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ANALYTIC NON-BOREL SETS AND VERTICES OF DIFFERENTIABLE CURVES IN THE PLANE

Abstract

The purpose of this paper is to show that given any non-zero cardinal number $n \leq \aleph_0$, the set of differentiable paths of class C^2 and of unit length in the plane having their arc length as the parameter in $[0,1]$ and tracing curves which have at least n vertices is analytic non-Borel, while for any $r \in (\mathbb{N} \cup \{\infty\}) \setminus \{0,1,2\}$, the set of differentiable paths of class C^r and of unit length in the plane having their arc length as the parameter in $[0,1]$ and tracing curves which have at least n vertices is F_σ if $n < \aleph_0$ and $F_{\sigma\delta}$ if $n = \aleph_0$.

1 Introduction

It is common knowledge in Mathematics that there is no unified definition of the notion of a curve (see, for example, Paragraph 51.I on page 275 of [19], 4.3 on page 45 of [31], 10.8 on page 200 of [25], and 2.6 on page 7 of [6]), and there is no unified definition of the notion of a vertex of a curve (see, for example, Paragraph 10 on page 33 of [2], 111E on page 414 of [12], and page 332 of [23]), even though it is common ground what is meant by the term *vertex theorems* (see **http://www.ams.org/msc/51Lxx.html**).

The first effort to give an exact definition of the notion of a curve using analytic methods was made by C. Jordan (see, for example, 93A and 93B on pages 345–346 of [12] or [17]). For the purpose of this paper, following the approach in [12] (see 93B on page 346), for any $r \in \mathbb{N} \cup \{\infty\}$, we define a plane *curve of class C^r* to be the image of a mapping $[0,1] \to E^2$ of class

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C^r sending the closed unit interval $[0,1]$ into the Euclidean plane E^2, and, following the approach of K. Kuratowski (see Paragraph 62.XI on page 584 of [19]), we call the mapping in question the *path of class* C^r which *traces* the curve. Identifying E^2 with $\mathbb{R}^2$, say by choosing a coordinate system, for any $r \in (\mathbb{N} \cup \{\infty\}) \setminus \{0\}$, we view the Polish space (i.e., the separable completely metrizable space)

$$\mathcal{P}^r = \left\{ (x,y) \in C^r([0,1],\mathbb{R}^2) : \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \right\}$$

as the space of all differentiable paths of class C^r and of unit length in the plane which have their arc length as the parameter in $[0,1]$, where $C^r([0,1],\mathbb{R}^2)$ is equipped with the *Whitney topology*, (See, for example, 1. on page 35 of [11].) a complete compatible metric for which is given by the formula

$$\begin{aligned} d_{C^r([0,1],\mathbb{R}^2)}((x,y),(u,v)) = &\sum_{0 \le i \le r; i \in \mathbb{N}} 2^{-i} \frac{\|x^{(i)} - u^{(i)}\|_\infty}{1 + \|x^{(i)} - u^{(i)}\|_\infty} \\ &+ \sum_{0 \le i \le r; i \in \mathbb{N}} 2^{-i} \frac{\|y^{(i)} - v^{(i)}\|_\infty}{1 + \|y^{(i)} - v^{(i)}\|_\infty} \end{aligned}$$

where (x,y), (u,v) are in $C^r([0,1],\mathbb{R}^2)$. (See, for example, page 8 of [27] and Section 5 on pages 148–149 of [24].) We should mention that $\mathcal{P}^r$ is a Polish space since it is a G_δ subspace of the Polish space $C^r([0,1],\mathbb{R}^2)$ in the Whitney topology (See, for example, 3.11 on page 17 of [18]).

What we are concerned with here is the notion of a vertex of a curve, which presupposes the notion of curvature. Thus, if (x,y) is any path in $\mathcal{P}^r$, where $r \in (\mathbb{N} \cup \{\infty\}) \setminus \{0,1\}$, then the *curvature* κ of the curve traced by (x,y) is given by the formula

$$\kappa = \frac{dx}{ds} \cdot \frac{d^2y}{ds^2} - \frac{dy}{ds} \cdot \frac{d^2x}{ds^2}$$

(See, for example, Section 11 on page 26 of [30].) and depends at least continuously on the arc length $s \in [0,1]$. For the purpose of this paper, following the approach in [12] (See 111E on page 414), we call a point A on the curve traced by (x,y) a *vertex* if $\left(\frac{d\kappa}{ds}\right)_A = 0$, in other words, if $\left(\frac{d\kappa}{ds}\right)_{s=a} = 0$, where a is the value of the parameter for which $(x(a), y(a))$ constitutes the pair of Cartesian coordinates of the point A in the Euclidean plane E^2. We should mention though that when considering curves of class C^2, a vertex is also defined as a local maximum or minimum of the curvature, (See, for example, page 332 of [23].) since the derivative of the curvature can not be defined apart from the case when the curvature is of bounded variation, and hence its derivative exists almost everywhere. (See, for example, Section 2 on pages 102–104 of [24].)

Theorem. *For any non-zero cardinal number $n \leq \aleph_0$, the set of paths in $\mathcal{P}^2$ tracing curves which have at least n vertices is analytic non-Borel in $\mathcal{P}^2$, while for any $r \in (\mathbb{N} \cup \{\infty\}) \setminus \{0, 1, 2\}$, the set of paths in $\mathcal{P}^r$ tracing curves which have at least n vertices is F_σ in $\mathcal{P}^r$ if $n < \aleph_0$, and $F_{\sigma\delta}$ in $\mathcal{P}^r$ if $n = \aleph_0$.*

Thus, for any non-zero cardinal number $n \leq \aleph_0$, there exist no Borel measurable necessary and sufficient conditions on a differentiable path of class C^2 and of unit length having its arc length as the parameter in $[0, 1]$ and tracing a curve in the plane, which can assert the existence of at least n vertices. In other words, for any non-zero cardinal number $n \leq \aleph_0$, any necessary and sufficient conditions for existence of at least n vertices of a differentiable path of class C^2 and of unit length having its arc length as the parameter in $[0, 1]$ and tracing a curve in the plane can not be expressed analytically through explicit formulas, simple analytic expressions, and the like.

