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TYPICAL PROPERTIES OF LIPSCHITZ FUNCTIONS

Abstract

In terms of Baire category, a typical real-valued Lipschitz function on a finite dimensional space has a local minimum at every point of a dense subset of the domain, and a Dini subdifferential that is either singleton or empty at all points. Moreover, its Dini subdifferential is empty outside a set of first category. Hence a typical Lipschitz function has no points of subdifferential regularity.

1 Introduction

In [8], Sciffer constructs a real-valued Lipschitz function on the line that is nowhere Clarke regular. Here we describe a natural setting in which this behavior can be regarded as typical, even for a function of several variables. In fact, we show that for a typical Lipschitz function,

- the Dini subdifferential never contains more than one element,
- the Dini subdifferential is empty outside a set of first category,
- local minimizers are dense in the domain, and
- the Clarke subdifferential is maximal at every point.

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Throughout this paper we deal with a fixed finite-dimensional normed space X with closed unit ball $\mathbb{B}$, topological dual X^*, and closed dual ball $\mathbb{B}^*$, and with a fixed compact convex set $C \subset X^*$ such that $\operatorname{int} C \neq \emptyset$. We write

$$\mathrm{Lip}_C := \{f\colon X \to R \,:\, f(x) - f(y) \le \sigma_C(x-y) \text{ for all } x, y \in X\},$$

where $\sigma_C(x) := \max\{\langle c, x\rangle \,:\, c \in C\}$ is the support function of C. The boundedness of C implies that every function in Lip_C is Lipschitzian in the usual sense. Indeed, when $C = K\mathbb{B}^*$ for some $K > 0$, Lip_C is the usual set of K-Lipschitz functions on X. We define a complete metric space (Lip_C, ρ) by setting $\rho_n(f,g) := \sup\{|f(x) - g(x)| \,:\, x \in n\mathbb{B}\}$ and

$$\rho(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{\rho_n(f,g), 1\}, \qquad f, g \in \mathrm{Lip}_C.$$

Note that a sequence of functions $\langle f_k\rangle$ in Lip_C converges to f in the metric ρ if and only if f_k converges to f uniformly on each compact subset of X. We call a property *typical in* Lip_C when it holds throughout some subset of Lip_C whose complement has first category.

Subgradients. For $f \in \mathrm{Lip}_C$ and $x \in X$, the lower Dini directional derivative and Dini subgradient are defined by

$$f_+(x;v) := \liminf_{t \to 0^+} \frac{f(x+tv) - f(x)}{t},$$
$$\widehat{\partial} f(x) := \{x^* \in X^* \,:\, f_+(x;v) \ge \langle x^*, v\rangle \ \ \forall v \in X\}.$$

(Other authors call $\widehat{\partial} f$ the Fréchet subdifferential or the regular subdifferential.) The set $\widehat{\partial} f(x)$ is closed, but may well be empty. Thus we must contend with the set $\mathrm{dom}(\widehat{\partial} f) := \left\{x \in X \,:\, \widehat{\partial} f(x) \neq \emptyset\right\}$. We will also refer to the upper Dini directional derivative $f^+ = -(-f)_+$, given explicitly by

$$f^+(x;v) := \limsup_{t \to 0^+} \frac{f(x+tv) - f(x)}{t}.$$

The general subgradient of f at x is, by definition,

$$\partial f(x) := \bigcap_{\delta > 0} \mathrm{cl}\left\{z^* \,:\, z^* \in \widehat{\partial} f(z),\ \|z - x\| < \delta\right\}.$$

The Clarke directional derivative and subgradient of f at x are

$$f^\circ(x;v) := \limsup_{y\to x, t\downarrow 0} \frac{f(y+tv)-f(y)}{t},\ v \in X,$$
$$\partial^\circ f(x) := \{x^* \in X^* : \langle x^*, v\rangle \le f^\circ(x;v)\ \forall v \in X\}.$$

Since $f \in \mathrm{Lip}_C$, we have $\partial^\circ f \equiv \mathrm{co}\, \partial f$. (See [3, 6] for details and discussion.) Recall that $f \in \mathrm{Lip}_C$ is called *subdifferentially regular* [or *Clarke regular*] at x when $\widehat{\partial} f(x) = \partial^\circ f(x)$.

