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OSCILLATION AND ω-PRIMITIVES

Abstract

We extend the results of [2], [6] in the case of topological spaces. It is shown that given an upper semicontinuous (USC) function $f : X \to [0, \infty)$ where X is a massive first countable T_1-space satisfying some "neighborhood conditions", there exists $F : X \to [0, \infty)$ whose oscillation equals f everywhere on X (Theorem 2.1). The analogous result holds for USC functions $f : X \to [0, \infty]$ if, in addition, X is a normal space (Theorem 2.4). A special metrizability criterion is established (Theorem 1.1). This is to show, by exhibiting corresponding examples, that the neighborhood conditions and massiveness do not imply that X is Baire or metrizable. Among other related topics, sequences of ω-primitives are discussed.

1 Preliminaries, Basic Definitions and Auxiliary Results

Let X be a topological space and $f : X \to [0, \infty]$ an upper semicontinuous (USC) function. The question we study is this. Does there exist a function $F : X \to \mathbb{R}$ whose oscillation $\omega(F, x)$ equals $f(x)$ at each $x \in X$? If such an F exists we call it an ω-primitive for f (cf. [2]). Note that we define an ω-primitive to be finite. The positive answer to this problem was given in [6] in the case when X is a metric Baire space, and in [2] in the case of so-called massive metric spaces. The notion of a massive space appeared in a natural way in connection with the method of proofs based on Teichmüller-Tukey's lemma.

It should be noted that massive metric spaces form a larger class than dense in itself Baire metric spaces (see Corollaries 1.2 and Examples 1.2, 1.3). We will show that in the case of a massive topological space X satisfying some conditions (which are not strong enough to imply the metrizability) there always exists a nonnegative ω-primitive F for each USC function $f : X \to [0, \infty)$.

Key Words: first countable space, σ-discrete set, Teichmüller-Tukey's lemma, upper semicontinuous function, massive space, oscillation, ω-primitive, quasi-uniform convergence.
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It is also worth mentioning that the problem of existence of a separately continuous ω-primitive $F : X_1 \times X_2 \to \mathbb{R}$ was studied in [9].

We confine our study to spaces dense in themselves. In fact, the ω-primitive F obviously does not exist if f is positive at an isolated point. So in what follows we consider a first countable dense in itself topological space $X = (X, \tau)$, whenever the existence of an ω-primitive is concerned.

For each $x \in X$ denote by

$$\mathcal{N}(x) = \{U_n(x) : n \in \mathbb{N}\} \tag{1}$$

an open countable neighborhood base for the topology τ at x. If $\mathcal{N}(x)$ is fixed for each $x \in X$ then we let

$$\mathcal{N} =: \{\mathcal{N}(x) : x \in X\}. \tag{2}$$

We call $\mathcal{N}$ a neighborhood system on X. Given any nonempty subset A of X for all $n \in \mathbb{N}$ let $U_n(A) := \bigcup_{x \in A} U_n(x)$ and inductively

$$U_n^2(A) := U_n(U_n(A)), \dots, U_n^{m+1}(A) := U_n(U_n^m(A)), \dots.$$

For a singleton we write simply $U_n^m(x) = U_n^m(\{x\})$. So we have

$$U_n^2(x) = \bigcup_{y \in U_n(x)} U_n(y), \dots, U_n^{m+1}(x) = \bigcup_{y \in U_n^m(x)} U_n(y), \dots. \tag{3}$$

The following properties of neighborhood systems $\mathcal{N}$ will be important for us. We call them "neighborhood conditions".

(N1). $\forall n \, \forall x \in X : U_{n+1}(x) \subset U_n(x)$.

(N2). There exists a function $s : \mathbb{N} \to \mathbb{N}$, $s(n) \geq n$, such that

$$\forall x, y \in X \; \forall n : x \in U_{s(n)}(y) \Rightarrow y \in U_n(x).$$

(N3). There exists a function $t : \mathbb{N} \to \mathbb{N}$, $t(n) > n$, such that

$$\forall x \in X \; \forall n : U_{t(n)}^2(x) \subset U_n(x).$$

Remark. We use function notation for the sequences s, t only for technical reasons.

Of course, a given neighborhood system may satisfy only some or none of these properties. Regarding of (N2) we should note that this is a stronger form of a similar property introduced in [1].

(N2*) $\forall y \in X \; \forall n \; \exists k(n, y) \in \mathbb{N} \; \forall x \in X : x \in U_{k(n,y)}(y) \Rightarrow y \in U_n(x)$.

Examples 1.1. (a) Let $\mathbb{R} \times [0, \infty)$ be the Niemytzki plane, $\{U\}$ being its standard neighborhood system. Define $\mathcal{N}$ to be the subfamily of $\{U\}$ when taking into account only neighborhoods corresponding to radii 2^{-n}, $n \in \mathbb{N}$. Then it is easy to check that $\mathcal{N}$ is a neighborhood system on $\mathbb{R} \times [0, \infty)$ satisfying (N1) and (N2) but not (N3).

