Real Analysis Exchange
Vol. (), , pp. 795–810

S. Graf, Fakultät für Matematik und Informatik, Universität Passau, 94030 Passau, Germany. e-mail: `graf@fmi.uni-passau.de`

H. Luschgy, Fachbereich Mathematik IV, Universität Trier, 54286 Trier, Germany. e-mail: `luschgy@uni-trier.de`

ASYMPTOTICS OF THE QUANTIZATION ERRORS FOR SELF-SIMILAR PROBABILITIES

Abstract

The formulae for determining the quantization dimensions of self–similar probabilities satisfying the open set condition are proved by a new method. In addition, this method gives the exact order of convergence for the quantization errors.

1 Introduction

Given a Borel probability P on $\mathbb{R}^d$, a number $r \in [0, +\infty]$ and a natural number $n \in \mathbb{N}$ the n–th *quantization error* of order r for P is defined by

$$e_{n,r} = \begin{cases} \inf\{\exp \int \log d(x,\alpha) dP(x) | \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\} & \text{if } r = 0 \\ \inf\{\big(\int d(x,\alpha)^r dP(x)\big)^{1/r} | \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\} & \text{if } 0 < r < \infty \\ \inf\{\sup_{x \in \mathrm{supp}(P)} d(x,\alpha) | \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\} & \text{if } r = \infty \end{cases}$$

where $d(x, \alpha)$ denotes the distance of the point x to the set α with respect to a given norm $\| \ \|$ on $\mathbb{R}^d$. (One has to impose certain conditions on P to guarantee that the integrals and the supremum in the above expressions exist in $\mathbb{R}$.) The *quantization dimension* of order r for P is

$$D_r(P) = \lim_{n \to \infty} \frac{\log n}{-\log e_{n,r}}$$

if this limit exists. For self–similar probabilities P satisfying the open set condition and $0 < r < \infty$ it was shown in [6] that the quantization dimension

Key Words: self-similar probability, quantization
Mathematical Reviews subject classification: 28A80 (primary) and 60E99 (secondary)
Received by the editors October 27, 2000

$D_r(P)$ exists in $(0, +\infty)$ and a formula for its computation was derived. In the present note we give a new proof of these results and extend them to the cases $r = 0$ and $r = +\infty$. Moreover we will show that, for all $r \in [0, +\infty]$, $0 < \liminf_{n\to\infty} n e_{n,r}^{D_r} \leq \limsup_{n\to\infty} n e_n^{D_r} < +\infty$.

2 Basic Notation and Definitions

In what follows N is always a natural number ≥ 2 and $S_1, \ldots, S_N$ are contractive similitudes from $\mathbb{R}^d$ into itself. Let s_i be the contraction number of S_i; i.e., $s_i \in (0,1)$ and $\|S_i x - S_i y\| = s_i \|x - y\|$ for all $x, y \in \mathbb{R}^d$. Sometimes the N–tuple $(S_1, \ldots, S_N)$ is called an *iterated function system* (IFS). Its *attractor* is the unique non–empty compact set A in $\mathbb{R}^d$ with

$$A = S_1(A) \cup \ldots \cup S_N(A).$$

For every probability vector $p = (p_1, \ldots, p_N)$ there exists a unique Borel probability P on $\mathbb{R}^d$ which satisfies the equation $P = \sum_{i=1}^{N} p_i P \circ S_i^{-1}$. P is called the *self–similar probability* corresponding to $(S_1, \ldots, S_N; p)$. If each component p_i of p is strictly positive, then the support of P equals A.

The IFS $(S_1, \ldots, S_N)$ is said to satisfy the *open set condition* (OSC) iff there is a non-empty open set U in $\mathbb{R}^d$ with $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all i, j with $i \neq j$. According to a result of Schief [7] U can be chosen to be bounded and such that $U \cap A \neq \emptyset$.
Let $\{1, \ldots, N\}^*$ be the set of finite words over the alphabet $\{1, \ldots, N\}$ including the empty word $\emptyset$. For $\sigma \in \{1, \ldots, N\}^*$ the length of σ is denoted by $|\sigma|$. For $n \in \mathbb{N}$, $\{1, \ldots, N\}^n$ is the set of all words of length n. A word $\sigma = \sigma_1 \ldots \sigma_n$ is said to be a *predecessor* of a word $\tau = \tau_1 \ldots \tau_m$, in symbols $\sigma \prec \tau$, iff $n \leq m$ an $\sigma_i = \tau_i$ for $i = 1, \ldots, n$. The empty word is the predecessor of every word. Words σ and τ are called incomparable if neither $\sigma \prec \tau$ nor $\tau \prec \sigma$. For $\sigma \in \{1, \ldots N\}^*$ set

$$S_\sigma = \begin{cases} id_{\mathbb{R}^d} & \text{if } \sigma = \emptyset \\ S_{\sigma_1} \circ \ldots \circ S_{\sigma_n} & \text{if } \sigma = \sigma_1 \ldots \sigma_n, \end{cases}$$

$$A_\sigma = S_\sigma(A),$$

$$s_\sigma = \begin{cases} 1 & \text{if } \sigma = \emptyset \\ s_{\sigma_1} \cdot \ldots \cdot s_{\sigma_n} & \text{if } \sigma = \sigma_1 \ldots \sigma_n, \end{cases}$$

and

$$p_\sigma = \begin{cases} 1 & \text{if } \sigma = \emptyset \\ p_{\sigma_1} \cdot \ldots \cdot p_{\sigma_n} & \text{if } \sigma = \sigma_1 \ldots \sigma_n. \end{cases}$$

If $(S_1, \ldots, S_N)$ satisfies the OSC, then $P(A_\sigma \cap A_\tau) = 0$ if σ and τ are incomparable and, moreover, $P(A_\sigma) = p_\sigma$ (see [2], Lemma 3.3).