Our purpose in this note is to show that while a typical function in Lip_C has a local minimum on a dense subset of points in its domain, making its Clarke subdifferential large everywhere, its Dini subdifferential is empty outside a set of first category, and never contains more than one element. Taken together, these statements imply that a typical Lipschitz function is nowhere subdifferentially regular.

2 Density of Local Minimizers

Given a continuous function $\phi\colon X \to \mathbb{R}$, let $\Sigma(\phi)$ denote the set of points in X where ϕ has a local minimum.

Theorem 1. *For each $c \in \mathrm{int}(C)$, the following set is residual in* (Lip_C, ρ).

$$G(c) := \{f \in \mathrm{Lip}_C : \Sigma(f-c) \text{ is dense in } X\}.$$

(*Here* $(f-c)(y) = f(y) - \langle c, y\rangle$.)

Proof. Fix $c \in \mathrm{int}\, C$. For each fixed $x \in X$, define a sequence of sets indexed by $n \in \mathbb{N}$.

$$G_x^n := \left\{ f \in \mathrm{Lip}_C : \Sigma(f-c) \cap \mathrm{int}\left(x + \frac{1}{n}\mathbb{B}\right) \ne \emptyset \right\}.$$

We claim that each set $\mathrm{int}(G_x^n)$ is dense in Lip_C. To prove this, choose an arbitrary $f \in \mathrm{Lip}_C$ and $\varepsilon \in (0,1)$. Define $h(y) := f(x) - \varepsilon + \sigma_C(y-x)$ for $y \in X$, and let $h_1 := \min\{f, h\}$. Clearly h and h_1 lie in Lip_C. Since $f(y) - f(x) \le \sigma_C(y-x)$ for every $y \in X$, we have

$$f(y) - \varepsilon \le f(x) - \varepsilon + \sigma_C(y-x) = h(y);$$

so $f \ge h_1 \ge f - \varepsilon$ and $\rho(h_1, f) \le \varepsilon$.

As $h(x) < f(x)$, by continuity there is some $\delta \in (0, 1/n)$ sufficiently small that $h(y) < f(y)$ for all y in $x + \delta\mathbb{B}$, whence $h_1(y) = h(y)$ for all $y \in x + \delta\mathbb{B}$.

This implies that $h_1 - c$ has a local minimum at x, since every $y \in x + \delta\mathbb{B}$ obeys

$$h_1(y) - \langle c, y\rangle = h_1(x) - \langle c, x\rangle + [\sigma_C(y-x) - \langle c, y-x\rangle] \tag{1}$$

and the inequality $\sigma_C(y-x) \geq \langle c, y-x\rangle$ holds because $c \in C$. Thus $h_1 \in G_x^n$.

Let us show that in fact $h_1 \in \operatorname{int} G_x^n$. By hypothesis, there is some $r > 0$ such that $c + r\mathbb{B}^* \subset C$. Thus (1) implies

$$m := \inf\{h_1(y) - \langle c, y\rangle \,:\, \|y - x\| = \delta\} \geq h_1(x) - \langle c, x\rangle + r\delta.$$

Let $\alpha := m - (h_1(x) - \langle c, x\rangle) > 0$ and choose $0 < \beta < \min\{\alpha/2, 1\}$. Also choose an integer $N > \|x\| + 1$, and note that every $g \in \mathrm{Lip}_C$ with $\rho(g, h_1) < \beta/2^N$ satisfies $\rho_N(g, h_1) < \beta$. When $\|y - x\| = \delta$ we have

$$\begin{aligned} g(y) - \langle c, y\rangle &= g(y) - h_1(y) + h_1(y) - \langle c, y\rangle \geq -\beta + m, \text{ and} \\ g(x) - \langle c, x\rangle &= g(x) - h_1(x) + h_1(x) - \langle c, x\rangle \leq \beta + h_1(x) - \langle c, x\rangle . \end{aligned}$$