(b) Let $\mathbb{R}_s$ be the Sorgenfrey line ([3], p. 39). For each $x \in \mathbb{R}$ let $\mathcal{N}(x) = \{[x, x + 2^{-n}) : n \in \mathbb{N}\}$. Then $\mathcal{N} = \{\mathcal{N}(x) : x \in \mathbb{R}\}$ is a neighborhood system on $\mathbb{R}_s$ which satisfies (N1) and (N3) but not (N2).

Proposition 1.1. *Let X be a first countable T_1-space with a neighborhood system satisfying (N1) and (N2). Then for each nonempty set $A \subset X$ its closure* $\operatorname{cl} A$ *can be represented in the form* $\operatorname{cl} A = \bigcap_{n=1}^{\infty} U_n(A) = \bigcap_{n=1}^{\infty} U_n(\operatorname{cl} A)$. *Thus X is a perfect space.*

PROOF. Let $x \in \operatorname{cl} A$, and s be from (N2). Since $U_{s(n)}(x) \cap A \neq \emptyset$ for each n, we may pick a point $a_n \in U_{s(n)}(x) \cap A$. Then by (N2) $x \in U_n(a_n) \subset U_n(A)$. This being true for each n, we get $\operatorname{cl} A \subset \bigcap_{n=1}^{\infty} U_n(A) \subset \bigcap_{n=1}^{\infty} U_n(\operatorname{cl} A)$. Now if $x \notin \operatorname{cl} A$, then $U_{n_0}(x) \cap \operatorname{cl} A = \emptyset$ for some $n_0 \geq 1$. Then an easy argument using (N2) yields $x \notin U_{s(n_0)}(\operatorname{cl} A)$. It follows immediately that $x \notin \bigcap_{n=1}^{\infty} U_n(\operatorname{cl} A)$, which shows that $\bigcap_{n=1}^{\infty} U_n(\operatorname{cl} A) \subset \operatorname{cl} A$. □

Next we will prove a metrizability criterion stated in terms of the above neighborhood conditions. But first we give some notation.

For any set $E \subset X$ and any open covering $\mathcal{A}$ of X put

$$St(E, \mathcal{A}) := \bigcup \{A \in \mathcal{A} : E \cap A \neq \emptyset\}. \tag{4}$$

To prove our metrizability criterion, we use Moore metrization theorem which we cite here.

Lemma 1.1. *([3], Theorem 5.4.2). A topological space X is metrizable if and only if it is a T_0-space and there exists a sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of open coverings of X such that $\forall x \in X\ \forall W(x)\ \exists V(x)\ \exists n :\ St(V(x), \mathcal{A}_n) \subset W(x)$ where $W(x), V(x)$ denote open neighborhoods of x.*

Theorem 1.1. *A first countable T_0-space X is metrizable if and only if there exists a neighborhood system $\mathcal{N}$ satisfying (N1), (N2) and (N3).*

PROOF. If X is metrizable then it is easily checked that the family of all open balls in X of radii $2^{-n}, n \in \mathbb{N}$, forms a neighborhood system satisfying all three conditions. Conversely, assume that a neighborhood system $\mathcal{N}$ satisfies (N1), (N2) and (N3) and let $s, t : \mathbb{N} \to \mathbb{N}$ be the functions from (N2), (N3) respectively. Then it is straightforward, from (3), (N1) and (N3), that

$$\forall x \in X\ \forall n : U^3_{t(t(n))}(x) \subset U_n(x). \tag{5}$$

Define the sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of open coverings of X by

$$\mathcal{A}_n = \{U_n(x) : x \in X\}.$$

Let $x \in X$ and $U_n(x)$ be fixed. Put $j = t(t(n))$ and let $y \in St(U_{s(j)}(x), \mathcal{A}_{s(j)})$. Then by (4) there is $z \in X$ such that $U_{s(j)}(z) \cap U_{s(j)}(x) \neq \emptyset$ and $y \in U_{s(j)}(z)$. Since $s(j) \geq j$ we also have by (N1) $y \in U_j(z)$. Fix any $p \in U_{s(j)}(z) \cap U_{s(j)}(x)$. By (N2) we get $z \in U_j(p)$. By (3) this yields that $z \in U_j^2(x)$. But then from (5) and $y \in U_j(z)$ we obtain that $y \in U_j^3(x) \subset U_n(x)$ which implies

$$St(U_{s(j)}(x), \mathcal{A}_{s(j)}) \subset U_n(x).$$

Thus we have shown that $\forall x \in X\ \forall n : St(U_{s(t(t(n)))}(x), \mathcal{A}_{s(t(t(n)))}) \subset U_n(x)$. So all conditions of Lemma 1.1 are fulfilled. Whence the metrizability of X. □

Our main tool is Teichmüller-Tukey's lemma. It is equivalent to Zorn's lemma, but technically it turned out to be useful and works well enough in our proofs. For convenience of the reader we recall necessary formulations. Suppose we are given a set S and a property P pertaining to subsets of S. We shall say that P is of finite character (on S) if the following holds

$$E \subset S \text{ has the property } P \iff \text{each finite subset } A \subset E \text{ has the property } P.$$

Lemma 1.2. *([3], p.22). Let P be a property of finite character on S. Then each set $A \subset S$ with the property P is contained in a maximal (with respect to the inclusion relation) set $B \subset S$ which also has the property P.*

A maximal set will be called P-maximal (to note that a P-maximal set need not be unique).