Geometric properties, though of a topological nature, that give rise to analytic or co-analytic non-Borel sets were also given by O. Nikodym and W. Sierpinski, (See [21], [22], and 27.18 on page 216 of [18].) and relatively recently by H. Becker (See 33.17 on page 256 of [18] or [3]). Moreover, the subject of vertex theorems occupied the mind of several generations of mathematicians starting with the proof in 1909 of the original *Four Vertex Theorem*, concerning local extrema of the curvature, by the Indian mathematician S. Mukhopadhaya (see [20]). The **MathSciNet** contains several items concerning vertex theorems from which, according to their reviews, we selectively refer the reader to [1], [2], [4], [5], [7], [8], [9], [10], [13], [14], [15], [16], [23], [26], [28], [29], [32], and [33].

2 Elements from Descriptive Set Theory

Descriptive set theory is the study of definable sets in Polish spaces, which are defined as separable, completely metrizable spaces. In this theory sets are classified in the Borel and the projective hierarchy according to the complexity of their definition. Given a Polish space X, the first level of the Borel hierarchy that corresponds to X consists of the class of its $\mathbf{\Sigma}^0_1$-sets or G-sets, which is by definition its open sets, and the class of its $\mathbf{\Pi}^0_1$-sets or F-sets, which is by definition its closed sets; the second level consists of the class of its $\mathbf{\Sigma}^0_2$-sets or F_σ-sets, which is defined as countable unions of its $\mathbf{\Pi}^0_1$-sets, and the class of its $\mathbf{\Pi}^0_2$-sets or G_δ-sets, which is defined as countable intersections of its $\mathbf{\Sigma}^0_1$-sets; the third level consists of the class of its $\mathbf{\Sigma}^0_3$-sets or $G_{\delta\sigma}$-sets, which is defined as countable unions of its $\mathbf{\Pi}^0_2$-sets, and the class of its $\mathbf{\Pi}^0_3$-sets or $F_{\sigma\delta}$-sets, which is defined as countable intersections of its $\mathbf{\Sigma}^0_2$-sets, etc. On the other hand, the first level of the projective hierarchy that corresponds to X consists of the class of its analytic or $\mathbf{\Sigma}^1_1$-sets, which is defined as continuous images of Polish spaces, and the class of its co-analytic or $\mathbf{\Pi}^1_1$-sets, which is defined as complements of its $\mathbf{\Sigma}^1_1$-sets; the second level consists of its $\mathbf{\Sigma}^1_2$-sets, which is defined as continuous images of $\mathbf{\Pi}^1_1$-sets, and the class of its $\mathbf{\Pi}^1_2$-sets, which is defined as complements of its $\mathbf{\Sigma}^1_2$-sets, etc. (See, for example, the Introduction, 11.B on pages 68–69, 25.A on pages 196–197, 32.A on pages 242–243, and 37.A on pages 313–315 of [18].)

Given a class $\mathbf{\Gamma}$ of sets in either the Borel or the projective hierarchy, if X and Y are any Polish spaces, then we call a $\mathbf{\Gamma}$-set $B \subseteq Y$ *Wadge reducible* to a set $A \subseteq X$, in symbols $B \leq_W A$, if there exists a continuous mapping $f : Y \to X$ such that $B = f^{-1}[A]$; moreover, we call A $\mathbf{\Gamma}$-*hard*, if for any Polish space Y and for any $\mathbf{\Gamma}$-set $B \subseteq Y$, we have $B \leq_W A$, and, in particular, we call A $\mathbf{\Gamma}$-*complete*, if it also constitutes a $\mathbf{\Gamma}$-set. A powerful technique to find a lower bound for the complexity of a given set is to show that it is $\mathbf{\Gamma}$-hard for some class $\mathbf{\Gamma}$ of sets in either the Borel or the projective hierarchy, usually by proving that another set which is known to be $\mathbf{\Gamma}$-hard is Wadge reducible to it, and by showing that it is $\mathbf{\Gamma}$-complete we compute its exact complexity. (See, for example, 21.13 on page 156, 22.B on pages 169–170, and 26.C on pages 206–207 of [18].)

3 Trees and Functions in L^1

Trees are basic combinatorial tools in descriptive set theory. A *tree* on $\mathbb{N}$ is a subset T of the set $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ of all finite sequences of natural numbers, which is closed under initial segments, and its *body* is $[T] = \{\alpha \in \mathbb{N}^{\mathbb{N}} : (\forall n \in \mathbb{N})(\alpha|n \in T)\}$, where $\alpha|n = (\alpha(0), \ldots, \alpha(n-1))$. A tree is usually viewed as an element of $2^{\mathbb{N}^{<\mathbb{N}}}$ by identifying it with its characteristic function, where $2^{\mathbb{N}^{<\mathbb{N}}}$ is equipped with the product topology with $2 = \{0, 1\}$ discrete, making it homeomorphic to the Cantor space, a closed subset of which is the set Tr of all trees on $\mathbb{N}$. Thus, Tr acquires the structure of a Polish space (i.e., a separable completely metrizable space), and it is partitioned into two characteristic subsets, the set $IF = \{T \in Tr : [T] \neq \emptyset\}$ of *ill-founded* trees on $\mathbb{N}$, which is $\mathbf{\Sigma}^1_1$-complete, and the set $WF = \{T \in Tr : [T] = \emptyset\}$ of *well-founded* trees on $\mathbb{N}$, which is $\mathbf{\Pi}^1_1$-complete. (See, for example, 2.A on pages 5–6, 4.32 on pages 27–28, 2.E on page 10, 27.1 on page 209, and 32.B on page 243 of [18].)