Thus

$$\begin{aligned} \inf\{g(y) - \langle c, y\rangle \,:\, \|y - x\| = \delta\} \geq& -\beta + m \\ >& \beta + h_1(x) - \langle c, x\rangle \geq g(x) - \langle c, x\rangle . \end{aligned}$$

Now the continuous function $g - c$ must attain its minimum over the compact set $x + \delta\mathbb{B}$, and the strict inequality above implies that the minimizing point cannot lie on the boundary. Hence this point must actually provide an unrestricted local minimum for $g - c$. Since $\delta < 1/n$, this shows that $g \in G_x^n$. The same conclusion holds for every g satisfying $\rho(g, h_1) < \beta/2^N$; so indeed $h_1 \in \operatorname{int} G_x^n$.

Since $\operatorname{int}(G_x^n)$ is open and dense in Lip_C, the set $G_x := \bigcap_{n=1}^{\infty} \operatorname{int}(G_x^n)$ is dense in Lip_C. If $f \in G_x$, then for every n there exists $x_n \in \operatorname{int}(x + n^{-1}\mathbb{B})$ such that $f - c$ attains a local minimum at x_n. That is, $f - c$ attains a local minimum in every neighborhood of x.

Finally, let $Q = \{x_k \,:\, k \in \mathbb{N}\}$ be a countable dense subset of X. Since each G_{x_k}, $k \in \mathbb{N}$, is a dense G_δ by the previous paragraph, the set $G := \bigcap_{k=1}^{\infty} G_{x_k}$ is a dense G_δ in Lip_C. Now consider any $f \in G$. Every open set U in X contains some x_k in Q, and since $f \in G_{x_k}$, the function $f - c$ attains a local minimum at some point in U. Hence the set of points at which $f - c$ attains a local minimum is dense in X. Since f is arbitrary in G, the dense G_δ set G defined here is a subset of the set $G(c)$ defined in the theorem statement. Hence $G(c)$ is residual. □

Corollary 2 ([1])**.** *The following set is residual in* (Lip_C, ρ).

$$\{f \in \mathrm{Lip}_C \,:\, \partial^\circ f(x) = \partial f(x) = C \;\forall x \in X\}.$$

PROOF. For each $c \in C$, define $G(c)$ as in Theorem 1. If $f \in G(c)$, then c lies in $\widehat{\partial} f(x)$ at every point where $f - c$ has a local minimum, and such points are dense in X. Consequently $c \in \partial f(x)$ for every $x \in X$. By taking a countable set $\{c_k : k \in \mathbb{N}\}$ dense in the interior of C we obtain a residual set $G := \bigcap_{k \in \mathbb{N}} G(c_k)$ such that every $f \in G$ obeys $c_k \in \partial f(x)$ for every $k \in \mathbb{N}$ and every $x \in X$. But $\partial f(x)$ is a closed set; so $\partial f(x) \supseteq C$. The reverse inclusion follows immediately from the defining property of Lip_C; so equality holds. Hence $\partial f(x) = C = \mathrm{co}\, C = \mathrm{co}\, \partial f(x) = \partial^\circ f(x)$. $\square$

3 Sparseness of the Dini Subdifferential

We include a simpler proof of the following lemma due to Giles and Sciffer [4] to make our exposition self-contained. Both the statement and the proof remain valid when X is replaced by an arbitrary separable Banach space.