In what follows we shall make use of a parameter ΔA characterizing in a way the "inner span" of a set A. Let X be a first countable T_1-space with a fixed neighborhood system $\mathcal{N}$ satisfying (N1). For each nonempty $A \subset X$ we let

$$N(A) = \{n : U_n(x) \cap A = \{x\} \text{ for each } x \in A\}.$$

Then for each nonempty $A \subset X$ put

$$\Delta A = \begin{cases} \sup\{1/n : n \in N(A)\} & \text{if } N(A) \neq \emptyset; \\ 0 & \text{if } N(A) = \emptyset \end{cases}$$

and by definition let $\Delta(\emptyset) = \infty$. Now for each $n \in \mathbb{N}$ define property P_n pertaining to subsets $A \subset X$ by saying A has property $P_n \iff \Delta A \geq \frac{1}{n}$. It is easy to check that P_n is indeed a property of finite character on X and that this property is hereditary. Note that if A has the P_n-property, then of course, it has the P_m-property for each $m \geq n$. Observe that if X is a first countable T_1-space with a fixed neighborhood system $\mathcal{N}$, then f or each $n \in \mathbb{N}$ there exists a P_n-maximal subset of X. In fact, for a singleton $\{x\}$ we obviously have $\Delta\{x\} = 1 \geq 1/n$. Whence by Lemma 1.2 our assertion follows.

A topological space X (or, what amounts to the same, its topology τ) is said to be (cf. [2]):

- σ-discrete at $x \in X$, if there is an open neighborhood W of x which is σ-discrete i.e. $W = \bigcup_{n=1}^{\infty} A_n$ where A_n is a discrete subset of X (The empty set is considered as discrete.);
- massive, if it is not σ-discrete at each $x \in X$.

Given a topological space X, define

$$\Sigma(X) := \{x \in X : X \text{ is } \sigma\text{−discrete at } x\}. \tag{6}$$

For $E \subset X$ we denote by E^d and $\operatorname{Int} E$ respectively the derived set and the interior of E.

The following properties are straightforward.
(a) $\Sigma(X)$ is an open subset of X.
(b) Each massive space is dense in itself; the converse, of course, generally being not true.
(c) If X is a massive space, then $\operatorname{card} X > \aleph_0$.

Proposition 1.2. *A topological space X is massive if and only if each σ-discrete set in X is a boundary set.*

PROOF. Suppose that there is a non-boundary σ-discrete $A \subset X$. Then $\operatorname{Int} A \neq \emptyset$ is contained in $\Sigma(X)$ which is impossible. Conversely, assume $\Sigma(X) \neq \emptyset$. Then each $x \in \Sigma(X)$ has a σ-discrete open neighborhood which obviously cannot be a boundary set. □

Next assertion describes some important properties which will be referred to in the sequel.

Theorem 1.2. *Let Z be a first countable T_1-space with a neighborhood system $\mathcal{N}$ satisfying (N1), (N2). Then the following holds.*

(i) If $A \subset Z$ and $\Delta A > 0$, then $A^d = \emptyset$ (so that A is closed and discrete). If, in addition, $Z = Z^d$, then A is nowhere dense.

(ii) There exists a sequence $\{E_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of Z such that E_1 is P_1-maximal in Z and E_n is P_n-maximal in $Z \setminus (E_1 \cup \ldots \cup E_{n-1})$ for each $n > 1$.

(iii) The set $E := \bigcup_{n=1}^{\infty} E_n$ (where E_n are defined in (ii)) is σ-discrete, $\mathcal{F}_\sigma$ and dense in Z. If, in addition, Z is massive, then E is a boundary set.

(iv) Each σ-discrete set $A \subset Z$ can be written in the form

$$A = \bigcup_{n=1}^{\infty} A_n \tag{7}$$

where A_n are pairwise disjoint and $\Delta A_n > 0$.

Proof. (i) We eliminate the trivial case $A = \emptyset$. So let $A \neq \emptyset$ and $\Delta A \geq 1/n$; i.e., A has the P_n-property. Suppose that $A^d \neq \emptyset$. Then there exists a convergent sequence $\{z_j : j \in \mathbb{N}\}$, $z_j \in A$, $z_{j'} \neq z_{j''}, j' \neq j''$. Put $a \in \lim z_j$. Fix a $z_{j_0} \in U_{s(n)}(a)$. Then by (N2) $a \in U_n(z_{j_0})$. Whence $z_j \in U_n(z_{j_0})$ for almost all j. But this contradicts the P_n-property of A. Thus $A^d = \emptyset$.