3 Statement of the Main Result

Let D_∞ be the unique real number with $\sum_{i=1}^{N} s_i^{D_\infty} = 1$. Then D_∞ is called the *similarity dimension* of $(S_1, \ldots, S_N)$. For $r \in (0, +\infty)$ there exists a unique $D_r \in (0, +\infty)$ satisfying $\sum_{i=1}^{N} (p_i s_i^r)^{\frac{D_r}{r+D_r}} = 1$. (see [5], Lemma 14.4). Let

$$D_0 = \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log s_i}$$

where $(0 \log 0 := 0)$.

Theorem 3.1. *Let $(S_1, \ldots, S_N)$ have the OSC, $p = (p_1, \ldots, p_N)$ with $p_i > 0$ for all i, and let P be the self–similar probability corresponding to $(S_1, \ldots, S_N; p)$. Let D_r be as above. Then, for every $r \in [0, +\infty]$,*

$$0 < \liminf_{n\to\infty} n e_{n,r}^{D_r} \le \limsup_{n\to\infty} n e_{n,r}^{D_r} < +\infty;$$

in particular $\lim_{n\to\infty} \dfrac{\log n}{-\log e_{n,r}} = D_r$.

Remark 3.2.

a) If $p = (s_1^{D_\infty}, \ldots, s_N^{D_\infty})$, then $D_r = D_\infty$ for all $r \in [0, +\infty]$.

b) If $p \neq (s_1^{D_\infty}, \ldots, s_N^{D_\infty})$, then the function $[0, +\infty] \to (0, +\infty)$, $r \to D_r$ is strictly increasing and continuous.

PROOF. That $(0, +\infty] \to (0, +\infty), r \to D_r$ is strictly increasing and continuous follows from Lemma 14.16 and the proof of Theorem 14.15 in [5]. (Actually the results there are stated for $r \ge 1$ only but the proofs work unchanged for

$r > 0$.) It remains to show that $\lim_{r\downarrow 0} D_r = D_0$. Let $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by $F(q,t) = \sum_{i=1}^{N} p_i^q s_i^t - 1$. Then for every $q\in\mathbb{R}$ there exists a unique $\beta(q)\in\mathbb{R}$ with $F(q,\beta(q)) = 0$. By implicit differentiation the function $\mathbb{R}\to\mathbb{R}, q\to\beta(q)$ is differentiable with derivative

$$\beta'(q) = -\frac{\sum_{i=1}^{N} p_i^q s_i^{\beta(q)} \log p_i}{\sum_{i=1}^{N} p_i^q s_i^{\beta(q)} \log s_i}.$$

Also, β is strictly decreasing with $\lim_{q\to-\infty}\beta(q) = +\infty$ and $\lim_{q\to+\infty}\beta(q) = -\infty$ (see for instance, Falconer [1], p. 193). From the definitions we deduce that for $0 < r < +\infty$, $\beta(\frac{D_r}{r+D_r}) = r\frac{D_r}{r+D_r}$. Since $\beta(1) = 0$, we get

$$\frac{\beta(\frac{D_r}{r+D_r}) - \beta(1)}{\frac{D_r}{r+D_r} - 1} = \frac{r\frac{D_r}{r+D_r}}{-\frac{r}{r+D_r}} = -D_r.$$

Thus $\lim_{r\downarrow 0} D_r = -\beta'(1) = \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log s_i} = D_0$ if we can show that $\lim_{r\downarrow 0}\frac{D_r}{r+D_r} = 1$. Since $0 < \frac{D_r}{r+D_r} < 1$ for all $r\in(0,+\infty)$ the claim is proved provided that, for every $r_n \downarrow 0$ for which $(\frac{D_{r_n}}{r_n+D_{r_n}})_{n\in\mathbb{N}}$ converges to some a, it follows that $a = 1$. But this obviously holds because $1 = \lim_{n\to\infty}\sum_{i=1}^{N}(p_i s_i^{r_n})^{\frac{D_{r_n}}{r_n+D_{r_n}}} = \sum_{i=1}^{N} p_i^a$. $\square$

c) It is an interesting question under what conditions the limit $\lim_{n\to\infty} n e_{n,r}^{D_r}$ exists. If $S_1 \ldots S_4\colon \mathbb{R}^2\to\mathbb{R}^2$ are defined by $S_i x = \frac{1}{2}x + x_i$ with $x_1 = (0,0)$, $x_2 = (\frac{1}{2},0)$, $x_3 = (0,\frac{1}{2})$, $x_4 = (\frac{1}{2},\frac{1}{2})$, and $p = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$, then the corresponding self–similar probability P is the uniform distribution on the square $[0,1]^2$. In this case $D_r = 2$ and $\lim_{n\to\infty} n e_{n,r}^{D_r}$ exists for all $r\in[0,+\infty]$ (see [5], Theorem 6.2 and Theorem 10.7 and [4], Theorem 3.2). If $S_1, S_2\colon \mathbb{R}\to\mathbb{R}$ are defined by $S_1 x = \frac{1}{3}x$ and $S_2 x = \frac{1}{3}x + \frac{2}{3}$ and $p = (\frac{1}{2},\frac{1}{2})$, then the corresponding self–similar probability P is the uniform distribution on the classical Cantor set, the quantization dimension of order 2 is $D_2 = \frac{\log 2}{\log 3}$, and the sequence $(n e_{n,2}^{D_2})_{n\in\mathbb{N}}$ does not converge (see [3], Theorem 6.3).

d) For general relationships between Hausdorff and box dimension of a probability P and the quantization dimensions of P the reader is referred to [4] and [5]. There he will also find a definition of upper and lower quantization dimensions together with their basic properties.