Let $1 \leq n \leq \aleph_0$ and let $-\infty < \alpha < \beta < \infty$. For any $i < n$, we set

$$I^{(i)}_{n;\emptyset} = \begin{cases} \left[\alpha + i\frac{\beta-\alpha}{n}, \alpha + (i+1)\frac{\beta-\alpha}{n}\right] & \text{if } n < \aleph_0 \\ \left[\alpha + \sum_{j=1}^{i} \frac{\beta-\alpha}{2^j}, \alpha + \sum_{j=1}^{i+1} \frac{\beta-\alpha}{2^j}\right] & \text{if } n = \aleph_0 \end{cases}$$

and if $s \in \mathbb{N}^{<\mathbb{N}}$ is such that the intervals $I^{(i)}_{n;s}$ are already defined, then we define the intervals $I^{(i)}_{n;s^\frown k}$, where $s^\frown k = (s(0), \dots, s(\text{length}(s)-1), k)$, as follows. If $I^{(i)}_{n;s} = [a, b]$, then for any $k \in \mathbb{N}$, we set $I^{(i)}_{n;s^\frown k} = \left[a + \sum_{j=1}^{2k+1} \frac{b-a}{2^j}, a + \sum_{j=1}^{2k+2} \frac{b-a}{2^j}\right]$. Moreover, if for any tree T on $\mathbb{N}$, we set

$$\phi_{n;T} = \prod_{i<n} \left(1 - \sum_{s \in T \setminus \{\emptyset\}} 2^{-\text{length}(s)} \chi_{I^{(i)}_{n;s}}\right)$$

and $\kappa_{n;T} = \int_\alpha^x \phi_{n;T}(t)\, dt$ for every $x \in [\alpha, \beta]$, then the following result holds.

Theorem 3.1. *Given any non-zero cardinal number $n \leq \aleph_0$ and any tree T on $\mathbb{N}$, the following are true.*

- *If $(\kappa_{n;T})'_-(\beta)$ exists, then $(\kappa_{n;T})'_-(\beta) \geq \dfrac{1}{6}$.*
- $T \in IF \Rightarrow (\forall i < n)\left(\exists \alpha_i \in \text{Int}(I^{(i)}_{n;\emptyset})\right)(\kappa'_{n;T}(\alpha_i) = 0)$.
- $T \in WF \Rightarrow (\forall x \in [\alpha, \beta))((\kappa_{n;T})'_+(x) > 0)$.

Moreover, both mappings $Tr \ni T \mapsto \phi_{n;T} \in L^1([\alpha, \beta], m)$, where m stands for the Lebesgue measure on $\mathbb{R}$, and $Tr \ni T \mapsto \kappa_{n;T} \in C([\alpha, \beta], \mathbb{R})$ are well-defined and continuous.

PROOF. Fix a tree T on $\mathbb{N}$. For simplicity set $\phi^{(i)}_{n;T} = 1 - \sum_{s \in T \setminus \{\emptyset\}} 2^{-\text{length}(s)} \chi_{I^{(i)}_{n;s}}$ for every $i < n$. We remark that if $i < n$, $j < n$ and $i \neq j$, then $\phi^{(i)}_{n;T} = 1$ on $I^{(j)}_{n;\emptyset}$, which implies that $\phi_{n;T} = \phi^{(i)}_{n;T}$ on $I^{(i)}_{n;\emptyset}$, while our construction implies that for any $x \in I^{(i)}_{n;\emptyset}$ and any $s \in \mathbb{N}^{<\mathbb{N}}$, there exists at most one $k \in \mathbb{N}$ for which $x \in I^{(i)}_{n;s^\frown k}$, which implies in its turn that $\phi^{(i)}_{n;T}(x) \geq 1 - \sum_{k=1}^{\infty} 2^{-k} = 0$, and $\phi^{(i)}_{n;T}(x) = 0$ if and only if there exists $\alpha \in [T]$ such that $x \in \bigcap_{k \in \mathbb{N}} I^{(i)}_{n;\alpha|k}$.

In addition, for any positive integer i, we have

$$\begin{aligned}
\frac{\kappa_{n;T}(\beta)-\kappa_{n;T}\left(\alpha+\sum\limits_{j=1}^{2i}\frac{\beta-\alpha}{2^j}\right)}{\beta-\left(\alpha+\sum\limits_{j=1}^{2i}\frac{\beta-\alpha}{2^j}\right)} &= \frac{1}{\beta-\left(\alpha+\sum\limits_{j=1}^{2i}\frac{\beta-\alpha}{2^j}\right)}\cdot\int_{\alpha+\sum\limits_{j=1}^{2i}\frac{\beta-\alpha}{2^j}}^{\beta}\phi_{n;T}(t)\,dt \\
&\geq \frac{1}{\beta-\left(\alpha+\sum\limits_{j=1}^{2i}\frac{\beta-\alpha}{2^j}\right)}\cdot\sum_{k=1}^{\infty}\frac{\beta-\alpha}{2^{2i+2k+1}}=\frac{1}{6},
\end{aligned}$$