Lemma 3. *For every $f \in \mathrm{Lip}_C$, there exists a dense G_δ subset of X in which every point x obeys $f^\circ(x; v) = f^+(x; v)$ $\forall v \in X$.*

PROOF. For $k \in \mathbb{N}$ and $v \in X$, define

$$D_k^v := \left\{ x \in X : \frac{f(x + t_x v) - f(x)}{t_x} - f^\circ(x; v) > -\frac{1}{k} \text{ for some } t_x \in \left(0, \frac{1}{k}\right) \right\}.$$

Because f is Lipschitz and $-f^\circ(\cdot; v)$ is l.s.c., D_k^v is open. To show that D_k^v is dense, let $x \in X$ and $\varepsilon > 0$ be given. Choose $y \in X$ such that $\|y - x\| < \varepsilon/2$ and $f^\circ(\cdot; v)$ is continuous at y. (This is possible because $f^\circ(\cdot; v)$ is u.s.c., so $f^\circ(\cdot; v)$ is continuous on a residual subset of X—see [7, Exercise 7.43].) Then choose $\delta \in (0, \varepsilon/2)$ so small that

$$f^\circ(y; v) - f^\circ(z; v) > -\frac{1}{2k} \quad \text{whenever } \|z - y\| < \delta. \tag{2}$$

The definition of $f^\circ(y; v)$ provides points z and t_z satisfying $\|z - y\| < \delta$, $0 < t_z < 1/k$, and

$$\frac{f(z + t_z v) - f(z)}{t_z} > f^\circ(y; v) - \frac{1}{2k}.$$

In conjunction with (2), this implies

$$\frac{f(z + t_z v) - f(z)}{t_z} > f^\circ(z; v) - \frac{1}{k}.$$

Thus $z \in D_k^v$; also $\|z-x\| < \varepsilon$; so D_k^v is dense. It follows that each set $D(v) := \bigcap_{k\in\mathbb{N}} D_k^v$ is a dense G_δ in X. Moreover, for every $x \in D(v)$, each $k \in \mathbb{N}$ has some $0 < t_k < 1/k$ for which

$$\frac{f(x+t_k v)-f(x)}{t_k} - f^\circ(x;v) > -\frac{1}{k}.$$

As $k \to \infty$ here, we obtain $f^+(x;v) \geq f^\circ(x;v)$. But $f^\circ(x;v) \geq f^+(x;v)$ is obvious, so equality holds.

Now take a countable dense subset $\{v_n : n \in \mathbb{N}\}$ of X, and consider $D := \bigcap_{n\in\mathbb{N}} D(v_n)$. This is a dense G_δ in X, and for every $x \in D$ we have $f^+(x;v_n) = f^\circ(x;v_n)$ for all $n \in \mathbb{N}$. But both $f^+(x,\cdot)$ and $f^\circ(x,\cdot)$ are Lipschitz, so in fact $f^+(x;\cdot) \equiv f^\circ(x;\cdot)$, as required. □

Theorem 4. *The following subset of* (Lip_C, ρ) *is residual.*

$$G := \left\{ g \in \mathrm{Lip}_C \,:\, \mathrm{dom}(\widehat{\partial} g) \textit{ has first category in } X \right\}.$$

PROOF. We will show $G \supseteq A$, where $A := \{g \in \mathrm{Lip}_C \,:\, \partial^\circ g(x) = C \ \forall x \in X\}$. Recall that A is residual in (Lip_C, ρ), by Corollary 2. Fix $g \in A$. Then $\partial^\circ(-g) \equiv -C$ on X, and the set

$$S := \left\{ x \in X \,:\, (-g)^+(x;\cdot) \equiv (-g)^\circ(x;\cdot) \right\}$$

is residual in X by Lemma 3. For fixed x in S, the definition of the Clarke subgradient gives $(-g)^+(x;v) = (-g)^\circ(x;v) = \sigma_{-C}(v)$; i.e., $g_+(x;v) = -\sigma_C(-v)$, for all $v \in X$. Now since C has nonempty interior, the sublinear function σ_C cannot be dominated by a linear function. Therefore the set

$$\widehat{\partial} g(x) = \{x^* \in X^* \,:\, \langle x^*, v\rangle \leq g_+(x;v) = -\sigma_C(-v) \ \forall v \in X\}$$

must be empty. This shows that $\mathrm{dom}(\widehat{\partial} g)$ is a subset of $X \setminus S$, which implies that $\mathrm{dom}(\widehat{\partial} g)$ has first category in X. Hence $g \in G$, as required. □