4 Proof of the Main Result

In this section we always assume that the assumptions of Theorem 3.1 are satisfied. Moreover, let U be a bounded open subset of $\mathbb{R}^d$ with $A \cap U \neq \emptyset$, $S_i(U) \subset U$, and $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$. That $\limsup_{n\to\infty} n e_{n,r}^{D_r} < +\infty$ is shown in [5], Proposition 14.5 and 14.6 for $r \in (0, +\infty]$ and in [4], Theorem 5.3 for $r = 0$. (Strictly speaking [5] only deals with $r \in [1, +\infty]$ but the results extend to $r \in (0, 1)$ without change of proof). Here the OSC need not be assumed. Proposition 14.13 in [5] shows that $0 < \liminf_{n\to\infty} n e_{n,\infty}^{D_\infty}$ and relies on the open set condition. That $0 < \liminf_{n\to\infty} n e_{n,r}^{D_r} \leq \limsup_{n\to\infty} n e_{n,r}^{D_r} < \infty$ implies $\lim_{n\to\infty} \frac{\log n}{-\log e_{n,r}} = D_r$ is shown in [5], Corollary 11.4 (b). To prove Theorem 3.1 it, therefore, remains to verify that $0 < \liminf_{n\to\infty} n e_{n,r}^{D_r}$ for all $r \in [0, +\infty)$. To establish this inequality we will need a series of lemmas.

Lemma 4.1. *For every finite set $\alpha \subset \mathbb{R}^d$ the function $\mathbb{R}^d \to [-\infty, +\infty]$, $x \to \log d(x, \alpha \cup U^c)$ is P–integrable.* ($U^c := \mathbb{R}^d \backslash U$, $\log 0 := -\infty$).

Proof. For every $x \in \mathbb{R}^d$ we have

$$\log d(x, \alpha \cup U^c) = \min(\log d(x, \alpha), \log d(x, U^c)).$$

According to [2], Prop. 3.4 the map $x \mapsto \log d(x, U^c)$ is P–integrable. It follows from Proposition 5.1 b) and the proof of Lemma 2.6 in [4] that $x \mapsto \log d(x, \alpha)$ is P–integrable and the lemma is proved. □

Definition 4.2. For every natural number $n \geq 1$ define

$$u_{n,0} = \inf\{\exp \int \log d(x, \alpha \cup U^c)\, dP(x) | \alpha \subset \mathbb{R}, \operatorname{card}(\alpha) \leq n\}, \hat{u}_{n,0} = \log u_{n,0},$$

and, for $0 < r < +\infty$,

$$u_{n,r} = \inf\Big\{ \int d(x, \alpha \cup U^c)^r \, dP(x) | \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \leq n\Big\}.$$

Remark 4.3. Obviously we have $u_{n,0} \leq e_{n,0}$ and, for $0 < r < \infty$; also $u_{n,r} \leq e_{n,r}^r$. The main idea in the proof of $0 < \liminf_{n\to\infty} n e_{n,r}^{D_r}$ is to replace $e_{n,r}$ by $u_{n,r}^{\frac{1}{r}}$ and then to use the techniques developed in [4] and [5] for the proof of $\liminf_{n\to\infty} n e_{n,r}^{D_r} > 0$ in the case of strongly separated self–similar probabilities.

Lemma 4.4. *For every* $r \in [0,+\infty)$ *and every* $n \in \mathbb{N}$ *there exists a set* $\alpha_n \subset \mathbb{R}^d$ *with* $\operatorname{card}(\alpha_n) \le n$ *and*

$$u_{n,r} = \begin{cases} \exp \int \log d(x, \alpha_n \cup U^c)\, dP(x) & \text{if } r = 0 \\ \int d(x, \alpha_n \cup U^c)^r\, dP(x) & \text{if } r > 0. \end{cases}$$

Proof. $r = 0$: Let $\bar{U}$ be the closure of U and define $f\colon \bar{U}^n \to \mathbb{R}$ by

$$f(x_1, \dots, x_n) = \int \log d(x, \{x_1, \dots, x_n\} \cup U^c)\, dP(x).$$

We will show that f is continuous. Let $(x_1, \dots, x_n) \in \bar{U}^n$ be arbitrary and let $((x_{1,k}, \dots, x_{n,k}))_{k\in\mathbb{N}}$ be an arbitrary sequence in $\bar{U}^n$ with

$$\lim_{k\to\infty} (x_{1,k}, \dots, x_{n,k}) = (x_1, \dots, x_n).$$

Since, for every $x \in A$ and every $(y_1, \dots, y_n) \in \bar{U}^n$,

$$\begin{aligned} \log d(x, \{y_1, \dots, y_n\} \cup U^c) &= \min(\{\log \|x - y_i\| \,|i = 1, \dots, n\} \cup \{\log d(x, U^c)\}) \\ &\le \log d(x, U^c) \end{aligned}$$

and since $x \mapsto \log d(x, U^c)$ is P–integrable (see [2], Prop. 3.4) we deduce from Lebesgue's dominated convergence theorem that

$$\begin{aligned} &\lim_{k\to\infty} \int \log_+ d(x, \{x_{1,k}, \dots, x_{n,k}\} \cup U^c)\, dP(x) \\ &= \int \log_+ d(x, \{x_1, \dots, x_n\} \cup U^c)\, dP(x). \end{aligned}$$

Since $\int g\, dP = \int\limits_0^\infty P(g \ge t)\, dt$ for every non–negative measurable $g\colon \mathbb{R}^d \to \mathbb{R}$, by an obvious substitution

$$\begin{aligned} &\int \log_- d(x, \{y_1, \dots, y_n\} \cup U^c)\, dP(x) \\ &= \int\limits_0^1 P(\{x \in A | d(x, \{y_1, \dots, y_n\} \cup U^c) \le s\}) \frac{ds}{s}. \end{aligned}$$