which implies that if $(\kappa_{n;T})'_-(\beta)$ exists, then $(\kappa_{n;T})'_-(\beta)\geq\frac{1}{6}$. So let $T\in WF$ and let $i<n$, while $x\in I^{(i)}_{n;\emptyset}$. Then there exists $s\in T$ of maximum length such that $x\in I^{(i)}_{n;s}$ and maximality implies that $x\in I^{(i)}_{n;s}\setminus\bigcup_{\nu\in\mathbb{N};s^\frown\nu\in T}I^{(i)}_{n;s^\frown\nu}$. If x is either the left endpoint or lies in the interior of $I^{(i)}_{n;s}$, then there exists $\epsilon>0$ such that for $x<y<x+\epsilon$, we have $\phi^{(i)}_{n;T}(y)=1-\sum\limits_{k=1}^{\mathrm{length}(s)}2^{-k}$, which implies that $(\kappa_{n;T})'_+(x)=1-\sum\limits_{k=1}^{\mathrm{length}(s)}2^{-k}$. So let x be the right endpoint of $I^{(i)}_{n;s}$. If $s\neq\emptyset$, then there is $\epsilon>0$ such that for $x<y<x+\epsilon$, we have $\phi^{(i)}_{n;T}(y)=1-\sum\limits_{k=1}^{\mathrm{length}(s)-1}2^{-k}$, which implies that $(\kappa_{n;T})'_+(x)=1-\sum\limits_{k=1}^{\mathrm{length}(s)-1}2^{-k}$, while if $s=\emptyset$ and $i+1<n$, then there exists $\epsilon>0$ such that for $x<y<x+\epsilon$, we have $\phi^{(i)}_{n;T}(y)=1$, which implies that $(\kappa_{n;T})'_+(x)=1$. We have thus proved that $T\in WF\Rightarrow(\forall x\in[\alpha,\beta))\big((\kappa_{n;T})'_+(x)>0\big)$.

So let $T\in IF$ and let $\alpha\in[T]$. If $i<n$ and α_i is the unique point contained in $\bigcap_{k\in\mathbb{N}}I^{(i)}_{n;\alpha|k}$, then we claim that $\kappa'_{n;T}(\alpha_i)=0$. Indeed, if $k\in\mathbb{N}$ and x, y are in $I^{(i)}_{n;\alpha|k}$, then

$$\begin{aligned}
|\phi_{n;T}(x)-\phi_{n;T}(y)| &= \left|\phi^{(i)}_{n;T}(x)-\phi^{(i)}_{n;T}(y)\right| \\
&\leq \sum_{s\in T\setminus\{\emptyset\}}2^{-\mathrm{length}(s)}\cdot\left|\chi_{I^{(i)}_{n;s}}(x)-\chi_{I^{(i)}_{n;s}}(y)\right| \\
&\leq 2\cdot\sum_{j>k}2^{-j}=2^{-k+1},
\end{aligned}$$

and hence if $x\neq\alpha_i$ lies in the interior of $I^{(i)}_{n;\alpha|k}$, while I stands for the interval defined by x and α_i, then, as $\phi^{(i)}_{n;T}(\alpha_i) = 0 \Rightarrow \phi_{n;T}(\alpha_i) = 0$, we obtain that

$$\begin{aligned} |\kappa_{n;T}(x) - \kappa_{n;T}(\alpha_i)| &= \int_I \phi_{n;T}(t)\,dt = \int_I (\phi_{n;T}(t) - \phi_{n;T}(\alpha_i))\,dt \\ &\leq 2^{-k+1} \cdot |x - \alpha_i| \Rightarrow \left|\frac{\kappa_{n;T}(x) - \kappa_{n;T}(\alpha_i)}{x - \alpha_i}\right| \leq 2^{-k+1}, \end{aligned}$$

and the claim follows. We have thus proved that

$$T \in IF \Rightarrow (\forall i < n)\left(\exists \alpha_i \in \mathrm{Int}(I^{(i)}_{n;\emptyset})\right)\left(\kappa'_{n;T}(\alpha_i) = 0\right).$$

What is left to show is that the mapping $Tr \ni T \mapsto \phi_{n;T} \in L^1([\alpha,\beta], m)$ is continuous, as the continuity of the mapping $Tr \ni T \mapsto \kappa_{n;T} \in C([\alpha,\beta],\mathbb{R})$ will then follow. Indeed, it suffices to note that for functions f, g in $L^1([\alpha,\beta], m)$ and $x \in [\alpha,\beta]$, we have $\left|\int_\alpha^x f(t)\,dt - \int_\alpha^x g(t)\,dt\right| \leq \int_\alpha^\beta |f(t) - g(t)|\,dt$. Given $i < n$, $s \in \mathbb{N}^{<\mathbb{N}}$ and $k \in \mathbb{N}$, it is not difficult to see that $m\left(I^{(i)}_{\aleph_0;\emptyset}\right) = \dfrac{\beta - \alpha}{2^{i+1}}$ and $m\left(I^{(i)}_{n;s^\frown k}\right) = \dfrac{m\left(I^{(i)}_{n;s}\right)}{4^{k+1}}$, while given T, T' in Tr, we have

$$\begin{aligned} \int_\alpha^\beta |\phi_{n;T'}(x) - \phi_{n;T}(x)|\,dx &= \sum_{i<n} \int_{I^{(i)}_{n;\emptyset}} |\phi_{n;T'}(x) - \phi_{n;T}(x)|\,dx \\ &= \sum_{i<n} \int_{I^{(i)}_{n;\emptyset}} \left|\phi^{(i)}_{n;T'}(x) - \phi^{(i)}_{n;T}(x)\right|\,dx, \end{aligned}$$