Now suppose that $Z = Z^d$. If A were not nowhere dense, there would exist an open neighborhood $U_n(z_0) \in \mathcal{N}(z_0)$, $U_n(z_0) \subset A$. But since $Z = Z^d$, the neighborhood $U_n(z_0)$ should contain infinitely many points of A, which contradicts the P_n-property of A. So A is nowhere dense.

(ii) It is obvious that the required sequence $\{E_n : n \in \mathbb{N}\}$ is easily constructed inductively. Clearly E_n are pairwise disjoint and $\Delta E_n \geq 1/n$.

(iii) Set $E := \bigcup_{n=1}^{\infty} E_n$. By (i) we have that E is σ-discrete and $\mathcal{F}_\sigma$. We claim that E is dense in Z. Suppose the contrary. Then there exists a neighborhood $U_m(p) \in \mathcal{N}$ disjoint from E. It follows by (N2) that $p \notin U_{s(m)}(z)$ for each $z \in E_{s(m)}$ (for otherwise we would get $U_m(p) \cap E_{s(m)} \neq \emptyset$ which is impossible since $U_m(p) \cap E = \emptyset$). This means that the set $E_{s(m)} \cup \{p\} \subset Z \setminus \bigcup_{i=1}^{s(m)-1} E_i$ has the $P_{s(m)}$-property contrary to the fact that its proper subset $E_{s(m)}$ is $P_{s(m)}$-maximal. Thus E is dense in Z. Now if, in addition, Z is massive, then by Proposition 1.2 E is a boundary set.

(iv) Let $A \subset X$ be σ-discrete. To prove our claim, it suffices to consider the case when A is discrete. By (ii), there exists a sequence $\{A_n : n \in \mathbb{N}\}$, $A_n \subset A$, such that A_1 is P_1-maximal in A and A_n is P_n-maximal in $A \setminus (A_1 \cup \ldots \cup A_{n-1})$ for $n > 1$. So we have that A_n are pairwise disjoint and $\Delta A_n \geq 1/n$. Now if we suppose that (7) does not hold, then there is a $z \in A \setminus \bigcup_{n=1}^{\infty} A_n$. Since each A_n is P_n-maximal, there exists $z_n \in A_n \cap U_n(z)$, $z_n \neq z$, and $z_n \neq z_m$, $n \neq m$. Hence we have $z_n \to z \in A$ which contradicts the discreteness of A, thereby proving (7). □

Corollary 1.1. *If X is a separable normal space and the neighborhood system $\mathcal{N}$ satisfies conditions (N1) and (N2), then each σ-discrete subset of X is countable.*

Remark. It is easy to see that claims (ii) and (iv) of Theorem 1.2 remain valid without condition (N2).

Proposition 1.3. *Let X be a first countable dense in itself T_1-space. Then the following holds.*

(i) Each discrete set $E \subset X$ is nowhere dense.

(ii) $\Sigma(X)$ is of first category.

PROOF. (i) Let $E \subset X$ be discrete. Take any $x \in E$. Then there is an open neighborhood V of x such that $V \cap E = \{x\}$. Since X is T_1 and dense in itself, the set $V \setminus \{x\}$ is open and nonempty. Because x is an arbitrary point of E, this obviously implies, that E is nowhere dense.

(ii) Let $x \in \Sigma(X)$. There is an open neighborhood $W(x)$ of x which is σ-discrete; i.e., $W(x) = \bigcup_{n=1}^{\infty} E_n$ where each E_n is discrete. It follows by (i) that $W(x)$ is first category. Since $\Sigma(X) = \bigcup_{x\in\Sigma(X)} W(x)$ we conclude by the Banach category theorem ([7], p. 82) that $\Sigma(X)$ is first category. □

Corollary 1.2. *Let $X = (X, \mathcal{N})$ be a first countable T_1-space. Then the following holds.*

(a) If X is a dense in itself, Baire space, then X is massive.

(b) If X is dense in itself and locally σ-discrete, then X is first category.

Proposition 1.4. *Each nontrivial T_1 topological vector space X is massive.*

PROOF. Let V be any balanced open neighborhood of $0 \in X$. Fix any $x \in V$, $x \neq 0$. Then the mapping $[0,1] \ni t \mapsto tx \in V$ is a continuous injection. Thus each balanced neighborhood V of the zero vector contains a subset $\{tx : t \in [0,1]\}$ homeomorphic with the interval [0,1]. It follows that if V were σ-discrete, then $[0,1]$ would also be σ-discrete; hence first category, which is impossible. □

Corollary 1.3. *Each nontrivial normed space is massive.*

The following examples show that a massive space need not be Baire.