Now we have

$$\begin{aligned} &P(\{x \in A | d(x, \{y_1, \dots, y_n\} \cup U^c) \le s\}) \\ =&P(\{x \in A | \exists \in 1, \dots, n\colon \|x - y_i\| \le s\} \cup \{x \in A | d(x, U^c) \le s\}) \\ \le& \sum_{i=1}^n P(B(y_i, s)) + P(\{x \in A | d(x, U^c) \le s\}) \end{aligned}$$

where $B(y_i, s)$ is the closed ball with radius s and center y_i. Since

$$\int_0^1 P(\{x \in A |\, d(x, U^c) \le s\}) \frac{ds}{s} = \int \log_- d(x, U^c)\, dP(x) < +\infty,$$

$\int (\sup_{y \in \mathbb{R}^d} P(B(y,s)) \frac{1}{s}) ds < +\infty$ (see [4], Prop. 5.1a) and for $\lambda - a.a.\ s \in [0, +\infty)$,

$$\begin{aligned} &\lim_{k\to\infty} P(\{x \in A | d(x, \{x_{1,k}, \ldots, x_{n,k}\} \cup U^c) \le s\}) \\ =& P(\{x \in A | d(x, \{x_1, \ldots, x_n\} \cup U^c) \le s\}). \end{aligned}$$

Lebesgue's dominated convergence theorem implies

$$\begin{aligned} &\lim_{k\to\infty} \int_0^1 P(\{x \in A | d(x, \{x_{1,k}, \ldots, x_{n,k}\} \cup U^c) \le s\}) \frac{ds}{s} \\ =& \int_0^1 P(\{x \in A | d(x, \{x_1, \ldots, x_n\} \cup U^c) \le s\}) \frac{ds}{s}. \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{k\to\infty} \int \log_- d(x, \{x_{1,k}, \ldots, x_{n,k}\} \cup U^c)\, dP(x) \\ =& \int \log_- d(x, \{x_1, \ldots, x_n\})\, dP(x). \end{aligned}$$

Combining the preceding results yields the continuity of f. Since $\bar{U}^n$ is compact, f attains its minimum at some $(a_1, \ldots, a_n) \in \bar{U}^n$ and $\alpha_n = \{a_1, \ldots, a_n\}$ satisfies the conclusion of the lemma for $r = 0$.

$r > 0$: Define $f \colon \bar{U}^n \to \mathbb{R}$ by

$$f(x_1, \ldots, x_n) = \int d(x, \{x_1, \ldots, x_n\} \cup U^c)^r\, dP(x).$$

Using similar techniques as above one can see that f is continuous. Hence it attains its minimum at some point $(a_1, \ldots, a_n) \in \bar{U}^n$. Obviously this minimum equals $u_{n,r}$ and $\alpha_n = \{a_1, \ldots, a_n\}$ has the desired property. $\square$

4.1 Definition and Remark

For a finite set $\alpha \subset \mathbb{R}^d$ and $a \in \alpha$ the set

$$W(a|\alpha) = \{x \in \mathbb{R}^d | \ \ \|x - a\| = d(x, \alpha)\}$$

is called the *Voronoi cell* of a with respect to α. A partition $(B_a)_{a\in\alpha}$ of $\mathbb{R}^d$ into Borel sets is said to be a *Voronoi partition* w.r.t. α iff $B_a \subset W(a|\alpha)$ for all $a \in \alpha$. It is obvious that, for every finite set $\alpha \subset \mathbb{R}^d$, there exists a Voronoi partition w.r.t. α.

Although an analogous result holds for $r \in (0, +\infty)$ we will need (and formulate) the following lemma only for $r = 0$.

Lemma 4.5. *For $n \in \mathbb{N}$ let $\alpha_n \subset \mathbb{R}^d$ satisfy $\operatorname{card}(\alpha_n) \leq n$ and*

$$u_{n,0} = \exp \int \log d(x, \alpha_n \cup U^c) \, dP(x) \ \textit{(cf. Lemma 4.4).}$$

Moreover, let $C_n = \{x \in \mathbb{R}^d | d(x, \alpha_n) \geq d(x, U^c)\}$ and let $(B_a)_{a\in\alpha}$ be a Voronoi partition with respect to α_n. Let $\gamma_n = \{a \in \alpha_n | P(B_a \backslash C_n) > 0\}$. Then $\operatorname{card}(\gamma_n) = n$, in particular $\alpha_n \subset U$ and $\operatorname{card}(\alpha_n) = n$.

PROOF. First we will show that $\hat{u}_{n,0} = \int \log d(x, \gamma_n \cup U^c) \, dP(x)$. The inequality $\hat{u}_{n,0} \leq \int \log d(x, \gamma_n \cup U^c) \, dP(x)$ holds by the definition of $u_{n,0}$ and $\hat{u}_{n,0}$. To show the converse inequality note that, for every $a \in \gamma_n$ and every $x \in B_a \backslash C_n$,

$$d(x, \alpha_n) = \|x - a\| \geq d(x, \gamma_n) \geq d(x, \alpha_n).$$

Hence $d(x, \alpha_n) = d(x, \gamma_n)$. Using this fact we obtain

$$\begin{aligned}
\hat{u}_{n,0} &= \int \log d(x, \alpha_n \cup U^c) \, dP(x) \\
&= \sum_{a\in\gamma_n} \int\limits_{B_a \backslash C_n} \log d(x, \alpha_n) \, dP(x) + \int\limits_{C_n} \log d(x, U^c) \, dP(x) \\
&= \sum_{a\in\gamma_n} \int\limits_{B_a \backslash C_n} \log d(x, \gamma_n) \, dP(x) + \int\limits_{C_n} \log d(x, U^c) \, dP(x) \\
&= \int\limits_{C_n^c} \log d(x, \gamma_n) \, dP(x) + \int\limits_{C_n} \log d(x, U^c) \, dP(x) \\
&\geq \int \log d(x, \gamma_n \cup U^c) \, dP(x).
\end{aligned}$$