Although Rademacher's theorem asserts that every g in Lip_C is differentiable on a set of full Lebesgue measure, each g in the set G of Theorem 4 is differentiable at most on a set of first category in X.

4 Nonangularity

A function $f\colon X \to R$ is called *nonangular at x in direction v* when

$$f_+(x;v) \leq -f_+(x;-v) \text{ and } (-f)_+(x;v) \leq -(-f)_+(x;-v). \tag{3}$$

We call f *nonangular at* x when (3) holds for every $v \in X$. This definition extends a concept well known when $X = \mathbb{R}$: see Bruckner [2].

Theorem 5. *The subset of* Lip_C *consisting of functions nonangular at every point of* X *is a dense* G_δ.

PROOF. Let us fix $v \in X$ and show that

$$\begin{aligned} A(v) &:= \{f \in \mathrm{Lip}_C \,:\, f_+(x;v) > -f_+(x;-v) \text{ for some } x \in X\}\,, \\ B(v) &:= \{f \in \mathrm{Lip}_C \,:\, (-f)_+(x;v) > -(-f)_+(x;-v) \text{ for some } x \in X\}\,, \end{aligned}$$

are F_σ sets of first category in Lip_C. We treat the set $A(v)$, the arguments for $B(v)$ being similar. Since $A(v) = \bigcup_{m=1}^{\infty} A^m$, where

$$A^m := \{f \in \mathrm{Lip}_C : f_+(x;v) > -f_+(x;-v) \text{ for some } x \in m\mathbb{B}\},$$

it suffices to show that each A^m is an F_σ of first category in Lip_C. For fixed $p, q \in \mathbb{Q}$ and $n \in \mathbb{N}$, with $p < q$, let

$$A_{pqn} := \Bigg\{ f \in \mathrm{Lip}_C \;: \text{some } x \in m\mathbb{B} \text{ obeys} \\ \text{both } \frac{f(x+tv) - f(x)}{t} \le p \;\forall\, t \in \left(-\frac{1}{n}, 0\right) \\ \text{and } \frac{f(x+tv) - f(x)}{t} \ge q \;\forall\, t \in \left(0, \frac{1}{n}\right) \Bigg\}.$$

Clearly $A^m = \bigcup_{p,q,n} A_{pqn}$. We check that each A_{pqn} is closed and nowhere dense in Lip_C.

To see that A_{pqn} closed, let $\{f_k\}$ be any sequence in A_{pqn} converging in Lip_C to some function f. We must show that $f \in A_{pqn}$. For each k, there exists $x_k \in m\mathbb{B}$ such that

$$\frac{f_k(x_k + tv) - f_k(x_k)}{t} \le p \;\forall\, t \in \left(-\frac{1}{n}, 0\right), \tag{4}$$

$$\frac{f_k(x_k + tv) - f_k(x_k)}{t} \ge q \;\forall\, t \in \left(0, \frac{1}{n}\right). \tag{5}$$

By passing to a subsequence if necessary (we do not relabel), we may assume that $\langle x_k \rangle$ converges to some point $x \in m\mathbb{B}$. As $k \to \infty$ along this subsequence,

(4) and (5) yield

$$\frac{f(x+tv)-f(x)}{t} \le p \;\forall\, t \in \left(-\frac{1}{n}, 0\right),$$
$$\frac{f(x+tv)-f(x)}{t} \ge q \;\forall\, t \in \left(0, \frac{1}{n}\right).$$

Thus $f \in A_{pqn}$ and A_{pqn} is closed.