Examples 1.2. Let $1 \leq r < p$. Consider the space l^p with the usual norm $\|\cdot\|_p$. It was shown in [11] that l^r is an $\mathcal{F}_\sigma$ first category dense subset of l^p. The density of l^r implies that it is first category and dense in itself. Whence $(l^r, \|\cdot\|_p)$ is not a Baire space. Finally, the space $(l^r, \|\cdot\|_p)$ is massive in view of Corollary 1.3.

Examples 1.3. Let X be a Banach space. Denote by τ, τ_w respectively the norm topology and the weak topology on X. Since the space (X,τ) is massive and $\tau_w \subset \tau$ we have that (X,τ_w) is also massive. The sets $B_n = \{x \in X : \|x\| \le n\}$ being τ_w-nowhere dense (see [5]), we conclude that the space (X,τ_w) is τ_w-first category and therefore is not a Baire space.

2 Main Results

We adopt the usual convention when dealing with $\pm\infty \in \overline{\mathbb{R}}$:

$$-\infty + \infty = \infty - \infty = 0,\ \infty - (-\infty) = \infty,\ |\pm\infty| = \infty.$$

Let X be a topological space and (Y,d) a metric space. Given $F : X \to Y$, its oscillation at $x \in X$ is defined as (cf. [3], [7])

$$\omega(F,x) = \inf_U \sup_{x',x'' \in U} d(F(x'),F(x''))$$

where the infimum is taken for all neighborhoods U from a neighborhood base at x. It is well known that the oscillation $\omega(F,\cdot) : X \to \overline{\mathbb{R}}$ is an upper semicontinuous (simply USC) nonnegative function.

In the case of real-valued functions $F : X \to \mathbb{R}$ we will also use the equivalent definition of the oscillation expressed via upper and lower Baire functions $M(F,\cdot),\ m(F,\cdot)$ [8]:

$$\omega(F,x) := M(F,x) - m(F,x) \tag{8}$$

where

$$M(F,x) := \inf_U \sup_{z \in U} F(z) \text{ and } m(F,x) := \sup_U \inf_{z \in U} F(z), \tag{9}$$

the infimum and supremum being taken over all neighborhoods U from a neighborhood base at x. It is also well known that $M(F,\cdot),\ m(F,\cdot)$ are respectively upper and lower semicontinuous functions on X.

Now if X is a first countable space with a neighborhood system $\mathcal{N} = \{\mathcal{N}(x) : x \in X\}$ (cf. (1), (2)) and $F : X \to \mathbb{R}$, then we may write as well

$$M(F,x) = \inf_n \sup_{z \in U_n(x)} F(z) = \lim_{n\to\infty} \sup_{z \in U_n(x)} F(z);$$
$$m(F,x) = \sup_n \inf_{z \in U_n(x)} F(z) = \lim_{n\to\infty} \inf_{z \in U_n(x)} F(z).$$

Observe that these relations do not depend on the choice of a neighborhood system $\mathcal{N}$.

Theorem 2.1. *Let X be a massive first countable T_1-space with a neighborhood system $\mathcal{N}$ satisfying (N1) and (N2), and let $f : X \to [0,\infty)$ be a USC function. Then there exists an ω-primitive F for f on X. More specifically, there is a function $F : X \to [0,\infty)$ such that*

(a) $M(F,x) = f(x)$ and $m(F,x) = 0$ for all $x \in X$, so that we have indeed $\omega(F,\cdot) = f$.

(b) $F = f\varphi$, where φ is the characteristic function of a dense boundary set $D \subset X$.

PROOF. Keeping the previous notations unchanged (cf. (1), (2)) we let $G(f)$ be the graph of f equipped with the topology τ_g induced by the product topology in $X \times \mathbb{R}$. It is immediate that for each $z = (x, f(x)) \in G(f)$ the family

$$\mathcal{N}_g(z) = \{W_n(z) := (U_n(x) \times (f(x) - 1/n, f(x) + 1/n)) \cap G(f), n \in \mathbb{N}\}$$

forms an open countable neighborhood system for τ_g at $z = (x, f(x))$, which gives rise to the neighborhood system $\mathcal{N}_g := \{\mathcal{N}_g(z) : z \in G(f)\}$ on $G(f)$. Moreover, it is easy to check that since the neighborhood system $\mathcal{N}$ on X satisfies conditions (N1), (N2), then $\mathcal{N}_g$ satisfies these conditions too, the function s being the same as in (N2) for $\mathcal{N}$. By (ii),(iii) of Theorem 1.2 there exists a sequence $\{Y_n : n \in \mathbb{N}\}$, $Y_n \subset G(f)$, such that

Y_1 is P_1-maximal in $G(f)$;

Y_n is P_n-maximal in $G(f) \setminus \bigcup_{i=1}^{n-1} Y_i$ for $n > 1$;

$Y = \bigcup_{n=1}^{\infty} Y_n$ is dense in $G(f)$.