Next, we claim that for every $a \in \gamma_n$ there exists a $b \in B_a$ such that $P(\{x \in A | \|x-a\| > \|x-b\|\} \cap (B_a \backslash C_n)) > 0$. Let $a \in \gamma_n$ be arbitrary. Since $P(B_a \backslash C_n) > 0$ and P is continuous we have $P(B_a \backslash (\{a\} \cup C_n)) > 0$. Hence, there is a compact set $K \subset B_a \backslash (\{a\} \cup C_n)$ with $P(K) > 0$. The open sets $U_b = \{x \in \mathbb{R}^d | \ \|x-a\| > \|x-b\|\}$, $b \in K$ form a covering of K since, for every $b \in K$, we have $b \in U_b$. Thus, there exists a finite set $\beta \subset K$ with $K \subset \bigcup_{b \in \beta} U_b$ which implies $P(K \cap U_b) > 0$ for some $b \in \beta \subset K$ and proves our claim.

Finally we prove $\mathrm{card}(\gamma_n) = n$. Assume to the contrary that $\mathrm{card}(\gamma_n) < n$. Choose $a_0 \in \gamma_n$ and $b \in B_{a_0}$ with $P(U_b \cap (B_{a_0} \backslash C_n)) > 0$. Then we get

$$\begin{aligned}
\hat{u}_{n,0} &\leq \int \log d(x, \gamma_n \cup \{b\} \cup U^c)\, dP(x) \\
&= \sum_{a \in \gamma_n \backslash \{a_0\}} \int\limits_{B_a \backslash C_n} \log d(x, \gamma_n \cup \{b\} \cup U^c)\, dP(x) \\
&\quad + \int\limits_{B_{a_o} \backslash C_n} \log d(x, \gamma_n \cup \{b\} \cup U^c)\, dP(x) + \int\limits_{C_n} \log d(x, \gamma_n \cup \{b\} \cup U^c)\, dP(x) \\
&\leq \sum_{a \in \gamma \backslash \{a_0\}} \int\limits_{B_a \backslash C_n} \log d(x, \gamma_n \cup U^c)\, dP(x) + \int\limits_{(B_{a_0} \backslash C_n) \cap U_b} \log \|x-b\|\, dP(x) \\
&\quad + \int\limits_{(B_{a_0} \backslash C_n) \backslash U_b} \log \|x-a_0\|\, dP(x) + \int\limits_{C_n} \log d(x, \gamma_n \cup U^c)\, dP(x).
\end{aligned}$$

Since

$$\int\limits_{(B_{a_0} \backslash C_n) \cap U_b} \log \|x-b\|\, dP(x) < \int\limits_{(B_{a_0} \backslash C_n) \cap U_b} \log \|x-a_0\|\, dP(x)$$

and since

$$\int\limits_{B_{a_0} \backslash C_n} \log \|x-a_0\|\, dP(x) = \int\limits_{B_{a_0} \backslash C_n} \log d(x, \gamma_n)\, dP(x),$$

we deduce

$$\begin{aligned}
\hat{u}_{n,0} &< \sum_{a \in \gamma_n} \int\limits_{B_a \backslash C_n} \log d(x, \gamma_n)\, dP(x) + \int\limits_{C_n} \log d(x, \gamma_n)\, dP(x), \\
&\leq \hat{u}_{n,0},
\end{aligned}$$

a contradiction. Thus, the lemma is proved. □

Lemma 4.6. $\lim_{n\to\infty}(\hat{u}_{n,0} - \hat{u}_{n+1,0}) = 0.$

PROOF. Let $\alpha_{n+1} \subset \mathbb{R}^d$ satisfy $\operatorname{card}(\alpha_{n+1}) \leq n+1$ and

$$\hat{u}_{n+1,0} = \int \log d(x, \alpha_{n+1} \cup U^c)\, dP(x).$$

According to Lemma 4.6 we have $\operatorname{card}(\alpha_{n+1} \cap U) = n+1$. Let $(B_a)_{a\in\alpha_{n+1}}$ and C_{n+1} be as in Lemma 4.6. Then there exists an $a_0 \in \alpha_{n+1}$ with $P(B_{a_0}) \leq \frac{1}{n+1}$, and we get

$$\begin{aligned}
\hat{u}_{n,0} &\leq \int \log d(x, (\alpha_{n+1}\backslash\{a_0\}) \cup U^c)\, dP(x)\\
&\leq \sum_{a\in\alpha_{n+1}\backslash\{a_0\}} \int\limits_{B_a\backslash C_{n+1}} \log d(x, (\alpha_{n+1}\backslash\{a_0\}) \cup U^c)\, dP(x)\\
&\quad + \int\limits_{B_{a_0}\backslash C_{n+1}} \log d(x, U^c)\, dP(x) + \int\limits_{C_{n+1}} \log d(x, U^c)\, dP(x)\\
&= \sum_{a\in\alpha_{n+1}} \int\limits_{B_a\backslash C_{n+1}} \log \|x-a\|\, dP(x) - \int\limits_{B_{a_0}\backslash C_{n+1}} \log\|x-a_0\|\, dP(x)\\
&\quad + \int\limits_{B_{a_0}\backslash C_{n+1}} \log d(x, U^c)\, dP(x) + \int\limits_{C_{n+1}} \log d(x, U^c)\, dP(x).
\end{aligned}$$

Since A is bounded, there exists a $c \in (1, +\infty)$ with $\log d(x, U^c) \leq c$ for all $x \in A$. For every $x \in C_{n+1}$ we have $d(x, U^c) = d(x, \alpha_{n+1} \cup U^c)$ and

$$\sum_{a\in\alpha_{n+1}} \int\limits_{B_a\backslash C_{n+1}} \log\|x-a\|\, dP(x) = \int\limits_{C_{n+1}^c} \log d(x, \alpha_{n+1} \cup U^c)\, dP(x).$$