where for any $i < n$ and any $x \in I^{(i)}_{n;\emptyset}$, we have

$$\phi^{(i)}_{n;T'}(x) - \phi^{(i)}_{n;T}(x) = \sum_{s \in T \setminus \{\emptyset\}} 2^{-\mathrm{length}(s)} \chi_{I^{(i)}_{n;s}}(x) - \sum_{s \in T' \setminus \{\emptyset\}} 2^{-\mathrm{length}(s)} \chi_{I^{(i)}_{n;s}}(x),$$

which implies that $\left|\phi^{(i)}_{n;T'} - \phi^{(i)}_{n;T}\right| \leq 1$ and also that $\phi^{(i)}_{n;T'} - \phi^{(i)}_{n;T}$ vanishes on $I^{(i)}_{n;s} \setminus \bigcup_{k\in\mathbb{N}} I^{(i)}_{n;s^\frown k}$ for every $s \in \mathbb{N}^{<\mathbb{N}}$. Therefore, for $i < n$ and $s \in \mathbb{N}^{<\mathbb{N}}$, we have $\int_{I^{(i)}_{n;s}} \left|\phi^{(i)}_{n;T'}(x) - \phi^{(i)}_{n;T}(x)\right|\,dx = \sum_{k=0}^{\infty} \int_{I^{(i)}_{n;s^\frown k}} \left|\phi^{(i)}_{n;T'}(x) - \phi^{(i)}_{n;T}(x)\right|\,dx$, where for any $k \in \mathbb{N}$, we have $\int_{I^{(i)}_{n;s^\frown k}} \left|\phi^{(i)}_{n;T'}(x) - \phi^{(i)}_{n;T}(x)\right|\,dx \leq m\left(I^{(i)}_{n;s^\frown k}\right) = \dfrac{m\left(I^{(i)}_{n;s}\right)}{4^{k+1}}$. So let $T \in Tr$ be arbitrary but fixed and given $N \in \mathbb{N} \setminus \{0\}$, let

$$V_{T;N} = \left\{T' \in Tr : \left(\forall s \in \bigcup_{n=0}^{N} \{0, 1, \ldots, N-1\}^n\right)(s \in T' \iff s \in T)\right\}$$

It is not difficult to see that the sets $V_{T;N}$ form a fundamental system of open neighborhoods of T in Tr. So let $N \in \mathbb{N} \setminus \{0\}$ be arbitrary but fixed and let $T' \in V_{T;N}$. Then for any $i < n$ and any $s \in \mathbb{N}^{<\mathbb{N}}$, we have

$$\int_{I_{n;s}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx \leq \sum_{k=0}^{N-1} \int_{I_{n;s^\frown k}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx$$
$$+ \sum_{k=N}^{\infty} \frac{m\left(I_{n;s}^{(i)}\right)}{4^{k+1}} = \sum_{k=0}^{N-1} \int_{I_{n;s^\frown k}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx + m\left(I_{n;s}^{(i)}\right) \cdot \frac{1}{3 \cdot 4^N}$$

and hence we obtain that

$$\int_{I_{n;\emptyset}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx \leq \sum_{k_0=0}^{N-1} \left(\sum_{k_1=0}^{N-1} \left(\dots (\sum_{k_{N-1}=0}^{N-1} \int_{I_{n;(k_0,k_1,\dots,k_{N-1})}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx \right.\right.$$
$$\left.\left. + \; m(I_{n;\emptyset}^{(i)}) \cdot \frac{1}{4^{k_0+1}} \cdots \frac{1}{4^{k_{N-2}+1}} \cdot \frac{1}{3 \cdot 4^N}\right) \dots \right) + \; m\left(I_{n;\emptyset}^{(i)}\right) \cdot \frac{1}{4^{k_0+1}} \cdot \frac{1}{3 \cdot 4^N}\right) + m\left(I_{n;\emptyset}^{(i)}\right) \cdot \frac{1}{3 \cdot 4^N}$$

So let $(k_0, k_1, \dots, k_{N-1}) \in \{0, 1, \dots, N-1\}^N$ be arbitrary but fixed. Given $s \in \mathbb{N}^{<\mathbb{N}}$, if s is an initial segment of $(k_0, k_1, \dots, k_{N-1})$, then $s \in T \iff s \in T'$, and if s is not an initial segment of $(k_0, k_1, \dots, k_{N-1})$, then for any $x \in I_{n;(k_0,k_1,\dots,k_{N-1})}^{(i)}$, we have $\chi_{I_{n;s}^{(i)}}(x) = 0$. Therefore, for any $x \in I_{n;(k_0,k_1,\dots,k_{N-1})}^{(i)}$, we have $-\frac{1}{2^N} = -\sum_{k>N} 2^{-k} \leq \phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x) \leq \sum_{k>N} 2^{-k} = \frac{1}{2^N}$, which implies that

$$\int_{I_{n;(k_0,k_1,\dots,k_{N-1})}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx \leq \frac{1}{2^N} \cdot m\left(I_{n;\emptyset}^{(i)}\right) \cdot \frac{1}{4^{k_0+1}} \cdots \frac{1}{4^{k_{N-1}+1}}$$

and, as $\sum_{k=0}^{N-1} \frac{1}{4^{k+1}} = \frac{1}{3} \cdot \left(1 - \frac{1}{4^N}\right)$, we obtain that