Let $\pi : X \times \mathbb{R} \to X$ be the natural projection. We claim that each $\pi(Y_n)$ is a discrete subset of X. For if not, there is $x_0 \in (\pi(Y_n))^d \cap \pi(Y_n)$. Then there is a sequence $\{x_k : k \in \mathbb{N}\}, x_k \in \pi(Y_n)$, such that $x_k \neq x_m$ for $k \neq m$, and $x_0 \in \lim x_k$. Let $z_k = (x_k, f(x_k))$. Since $z_k \in Y_n$, we have $z_i \notin W_n(z_j), i \neq j$. Let s be the function from (N2). Since $x_k \to x_0$, there is k_0 such that $x_k \in U_{s(n)}(x_0)$ for $k \geq k_0$. Whence we may write by (N2) that $x_0 \in U_n(x_k)$ for $k \geq k_0$. It then readily follows that we may pick a subsequence $\{x_{k_i} : i \in \mathbb{N}\}$ such that $x_{k_j} \in U_n(x_{k_i})$ whenever $j \geq i$ (e.g., we may construct inductively $\{x_{k_i} : i \in \mathbb{N}\}$ so that $x_{k_{i+1}} \in U_n(x_{k_1}) \cap \cdots \cap U_n(x_{k_i})$). Since $z_{k_i} \notin W_n(z_{k_j}), i \neq j$, we conclude that $|f(x_{k_i}) - f(x_{k_j})| \geq 1/n$ whenever $i \neq j$. This means that f is not locally bounded at x_0, contrary to the assumption that f is USC and nonnegative. This contradiction shows that $\pi(Y_n)$ is discrete. This implies, in view of the density of Y in $G(f)$, that the set $\pi(Y) = \bigcup_{n=1}^{\infty} \pi(Y_n)$ is σ-discrete and dense in X. Consequently, since X is massive, we may conclude that $\pi(Y)$ is a boundary $\mathcal{F}_\sigma$-subset of X (by Theorem 1.2) and that $X \setminus \pi(Y)$ is a massive subspace of X. Therefore again by (ii) and(iii) of Theorem 1.2 there exists a sequence $\{C_n : n \in \mathbb{N}\}$, $C_n \subset X \setminus \pi(Y)$, such that

C_1 is P_1-maximal in $X \setminus \pi(Y)$;

C_n is P_n-maximal in $X \setminus (\pi(Y) \cup \bigcup_{i=1}^{n-1} C_i)$ for $n > 1$,

$C := \bigcup_{n=1}^{\infty} C_n \subset X \setminus \pi(Y)$ is a σ-discrete dense boundary subset of X.

Fix any set $A \subset X \setminus (\pi(Y) \cup C)$. Let $D = A \cup \pi(Y)$ and φ be the characteristic function of D. Put $F = f\varphi$. Since $C \cap D = \emptyset$, we have that $X \setminus D$ is dense in X. Whence $m(F, x) = 0$ for each $x \in X$.

Now take any $z = (x, f(x)) \in G(f)$. Since Y is dense in $G(f)$, there is a sequence $\{z_n : n \in \mathbb{N}\}$, $z_n = (x_n, f(x_n)) \in Y$, $\lim z_n = z$ (note that the case $z_n = z$ for all n is not excluded). Then we get

$$f(x) = \lim f(x_n) = \lim F(x_n) \leq M(F, x) \leq M(f, x) = f(x);$$

i.e., $M(F, x) = f(x)$ for all $x \in X$. It follows immediately by (8), and the equality $m(F, \cdot) = 0$, that $\omega(F, \cdot) = f$; i.e., F is an ω-primitive for f having the required properties. □

Corollary 2.1. *Under assumptions of Theorem 2.1 an ω- primitive $F = f\varphi$ can always be found in at most Baire class 2.*

PROOF. Indeed, letting $A = \emptyset$ or $A = X \setminus (\pi(Y) \cup C)$, the set $D = A \cup \pi(Y)$ will be $\mathcal{F}_\sigma$ or $\mathcal{G}_\delta$ respectively. Therefore the function φ is at most in Baire class 2, what was to be shown. □

Theorem 2.2. *Let X be a massive first countable T_1-space with a neighborhood system $\mathcal{N}$ satisfying (N1) and (N2), and let $f_1, f_2 : X \to [0, \infty)$ be two USC functions such that the set $D = \{x \in X : f_1(x) \neq f_2(x)\}$ is σ-discrete. Then there exist ω-primitives F_1, F_2 for f_1, f_2 respectively such that $\{x \in X : F_1(x) \neq F_2(x)\} = D$.*