Thus, we deduce

$$\hat{u}_{n,0} \leq \hat{u}_{n+1,0} - \int\limits_{B_{a_0}\backslash C_{n+1}} \log\|x-a_0\|\, dP(x) + cP(B_{a_0}\backslash C_{n+1}).$$

Now

$$\int_{B_{a_0}\setminus C_{n+1}} \log\|x-a_0\|\,dP(x) \geq \int_{(B_{a_0}\setminus C_{n+1})\cap\bar{B}(a_0,1)} \log\|x-a_0\|\,dP(x)$$
$$= -\int_0^1 P((B_{a_0}\setminus C_{n+1})\cap B(a_0,s))\frac{ds}{s}.$$

Let $p>1$ and q with $\frac{1}{p}+\frac{1}{q}=1$. Then Hölder's inequality yields

$$P((B_{a_0}\setminus C_{n+1})\cap B(a_0,s)) \leq P(B_{a_0}\setminus C_{n+1})^{\frac{1}{p}}P(B(a_0,s))^{\frac{1}{q}}.$$

By [4], Prop. 5.1 a), there is a $C\in\mathbb{R}$ and $t>0$ with, $P(B(x,s))\leq Cs^t$ for all $x\in\mathbb{R}^d$ and all $s\in[0,1]$. Hence we obtain

$$\begin{aligned}\hat{u}_{n,0} \leq& \hat{u}_{n+1,0} + \int_0^1 P((B_{a_0}\setminus C_{n+1}))^{\frac{1}{p}} C s^t \frac{ds}{s} + cP(B_{a_0}\setminus C_{n+1})\\ \leq& \hat{u}_{n+1,0} + (\frac{1}{n+1})^{\frac{1}{p}} c\frac{1}{t} + \frac{1}{n+1}c.\end{aligned}$$

Hence, the lemma is proved. □

Lemma 4.7. *Let $r\in[0,+\infty)$ and, for each $n\in\mathbb{N}$, let the set $\alpha_n\subset\mathbb{R}^d$ satisfy* $\operatorname{card}(\alpha_n)\leq n$ *and*

$$u_{n,r} = \begin{cases}\exp\int\log d(x,\alpha_n\cup U^c)\,dP(x) & \text{if } r=0\\ \int d(x,\alpha_n\cup U^c)^r\,dP(x) & \text{if } r>0.\end{cases}$$

Set $\delta_n = \max_{x\in A} d(x,\alpha_n\cup U^c)$. Then $\lim_{n\to\infty}\delta_n = 0$.

PROOF. Since P is continuous and $P(A\cap U)=1$ (follows from [2], Proposition 3.4), we have $\delta_n>0$ for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$ there is an $x_n\in A$ and an $a_n\in\alpha_n\cup U^c$ with $\|x_n-a_n\| = d(x_n,\alpha_n\cup U^c)=\delta_n$. For all $x\in A\cap B(x_n,\frac{1}{2}\delta_n)$ and all $a\in\alpha_n\cup U^c$ we have

$$(*)\qquad \|x-a\| \geq \|x_n-a\| - \|x_n-x\| \geq \|x_n-a_n\| - \frac{1}{2}\delta_n = \frac{1}{2}\delta_n.$$

$r=0$: Set $\beta_{n+1}=\alpha_n\cup\{x_n\}$. Then $x\in A\cap B(x_n,\frac{1}{2}\delta_n)$ implies

$$d(x,\beta_{n+1}\cup U^c) = \|x-x_n\|.$$

Thus we deduce

$$\begin{aligned}
\hat{u}_{n+1,0} &\leq \int \log d(x, \beta_{n+1} \cup U^c)\, dP(x) \\
&\leq \int\limits_{B(x_n, \frac{\delta_n}{2})} \log \|x - x_n\|\, dP(x) + \int\limits_{A \setminus B(x_n, \frac{\delta_n}{2})} \log d(x, \alpha_n \cup U^c)\, dP(x) \\
&= \int \log d(x, \alpha_n \cup U^c)\, dP(x) - \int\limits_{B(x_n, \frac{\delta_n}{2})} \log d(x, \alpha_n \cup U^c)\, dP(x) \\
&+ \int\limits_{B(x_n, \frac{\delta_n}{2})} \log \|x - x_n\|\, dP(x).
\end{aligned}$$

Since $d(x, \alpha_n \cup U^c) \geq \frac{\delta_n}{2}$ for all $x \in A \cap B(x_n, \frac{\delta_n}{2})$, we obtain

$$\int\limits_{B(x_n, \frac{\delta_n}{2})} \log d(x, \alpha_n \cup U^c)\, dP(x) \geq P(B(x_n \frac{\delta_n}{2})) \log \frac{\delta_n}{2}$$

and therefore,

$$\hat{u}_{n,0} - \hat{u}_{n+1,0} \geq P(B(x_n, \frac{\delta_n}{2})) \log \frac{\delta_n}{2} - \int\limits_{B(x_n, \frac{\delta_n}{2})} \log \|x - x_n\|\, dP(x).$$

If $\delta_n \leq 2$, then it follows that $\hat{u}_{n,0} - \hat{u}_{n+1,0} \geq \int\limits_0^{\frac{\delta_n}{2}} P(B(x_n, s)) \frac{ds}{s}$. If $\delta_n > 2$, then it follows that

$$\begin{aligned}
\hat{u}_{n,0} - \hat{u}_{n+1,0} \geq & P(B(x_n, \frac{\delta_n}{2})) \log \frac{\delta_n}{2} - \int\limits_{B(x_n, \frac{\delta_n}{2}) \setminus B(x_n, 1)} \log \frac{\delta_n}{2}\, dP(x) \\
& - \int\limits_{B(x_n, 1)} \log \|x - x_n\|\, dP(x) \geq \int\limits_0^1 P(B(x_n, s)) \frac{ds}{s}.
\end{aligned}$$

