB}_1^\omega$ since $D_f = C$ is uncountable. $\square$

Example 5.2. There exists $f \in \mathcal{DB}_1^\omega \setminus \mathcal{DB}_1^*$.

CONSTRUCTION. Let C be the Cantor set. For each $n \in \mathbb{N}$ let $(a_n, b_n) = \left(\dfrac{3^n - 2}{3^n}, \dfrac{3^n - 1}{3^n} \right)$. Define the auxiliary function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ 1 - \left| \dfrac{2x - a_n - b_n}{a_n - b_n} \right| & \text{if } x \in (a_n, b_n), \\ 0 & \text{otherwise.} \end{cases}$$

Define $f_1 : I \to I$ by $f_1(x) = \phi(3x)$ for $x \in [0, \frac{1}{2}]$ and by reflection across the point $\frac{1}{2}$ for $x \in [\frac{1}{2}, 1]$; namely, such that $f_1(x) = f_1(1 - x)$ for $x \in I$.
   Define $f_n : I \to I$, $n \geq 2$, by

$$f_n(x) = \begin{cases} f_{n-1}(3x) & \text{if } x \in [0, \frac{1}{3}], \\ 0 & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \\ f_{n-1}(3x - 2) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

Define $f : I \to I$ by $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x)$. Since $\mathcal{DB}_1$ is closed under uniform convergence, $f \in \mathcal{DB}_1$. Observe that f is zero at each bilateral limit point of C and non-zero at each unilateral limit point of C. Thus $f \upharpoonright C$ has no portion of continuity. Hence, f is not Darboux, Baire* one.
   Note that f is continuous on each component of $I \setminus C$ since it is a uniform sum of continuous function (actually a finite sum). Moreover, f is continuous

at each bilateral limit point, c, of C since for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\left|\sum_{n=N+1}^{\infty} 2^{-n} f_n\right| < \frac{\epsilon}{2}$. Then we may choose $\delta > 0$ such that $\left|\sum_{n=1}^{N} 2^{-n} f_n(x)\right| < \frac{\epsilon}{2}$ for $x \in (c - \delta, c + \delta)$. (Observe that $f(c) = 0$ for each such bilateral limit point of C.) Thus, the points of discontinuity of f are contained in the countable set of unilateral limit points of C, which shows that $f \in \mathcal{DB}_1^\omega$. $\qquad\square$

Example 5.3. There exists $f \in \mathcal{DB}_1^\omega \setminus \mathcal{M}_i^P$ when P is the property of being Darboux, Baire one.

CONSTRUCTION. Let C be the Cantor set and note that the lengths of the components of $I \setminus C$ are 3^{-n} for $n \in \mathbb{N}$. Letting f be from Example 5.2 we observe that the value of f at either end point of a component of length 3^{-n} is 2^{-n}.

Define $g : I \to I$ by

$$g(x) = \begin{cases} 1 - 2^{n+4}|x - 2^{-n}| & \text{if } |x - 2^{-n}| < 2^{-n-4}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in \mathcal{DB}_1$ and $g \circ f$ is not in $\mathcal{DB}_1$ since $g \circ f$ is zero for each bilateral limit point of C and one for each unilateral limit point of C which implies $(g \circ f) \upharpoonright C$ has no point of continuity. Thus $g \circ f \notin \mathcal{DB}_1$ and hence $f \notin \mathcal{M}_i^P$. $\qquad\square$

Remark 5.4. The previous example also shows that $\mathcal{DB}_1^\omega \circ \mathcal{DB}_1^\omega \neq \mathcal{DB}_1^\omega$ since the function g is continuous everywhere except at the origin and hence is in $\mathcal{DB}_1^\omega$.

Theorem 5.5. *Let $f, g : I \to I$ be Darboux, Baire$^\omega$ one functions. Then $g \circ f$ is has a fixed point.*

Our proof of this special case is long and technically difficult and hence is omitted.

6 Conclusion

Although the main problem addressed here was solved in [3] and [4], one problem which surfaced in our work remains unsolved.

Open Problem 6.1. Is the composition of any two Darboux, Baire one functions representable as the composition of a Darboux, Baire* one function on the outside and a Darboux, Baire one function on the inside.

An affirmative solution to this problem, when coupled with Corollary 4.4 would provide an alternative solution to Ciesielski's original question.

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