PROOF. Let $G(f_i)$ be the graph of f_i; let $z_i = (x, f_i(x))$, $i = 1, 2$. Proceeding much as in Theorem 2.1, we define the neighborhood system $\mathcal{N}_i$ in the space $G(f_i)$ as the collection of all sets of the form

$$W_{i,n}(z_i) = (U_n(x) \times (f_i(x) - 1/n, f_i(x) + 1/n)) \cap G(f_i),$$

i=1,2. If $\Delta A > 0$ for $A \subset X, A \neq \emptyset$, then obviously $\Delta A \geq 1/n$ with some n. Put $A(i) = \{z_i \in G(f_i) : x \in A\}$, i=1,2. Then for each $z_i \in A(i)$ we have $W_{i,n}(z_i) \cap A(i) = \{z_i\}$. Whence $\Delta A(i) \geq 1/n$. By (iv) of Theorem 1.2 we may write $D = \cup_{n=1}^{\infty} A_n$, where A_n are pairwise disjoint, $\Delta A_n > 0$,. Whence (recalling the properties of $A(i)$) we conclude that the sets $D_i := \{z_i : x \in D\}$ are σ-discrete in $G(f_i)$, $i = 1, 2$. By (ii), (iii) of Theorem 1.2 there exist sequences $\{Y_{i,n} : n \in \mathbb{N}\}$, $Y_{i,n} \subset G(f_i)$, $i = 1, 2$, such that

$Y_{i,1}$ is P_1-maximal in $G(f_i) \setminus D_i$;
$Y_{i,n}$ is P_n-maximal in $G(f_i) \setminus (D_i \cup \bigcup_{j=1}^{n-1} Y_{i,j})$ for $n > 1$;
$\pi(Y_{i,n})$ is discrete in X, $, n \geq 1$.

The set $Y_i = \bigcup_{n=1}^{\infty} Y_{i,n}$ is σ-discrete and dense in $G(f_i)$, $i = 1, 2$. Hence $\pi(Y_1)$, $\pi(Y_2)$ are σ-discrete and dense in X. Since X is massive, the set $D \cup \pi(Y_1) \cup \pi(Y_2)$, is dense and boundary in X. Let φ be the characteristic function of that set. Set $F_i = f_i\varphi$, $i = 1, 2$. Clearly $\{x \in X : F_1(x) \neq F_2(x)\} = D$, and the same argument that was used in the proof of Theorem 2.1, shows that $\omega(F_i, \cdot) = f_i$, $i = 1, 2$. □

Theorem 2.3. *Let X be a massive first countable T_1-space with a neighborhood system $\mathcal{N}$ satisfying (N1), (N2). Assume that a sequence $\{f_n : n \in \mathbb{N}\}$ of USC functions $f_n : X \to [0, \infty)$ converges pointwise to a USC function $f : X \to [0, \infty)$. Then there exist functions $F, F_n : X \to [0, \infty)$, $n \in \mathbb{N}$, such that $\omega(F, \cdot) = f$, $\omega(F_n, \cdot) = f_n$, $n \in \mathbb{N}$, and $F_n \to F$ pointwise.*

PROOF. Using the same technique as in the proof of Theorem 2.1, we construct the sets $E \subset G(f)$, $E_n \subset G(f_n)$ σ-discrete and dense (in $G(f), G(f_n)$ respectively) such that $\pi(E)$, $\pi(E_n)$ are σ-discrete, dense and boundary in X. Let φ, φ_n be the characteristic functions of the sets $C = \pi(E) \cup \bigcup_{i=1}^{\infty} \pi(E_i)$ and $C_n = \pi(E) \cup \bigcup_{i=1}^{n} \pi(E_i)$ respectively, $n \in \mathbb{N}$. The sets C, C_n are obviously dense and σ-discrete in X. Since X is massive, $X \setminus C$ and $X \setminus C_n$ are dense in X. Let $F = f\varphi$, $F_n = f_n\varphi_n$. As in the proof of Theorem 2.1, we obtain $\omega(F, \cdot) = f$, $\omega(F_n, \cdot) = f_n$. Finally, it is straightforward from our construction that $F_n \to F$ pointwise. □

Corollary 2.2. . *Let X be a massive metric space and $f : X \to [0, \infty)$ a USC function. Then there exist functions F, $F_n : X \to [0, \infty)$, $n \in \mathbb{N}$, such that $\omega(F, \cdot) = f$, $F_n \to F$, $\omega(F_n, \cdot) \to f$ and each $\omega(F_n, \cdot)$ is continuous.*

PROOF. X being metric, f is the limit of a decreasing sequence $\{f_n : n \in \mathbb{N}\}$ of continuous functions $f_n : X \to [0, \infty)$ (cf. e.g. [3], Problem 1.7.15 (c)). It remains to apply Theorem 2.3 and we are done. □

In the two theorems which follow, a USC function f may take on the value ∞.