Since, for every $w \in [0, 1]$, the map $x \mapsto \int_0^w P(B(x, s)) \frac{ds}{s}$ is continuous, we have $g(w) = \min\limits_{x \in A} \int\limits_0^w P(B(x, s))\, \frac{ds}{s} > 0$ for $w > 0$ and the function $g\colon [0, 1] \to \mathbb{R}$ is nondecreasing. We obtain $\hat{u}_{n,0} - \hat{u}_{n+1,0} \geq g(\min(1, \frac{\delta_n}{2}))$. Now because $\lim\limits_{n \to \infty} (\hat{u}_{n,0} - \hat{u}_{n+1,0}) = 0$ (see Lemma 4.7) this implies $\lim\limits_{n \to \infty} \delta_n = 0$.

$r > 0$: From $(*)$ we deduce

$$\begin{aligned} u_{n,r} &= \int d(x, \alpha_n \cup U^c)^r \, dP(x) \\ &\geq \int\limits_{B(x_n, \frac{\delta_n}{2})} (\frac{1}{2}\delta_n)^r \, dP(x) = (\frac{1}{2}\delta_n)^r P(B(x_n, \frac{1}{2}\delta_n)). \end{aligned}$$

Assume $\limsup\limits_{n\to\infty} \delta_n > \delta > 0$. Then $\delta_n > \delta$ for infinitely many n and hence $u_{n,r} \geq P(B(x_n, \frac{1}{2}\delta))(\frac{1}{2}\delta)^r$ for infinitely many n. Since $\min\limits_{x\in A} P(B(x, \frac{1}{2}\delta)) > 0$ this implies $\limsup\limits_{n\to\infty} u_{n,r} > 0$. Since $e^r_{n,r} \geq u_{n,r}$ and $\lim\limits_{n\to\infty} e_{n,r} = 0$ (see [5], Lemma 6.1). This yields a contradiction and the lemma is proved. □

Lemma 4.8. *Let $r \in [0, +\infty)$ be given. Then there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, there are $n_1(n), \ldots, n_N(n) \in \mathbb{N}$ with $n_i(n) \geq 1$, $\sum\limits_{i=1}^{N} n_i(n) \leq n$, and*

$$u_{n,r} \geq \begin{cases} \prod\limits_{i=1}^{N} (s_i u_{n_i(n),0})^{p_i} & \text{if } r = 0 \\ \sum\limits_{i=1}^{N} p_i s_i^r u_{n_i(n),r} & \text{if } r > 0. \end{cases}$$

PROOF. There is a $\tau \in \{1, \ldots, \mathbb{N}\}^*$ with $A_\tau \subset U$ (see, for instance, [2], proof of Lemma 3.3). Then $\varepsilon = d(A_\tau, U^c) > 0$. Set $s_{\min} = \min\{s_1, \ldots, s_N\}$. We deduce $d(S_i(A_\tau), S_i(U)^c) = s_i d(A_\tau, U^c) \geq s_{\min}\varepsilon$ and, hence, that $d(x, U^c) \geq d(x, S_i(U)^c) \geq s_{\min}\varepsilon$ for all $x \in S_i(A_\tau)$. Let α_n and δ_n be as in Lemma 4.8 and choose n_0 such that $\delta_n < s_{\min}\varepsilon$ for all $n \geq n_0$. Let $n \geq n_0$ and $x \in S_i(A_\tau)$ be fixed for the moment. Then there exists an $a \in \alpha_n \cup U^c$ with $\|x - a\| = d(x, \alpha_n \cup U^c) \leq \delta_n < s_{\min}\varepsilon$. Thus we get $a \in S_i(U) \subset U$ and, therefore, $S_i(U) \cap \alpha_n \neq \emptyset$.

Now define $\alpha_{n,i} = \alpha_n \cap S_i(U)$ and $n_i = \operatorname{card}(\alpha_{n,i})$. Then $n_i \geq 1$ and, since $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$, $\sum\limits_{i=1}^{N} n_i \leq \operatorname{card}(\alpha_n) \leq n$.

Using the self–similarity of P and the fact that $S_i(U) \subset U$, we obtain for $r = 0$

$$\begin{aligned} \hat{u}_{n,0} &= \sum_{i=1}^{N} p_i \int \log d(S_i x, \alpha_n \cup U^c) \, dP(x) \\ &\geq \sum_{i=1}^{N} p_i \int \log d(S_i x, \alpha_n \cup S_i(U)^c) \, dP(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^N p_i \int \log d(S_i x, \alpha_{n,i} \cup S_i(U)^c)\, dP(x) \\
&= \sum_{i=1}^N p_i \int \log(s_i d(x, S_i^{-1}(\alpha_{n,i}) \cup U^c)\, dP(x) \\
&= \sum_{i=1}^N p_i \log s_i + \sum_{i=1}^N p_i \int \log d(x, S_i^{-1}(\alpha_{n,i}) \cup U^c)\, dP(x) \\
&\geq \sum_{i=1}^N p_i \log s_i + \sum_{i=1}^N p_i \hat{u}_{n_i 0}
\end{aligned}$$

and, for $r > 0$,

$$\begin{aligned}
u_{n,r} &= \sum_{i=1}^N p_i \int d(S_i x, \alpha_n \cup U^c)^r\, dP(x) \\
&\geq \sum_{i=1}^N p_i s_i^r \int d(x, S_i^{-1}(\alpha_{n,i}) \cup U^c)^r\, dP(x) \geq \sum_{i=1}^N p_i s_i^r u_{n_i,r}.
\end{aligned}$$

Thus the lemma is proved. □

Lemma 4.9.

a) $\inf\{n^{\frac{1}{D_0}} u_{n,0} \colon n \in \mathbb{N}\} > 0$

b) $\inf\{n^{\frac{r}{D_r}} u_{n,r} \colon n \in \mathbb{N}\} > 0$ *for* $r \in (0, +\infty)$.