$$\int_{I_{n;\emptyset}^{(i)}} \left|\phi_{n;T'}^{(i)}(x) - \phi_{n;T}^{(i)}(x)\right| dx$$
$$\leq \sum_{k_0=0}^{N-1} \left(\sum_{k_1=0}^{N-1} \left(\dots (\sum_{k_{N-1}=0}^{N-1} \frac{1}{2^N} \cdot m(I_{n;\emptyset}^{(i)}) \cdot \frac{1}{4^{k_0+1}} \cdots \frac{1}{4^{k_{N-1}+1}} \right.\right.$$
$$\left.\left. + \; m(I_{n;\emptyset}^{(i)}) \cdot \frac{1}{4^{k_0+1}} \cdots \frac{1}{4^{k_{N-2}+1}} \cdot \frac{1}{3 \cdot 4^N}\right) \dots \right)$$

$$
\begin{aligned}
&+\ m\big(I^{(i)}_{n;\emptyset}\big)\cdot\frac{1}{4^{k_0+1}}\cdot\frac{1}{3\cdot 4^N}\big)+m\big(I^{(i)}_{n;\emptyset}\big)\cdot\frac{1}{3\cdot 4^N}\\
\leq&\frac{1}{2^N}\cdot m\big(I^{(i)}_{n;\emptyset}\big)\cdot\Big[\sum_{k_0=0}^{N-1}\Big(\sum_{k_1=0}^{N-1}\Big(\dots\big(\sum_{k_{N-1}=0}^{N-1}\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-1}+1}}\\
&+\ \frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-2}+1}}\big)\dots\Big)+\frac{1}{4^{k_0+1}}\Big)+1\Big]\\
&=\frac{1}{2^N}\cdot m\big(I^{(i)}_{n;\emptyset}\big)\cdot\sum_{k=0}^{N}\Big(\frac{1}{3}\cdot\big(1-\frac{1}{4^N}\big)\Big)^k\\
&<\frac{1}{2^N}\cdot m\big(I^{(i)}_{n;\emptyset}\big)\cdot\sum_{k=0}^{\infty}\Big(\frac{1}{3}\cdot\big(1-\frac{1}{4^N}\big)\Big)^k\leq\frac{3}{2^{N-1}}\cdot m\big(I^{(i)}_{n;\emptyset}\big)
\end{aligned}
$$

for every $i<n$, which implies that $\int_\alpha^\beta|\phi_{n;T'}(x)-\phi_{n;T}(x)|\,dx\leq 3\cdot(\beta-\alpha)\cdot 2^{-N+1}$ for every $T'\in V_{T;N}$. ☐

4 Analytic Non-Borel Sets, Tangents of Continuous Curves in the Plane and Tangent Hyperplanes of Graphs of Continuous Functions

Theorem 4.1. *Given any line in the plane and any non-zero cardinal number $n\leq\aleph_0$, the set of continuous paths in the plane tracing curves which have at least n tangents parallel to the given line is analytic, non-Borel.*

Proof. Once $1\leq n\leq\aleph_0$ is given, by appropriately choosing a coordinate system in the plane, it is enough to prove that the set of continuous paths in $\mathbb{R}^2$ tracing curves which have at least n tangents parallel to the real line is $\mathbf{\Sigma}^1_1$-complete in $C([0,1],\mathbb{R}^2)$, and therefore analytic, non-Borel. The fact that the set in question is $\mathbf{\Sigma}^1_1$-hard in $C([0,1],\mathbb{R}^2)$ follows immediately from Theorem 3.1. Indeed, for $\alpha=0$ and $\beta=1$ we need only consider the mapping that assigns to every tree T on $\mathbb{N}$ the continuous path $[0,1]\ni t\mapsto(t,\kappa_{n;T}(t))\in\mathbb{R}^2$. Thus, what is left to show is that the set in question is actually $\mathbf{\Sigma}^1_1$ in $C([0,1],\mathbb{R}^2)$ in case $n<\aleph_0$. (See, for example, 14.4 on page 86 of [18].) But this follows from the fact that given any $(x,y)\in C([0,1],\mathbb{R}^2)$, the path (x,y) traces a curve which has at least n tangents parallel to the real line if and only if there exists $(a_1,\dots,a_n,b_1,\dots,b_n)\in[0,1]^n\times\mathbb{R}^n$ with the properties $a_1<\dots<a_n$, $(\forall i\in\{1,\dots,n\})(b_i\neq 0)$ and $(\forall i\in\{1,\dots,n\})(\forall\epsilon\in\mathbb{Q}^*_+)(\exists\delta\in\mathbb{Q}^*_+)(\forall r\in[0,1]\cap\mathbb{Q})$

$$
\Big(0<|r-a_i|\Rightarrow\Big(\Big|\frac{x(r)-x(a_i)}{r-a_i}-b_i\Big|\leq\epsilon\ \wedge\ \Big|\frac{y(r)-y(a_i)}{r-a_i}\Big|\leq\epsilon\Big)\Big).
$$

(See, for example, 14.3 on page 86 of [18].) □

Theorem 4.2. *Given any positive integer N and any non-zero cardinal number $n \leq \aleph_0$, if $-\infty < \alpha < \beta < \infty$, then the set of all functions in $C([\alpha,\beta]^N,\mathbb{R})$ whose graph in $\mathbb{R}^{N+1}$ has at least n tangent N-dimensional hyperplanes parallel to $\mathbb{R}^N$ is $\mathbf{\Sigma}_1^1$-complete in $C([\alpha,\beta]^N,\mathbb{R})$, and therefore analytic non-Borel.*