To show that A_{pqn} is nowhere dense in Lip_C, it suffices to observe that any differentiable function in Lip_C must lie outside A_{pqn} and that differentiable functions are dense in Lip_C. Indeed, standard mollification methods show that C^∞ functions are dense in Lip_C. For example, following [6, page 409], let $\phi\colon X \to \mathbb{R}$ be a nonnegative function of class C^∞ supported in $\mathbb{B}$ and satisfying $\int_{\mathbb{B}} \phi(x)dx = 1$. For given $f \in \mathrm{Lip}_C$, define $v_k = 1/\int_X \phi(kz)dz$ and let

$$f_k(x) := v_k \int_X f(x-z)\phi(kz)\,dz = v_k \int_X f(z)\phi(k(x-z))\,dz,\;\; k \in \mathbb{N}.$$

Then $f_k \to f$ uniformly on compact subsets of X, and $f_k \in C^\infty$ for each k. Moreover, $f_k'(x) = v_k \int_X f'(x-z)\phi(kz)\,dz$. Since $f'(x-z) \in C$ almost everywhere (Rademacher's Theorem), while C is compact convex and $v_k \int_X \phi(kz)dz = 1$, we have $f_k'(x) \in C$ for every $x \in X$. This implies $f_k \in \mathrm{Lip}_C$.

Now let $\{v_k : k \in \mathbb{N}\}$ be a countable dense subset of X, and put

$$G = \mathrm{Lip}_C \setminus \bigcup_{k\in\mathbb{N}} [A(v_k) \cup B(v_k)].$$

This is a dense G_δ in Lip_C. For every $f \in G$, we have $f_+(x;v_k) \le -f_+(x;-v_k)$ and $(-f)_+(x;v_k) \le -(-f)_+(x;-v_k)$ for all $k \in \mathbb{N}$. But $f_+(x;\cdot)$ and $(-f)_+(x;\cdot)$ are Lipschitzian, so these inequalities extend to every v in X. Hence every $f \in G$ is nonangular in every direction, at every point of X. $\square$

Note that if f is nonangular at x and $x^* \in \widehat{\partial} f(x)$, then $\langle x^*, v\rangle = f_+(x;v)$ for all v, so $\widehat{\partial} f(x) = \{x^*\}$. Thus the set $\widehat{\partial} f(x)$ contains at most one point, and likewise for $\widehat{\partial}(-f)(x)$. Together with Corollary 2 and Theorem 5, this observation establishes the following generic complement to the explicit construction of Sciffer [8].

Corollary 6. *There is a residual subset G of Lip_C such that every $f \in G$ fails to be subdifferentially regular at every point of X, and the same is true for $-f$.*

Remark. When $X = \mathbb{R}$, analogues of Corollary 6 hold in the spaces of bounded continuous functions on X or bounded nondecreasing functions on X using the supremum norm.

Remark. When $X = \mathbb{R}$ and $C = [-1, 1]$, Preiss and Tiśer [5] have shown that the set

$$H := \left\{ f \in \mathrm{Lip}_C \,:\, \limsup_{y \to x} \left| \frac{f(y) - f(x)}{y - x} \right| = 1 \text{ for every } x \in [0, 1] \right\}$$

is residual. Every f in H has $f'(x) \in \{-1, 1\}$ for almost every $x \in \mathbb{R}$. In conjunction with Corollary 2 and [3, Theorem 2.5.1], this implies that there is a residual subset of Lip_C in which every f is such that both

$$D_+ = \{x \in \mathbb{R} \,:\, f'(x) = 1\} \quad \text{and} \quad D_- = \{x \in \mathbb{R} \,:\, f'(x) = -1\}$$

meet every open interval in a set of positive measure.

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