Proof.

a) It is enough to show $\inf\{\frac{1}{D_0}\log n + \hat{u}_{n,0} \colon n \in \mathbb{N}\} > -\infty$. It follows from Lemma 4.1 and Lemma 4.4 that $\hat{u}_{n,0} > -\infty$ for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ and, for $n \geq n_0$, $n_1(n), \ldots, n_N(n)$ be as in Lemma 4.9. Set

$$c = \min\{\frac{1}{D_0}\log n + \hat{u}_{n,0} \colon n \leq n_0\}.$$

If $n \geq n_0$ and $\frac{1}{D_0}\log k + \hat{u}_{k,0} \geq c$ for all $k \leq n-1$, then

$$\begin{aligned}\hat{u}_{n.0} &\geq \sum_{i=1}^{N} p_i \log s_i + \sum_{i=1}^{N} p_i \hat{u}_{n_i(n),0} \\ &\geq \sum_{i=1}^{N} p_i \log s_i + \sum_{i=1}^{N} p_i (c - \frac{1}{D_0}\log n_i(n)) \\ &\geq c - \frac{1}{D_0}\log n + \sum_{i=1}^{N} p_i \log s_i - \frac{1}{D_0}\sum_{i=1}^{N} p_i \log \frac{n_i(n)}{n}.\end{aligned}$$

Since $\sum\limits_{i=1}^{N} p_i \log \frac{n_i(n)}{n} \leq \sum\limits_{i=1}^{N} p_i \log p_i$, we get

$$\frac{1}{D_0}\log n + \hat{u}_{n,0} \geq c + \sum_{i=1}^{N} p_i \log s_i - \frac{1}{D_0}\sum_{i=1}^{N} p_i \log p_i = c.$$

By induction we obtain $\inf\{\frac{1}{D_0}\log n + \hat{u}_{n,0} | n \in \mathbb{N}\} \geq c > -\infty$.
b) Let α_n be as in Lemma 4.8. Since $P(A \cap (\alpha_n \cup U^c)) = 0$ and $d(x, \alpha_n \cup U^c) > 0$ for all $x \in A \backslash (\alpha_n \cup U^c)$ we get $u_{n,r} > 0$ for all $n \in \mathbb{N}$. Let n_0 and, for $n \geq n_0$, $n_1(n), \ldots, n_N(n)$ be as in Lemma 4.9. Set $c = \min\{n^{\frac{r}{D_r}} u_{n,r} \colon n \leq n_0\}$. Then we have $c > 0$. Let $n \geq n_0$ be such that $k^{\frac{r}{D_r}} u_{k,r} \geq c$ for all $k \leq n-1$. Using Lemma 4.9 we deduce

$$n^{\frac{r}{D_r}} u_{n,r} \geq n^{\frac{r}{D_r}} \sum_{i=1}^{N} p_i s_i^r n_i(n)^{-\frac{r}{D_r}} n_i(n)^{\frac{r}{D_r}} u_{n_i(n),r}.$$

Since $n_i(n) < n$, we obtain $n^{\frac{r}{D_r}} u_{n,r} \geq c \sum\limits_{i=1}^{N} p_i s_i^r (\frac{n_i(n)}{n})^{-\frac{r}{D_r}}$. Using Hölder's inequality (exponents less than 1) yields

$$\sum_{i=1}^{N} p_i s_i^r (\frac{n_i(n)}{n})^{-\frac{r}{D_r}} \geq \sum_{i=1}^{N} (p_i s_i^r)^{(\frac{D_r}{r+D_r})^{1+\frac{r}{d_r}}} \left(\sum_{i=1}^{N} (\frac{n_i(n)}{n})^{(-\frac{r}{D_r})\cdot(-\frac{D_r}{r})} \right)^{-\frac{r}{D_r}} = 1.$$

By induction we get $n^{\frac{r}{D_r}} u_{n,r} \geq c$ for all $n \in \mathbb{N}$ and the lemma is proved. □

Proof of Theorem 3.1. According to the considerations at the beginning of this section the theorem is proved if one can establish that for all $r \in [0, +\infty)$

$0 < \liminf_{n\to\infty} n e_{n,r}^{D_r}$. We know that

$$u_{n,r} \leq \begin{cases} e_{n,r} & \text{if } r = 0 \\ e_{n,r}^r & \text{if } r > 0 \end{cases}$$

and Lemma 4.10 immediately implies $\liminf_{n\to\infty} n u_{n,0}^{D_0} > 0$ and $\liminf_{n\to\infty} n u_{n,r}^{\frac{D_r}{r}} > 0$ for $r \in (0, +\infty)$. Thus the theorem is proved. □

References

[1] K. Falconer, *Techniques in fractal geometry*, John Wiley&Sons, 1997.

[2] S. Graf, *On Bandt's tangential distribution for self-similar measures*, Mh. Math. **120** (1995), 223–246.

[3] S. Graf and H. Luschgy, *The quantization of the Cantor distribution*, Math. Nachr. **183** (1997), 113–133.

[4] S. Graf and H. Luschgy, *Quantization for probability measures with respect to the geometric mean error*, Preprint 1999.

[5] S. Graf and H. Luschgy, *Foundations of quantization for probability distributions*, Lect. Notes Math. **1730**, Springer 2000.

[6] S. Graf and H. Luschgy, *The quantization dimension of self–similar probabilities*, Math. Nachr., to appear.

[7] A. Schief, *Separation properties of self-similar sets*, Proc. Amer. Math. Soc. **122** (1994), 111–116.