PROOF. We will first prove that the set in question is $\mathbf{\Sigma}_1^1$-hard in $C([\alpha,\beta]^N,\mathbb{R})$. To this end we consider the mapping that assigns to every tree T on $\mathbb{N}$ the continuous function $f_T : [\alpha,\beta]^N \ni (x_1,\dots,x_N) \mapsto \kappa_{n;T}(x_1)+\dots+\kappa_{n;T}(x_N) \in \mathbb{R}$. We remark that given $(a_1,\dots,a_N) \in [\alpha,\beta]^N$, the graph of f_T in $\mathbb{R}^{N+1}$ has a tangent N-dimensional hyperplane at the point $(a_1,\dots,a_N,f_T(a_1,\dots,a_N))$ if and only if f_T is differentiable at the point $(a_1,\dots,a_N)$ or (equivalently) $\kappa_{n;T}$ is differentiable at the points $a_1, \dots, a_N$. Moreover, the tangent N-dimensional hyperplane in question, if it exists, is perpendicular to the vector $(-\nabla f_T(a_1,\dots,a_N),1) = \big(-\kappa'_{n;T}(a_1),\dots,-\kappa'_{n;T}(a_N),1\big)$, and consequently it is parallel to $\mathbb{R}^N$ if and only if $\kappa'_{n;T}(a_1) = \dots = \kappa'_{n;T}(a_N) = 0$. Therefore, an application of Theorem 3.1 shows that the set in question is $\mathbf{\Sigma}_1^1$-hard in $C([\alpha,\beta]^N,\mathbb{R})$, and what is left to show is that it is actually $\mathbf{\Sigma}_1^1$ in $C([\alpha,\beta]^N,\mathbb{R})$ in case $n < \aleph_0$. (See, for example, 14.4 on page 86 of [18].) But again this follows from the fact that given $f \in C([\alpha,\beta]^N,\mathbb{R})$, the graph of f in $\mathbb{R}^{N+1}$ has at least n tangent N-dimensional hyperplanes parallel to $\mathbb{R}^N$ if and only if there is $(\mathbf{a}^1,\dots,\mathbf{a}^n) \in ([\alpha,\beta]^N)^n$ satisfying $1 \leq i < j \leq n \Rightarrow \mathbf{a}^i \neq \mathbf{a}^j$ and

$$(\forall (i,\nu) \in \{1,\dots,n\}\times\{1,\dots,N\})(\forall \epsilon \in \mathbb{Q}_+^*)(\exists \delta \in \mathbb{Q}_+^*)(\forall r \in [\alpha,\beta]\cap\mathbb{Q})$$

$$\left(0 < \left|r - a_\nu^i\right| < \delta \Rightarrow \left|\frac{f(\mathbf{a}^i + (r-a_\nu^i)\mathbf{e}_\nu) - f(\mathbf{a}^i)}{r - a_\nu^i}\right| \leq \epsilon\right),$$

(See, for example, 14.3 on page 86 of [18].) where $\mathbf{e}_1, \dots, \mathbf{e}_N$ denote the standard basis vectors in $\mathbb{R}^N$. □

5 Analytic Non-Borel Sets and Vertices of Differentiable Curves in the Plane

Theorem 5.1. *For any non-zero cardinal number $n \leq \aleph_0$, the set of paths in $\mathcal{P}^2$ tracing curves which have at least n vertices is $\mathbf{\Sigma}_1^1$-complete in $\mathcal{P}^2$, and therefore analytic, non-Borel in $\mathcal{P}^2$, while for any $r \in (\mathbb{N}\cup\{\infty\})\setminus\{0,1,2\}$, the set of paths in $\mathcal{P}^r$ tracing curves which have at least n vertices is $\mathbf{\Sigma}_2^0$ in $\mathcal{P}^r$, in other words, F_σ in $\mathcal{P}^r$ if $n < \aleph_0$, and $\mathbf{\Pi}_3^0$ in $\mathcal{P}^r$, in other words, $F_{\sigma\delta}$ in $\mathcal{P}^r$ if $n = \aleph_0$.*

Proof. We will first prove that the set of paths in $\mathcal{P}^2$ tracing curves which have at least n vertices is $\mathbf{\Sigma}^1_1$-hard in $\mathcal{P}^2$. To this end for $\alpha = 0$ and $\beta = 1$ we consider the mapping that assigns to every tree T on $\mathbb{N}$ the path in $\mathcal{P}^2$ defined by $x_{n;T}(s) = \int_0^s \cos(\psi_{n;T}(\xi))d\xi$ and $y_{n;T}(s) = \int_0^s \sin(\psi_{n;T}(\xi))d\xi$ for every $s \in [0,1]$, where $\psi_{n;T}(s) = \int_0^s \kappa_{n;T}(\xi)d\xi$ for every $s \in [0,1]$. It is not difficult to verify that the mapping $Tr \ni T \mapsto (x_{n;T}, y_{n;T}) \in \mathcal{P}^2$ is well-defined, and given any $T \in Tr$, we have $T \in IF$ if and only if $(x_{n;T}, y_{n;T})$ traces a curve having at least n vertices. This follows from Theorem 3.1 and the fact that for any $T \in Tr$, the curvature of the curve traced by $(x_{n;T}, y_{n;T})$ is given by the function $\kappa_{n;T}$, as it follows from the proof of the theorem on the existence of a plane curve with given curvature. (See, for example, Section 12 on pages 27–28 of [30].) What we need to show is that the mapping $Tr \ni T \mapsto (x_{n;T}, y_{n;T}) \in \mathcal{P}^2$ is continuous. By Theorem 3.1, if ϕ is either the identity, the sine or the cosine function, it suffices to show that the mappings $\Phi_1 : C([0,1],\mathbb{R}) \to C^1([0,1],\mathbb{R})$ and $\Phi_2 : C^1([0,1],\mathbb{R}) \to C^2([0,1],\mathbb{R})$, defined by $\Phi_1(f)(x) = \int_0^x \phi(f(t))\,dt$ for every $x \in [0,1]$ and every $f \in C([0,1],\mathbb{R})$, and $\Phi_2(f)(x) = \int_0^x \phi(f(t))\,dt$ for every $x \in [0,1]$ and every $f \in C^1([0,1],\mathbb{R})$, are continuous.

The proof of the continuity of Φ_1 is left to the reader and since, for complete metric spaces, uniform convergence on compacts is equivalent to continuous convergence, (See, for example, Problem 40 on page 162 of [24].) if $f_k \to f$ in $C^1([0,1],\mathbb{R})$ and $x_k \to x$ in $[0,1]$ as $k \to \infty$, it is enough to show that $\Phi_2(f_k)(x_k) \to \Phi_2(f)(x)$, $\Phi_2(f_k)'(x_k) \to \Phi_2(f)'(x)$ and $\Phi_2(f_k)''(x_k) \to \Phi_2(f)''(x)$ as $k \to \infty$. Indeed, the continuity of both ϕ and ϕ', the Lebesgue Dominated Convergence Theorem (See, for example, 1.34 on page 26 of [25]) and the fact that both $f_k(x_k) \to f(x)$ and $f_k'(x_k) \to f'(x)$ as $k \to \infty$ are easily seen to imply that

$$\begin{aligned}\Phi_2(f_k)(x_k) &= \int_0^1 \phi(f_k(t))\chi_{[0,x_k]}(t)\,dt \to \int_0^1 \phi(f(t))\chi_{[0,x]}(t)\,dt \\ &= \Phi_2(f)(x),\ \Phi_2(f_k)'(x_k) = \phi(f_k(x_k)) \to \phi(f(x)) = \Phi_2(f)'(x)\end{aligned}$$

and

$$\Phi_2(f_k)''(x_k) = \phi'(f_k(x_k)) \cdot f_k'(x_k) \to \phi'(f(x)) \cdot f'(x) = \Phi_2(f)''(x) \text{ as } k \to \infty$$

Our next step is to show that the set of paths in $\mathcal{P}^2$ tracing curves which have at least n vertices is $\mathbf{\Sigma}^1_1$ in $\mathcal{P}^2$ in case $n < \aleph_0$. (See, for example, 14.4 on page 86 of [18].) Indeed, we need only remark that given any $(x,y) \in \mathcal{P}^2$, the curve traced by (x,y) has at least n vertices if and only if there exists

$(a_1, \ldots, a_n) \in [0,1]^n$ with the properties $a_1 < \cdots < a_n$ and

$$(\forall i \in \{1, \ldots, n\})(\forall \epsilon \in \mathbb{Q}_+^*)(\exists \delta \in \mathbb{Q}_+^*)(\forall r \in [0,1] \cap \mathbb{Q})\, (0 < |r - a_i| < \delta$$

$$\Rightarrow \left| \frac{x'(r)y''(r) - y'(r)x''(r) - x'(a_i)y''(a_i) + y'(a_i)x''(a_i)}{r - a_i} \right| \leq \epsilon \Big).$$

(See, for example, 14.3 on page 86 of [18].)

Finally, given $r \in (\mathbb{N} \cup \{\infty\}) \setminus \{0,1,2\}$, we will prove that the set of paths in $\mathcal{P}^r$ tracing curves which have at least n vertices is $\mathbf{\Sigma}_2^0$ in $\mathcal{P}^r$ if $n < \aleph_0$, and $\mathbf{\Pi}_3^0$ in $\mathcal{P}^r$ if $n = \aleph_0$. To this end, given any positive integer N, it is enough to prove that the set

$$\mathcal{P}_N^r = \big\{(x,y) \in \mathcal{P}^r : (\exists (a_1, \ldots, a_n) \in [0,1]^n)(1 \leq i < j \leq n \Rightarrow |a_i - a_j| \geq N^{-1}$$

$$\wedge\ (\forall i \in \{1, \ldots, n\})(x'(a_i)y'''(a_i) - y'(a_i)x'''(a_i) = 0))\}$$

is closed in $\mathcal{P}^r$ if $n < \aleph_0$. So let $(x_k, y_k) \to (x,y)$ in $\mathcal{P}^r$ as $k \to \infty$ and let $(x_k, y_k) \in \mathcal{P}_N^r$, whenever $k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$, there exists $(a_1^k, \ldots, a_n^k) \in [0,1]^n$ such that $1 \leq i < j \leq n \Rightarrow \left|a_i^k - a_j^k\right| \geq N^{-1}$ and $x_k'(a_i^k)y_k'''(a_i^k) - y_k'(a_i^k)x_k'''(a_i^k) = 0$ for every $i \in \{1, \ldots, n\}$. The compactness of $[0,1]^n$ implies that there exists a subsequence $((a_1^{k_j}, \ldots, a_n^{k_j}))_{j \in \mathbb{N}}$ of $((a_1^k, \ldots, a_n^k))_{k \in \mathbb{N}}$ which converges to some point $(a_1, \ldots, a_n)$ in $[0,1]^n$, and it is not difficult to prove that $1 \leq i < j \leq n \Rightarrow |a_i - a_j| \geq N^{-1}$. Moreover, as, for complete metric spaces, uniform convergence on compacts is equivalent to continuous convergence, (See, for example, Problem 40 on page 162 of [24].) we deduce that $x'(a_i)y'''(a_i) - y'(a_i)x'''(a_i) = 0$ for every $i \in \{1, \ldots, n\}$, and consequently $(x,y) \in \mathcal{P}_N^r$. □

Open Problem. *Given $r \in (\mathbb{N} \cup \{\infty\}) \setminus \{0,1,2\}$, is the set of paths in $\mathcal{P}^r$ that trace curves having infinitely many vertices, $\mathbf{\Pi}_3^0$-complete in $\mathcal{P}^r$?*

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