Theorem 2.4. *Let X be a massive first countable normal space with a neighborhood system $\mathcal{N}$ satisfying conditions (N1) and (N2) and let $f : X \to [0, \infty]$ be USC. Then there exists an ω-primitive $F : X \to [0, \infty)$ for f. In particular, one can always choose F from at most Baire class 2.*

PROOF. Let $H = \{x \in X : f(x) = \infty\}$. Without loss of generality we may assume $X \setminus H \neq \emptyset \neq \operatorname{Int} H$. Since the open subspace $X \setminus H$ and $f|(X \setminus H)$ satisfy assumptions of Theorem 2.1, there exists an ω-primitive $F_1 : X \setminus H \to [0, \infty)$ for $f|(X \setminus H)$ which is at most in Baire class 2 (see Corollary 2.1).

The open subspace $\operatorname{Int} H$ being massive, according to (ii), (iii) of Theorem 1.2, there exists a sequence $\{E_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of $\operatorname{Int} H$ such that the set $E := \bigcup_{n=1}^{\infty} E_n$ is $\mathcal{F}_\sigma$, dense and boundary in $\operatorname{Int} H$. Let $F_2 : \operatorname{Int} H \to [0, \infty)$ letting $F_2(x) = n$ if $x \in E_n$ and $F_2(x) = 0$ if $x \in \operatorname{Int} H \setminus E$. It is immediate that $\omega(F_2, x) = \infty$ at each $x \in \operatorname{Int} H$; i.e., F_2 is an ω-primitive for $f|\operatorname{Int} H$. It is clear that F_2 is in Baire class 2. It follows from our assumptions and Proposition 1.1 that X is a perfect normal space, therefore there exists a continuous function $g : X \to [0, 1]$ such that $H \setminus \operatorname{Int} H = g^{-1}(0)$. Put

$$F(x) = \begin{cases} F_1(x) + (g(x))^{-1} & \text{if } x \in X \setminus H \\ F_2(x) & \text{if } x \in \operatorname{Int} H \\ 0 & \text{if } x \in H \setminus \operatorname{Int} H. \end{cases}$$

It follows at once from this definition that F is indeed an ω-primitive for f and is at most in Baire class 2. Note that the continuous function $x \mapsto (g(x))^{-1}$ is added to compensate possible boundedness of F_1 near the set $H \setminus \operatorname{Int} H$, thereby ensuring infinite oscillation of F at points of this set (cf. Theorem 2 in [2]). □

We recall that by our definition an ω-primitive is a finite function. The next theorem shows that if we give up on this restriction, then the assumption of normality in Theorem 2.4 becomes redundant.

Theorem 2.5. *Let X be a massive T_1-space with a neighborhood system $\mathcal{N}$ satisfying (N1), (N2) and let $f : X \to [0, \infty]$ be USC. Then there exists a function $F : X \to [0, \infty]$ such that $\omega(F, \cdot) = f$ and the set $\{x \in X : F(x) = \infty\}$ is nowhere dense.*

PROOF. We repeat the proof of Theorem 2.4 (using the same notations) up to the point where the function g appears. This time there is no clear evidence that under our assumptions such a function exists. But since the value ∞ is now admissible for F, we may let

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in X \setminus H \\ F_2(x) & \text{if } x \in \operatorname{Int} H \\ \infty & \text{if } x \in H \setminus \operatorname{Int} H. \end{cases}$$

Obviously it suffices to check the equality $\omega(F, x) = \infty$ at points $x \in H \setminus \operatorname{Int} H$. But this is immediate since $F|(H \setminus \operatorname{Int} H) = \infty$ whereas $m(F_1, \cdot) = 0$, and $m(F_2, \cdot) = 0$. □

In the next theorem stated for Baire spaces, massiveness and conditions (N1), (N2) may be dropped. But in that case we know nothing about possible Baire class of an ω-primitive.

Lemma 2.1 ([4], p. 331)**.** *Given a first countable dense in itself T_0-space X, there exists a dense boundary set $A \subset X$.*

Theorem 2.6. *Let X be a first countable T_1 dense in itself Baire space and a USC function $f : X \to [0, \infty)$. Then there exists an ω-primitive F for f.*

PROOF. Our proof is similar to the proof of Theorem 5 in [6]. Denote $C(f) = \{x \in X : \ f \text{ is continuous at } x\}$. Since the T_1-space X is Baire and $X = X^d$, we have that the $\mathcal{G}_\delta$-set $C(f)$ is a T_1-subspace dense in X. By Lemma 2.1 there exists a set $A \subset C(f)$ dense and boundary in $C(f)$ and hence in X. Define $F = f\varphi$ where φ is the characteristic function of $X \setminus A$. We claim that F is an ω-primitive for f. Since $X \setminus A$ is dense in X, we have $m(F, \cdot) = 0$ on X. So it remains to show (cf. (8)) that $M(F, \cdot) = f$ on X.

First let $x \in A$. The set $C(f) \setminus A$ being dense in $C(f)$, there is a sequence $\{x_n : n \in \mathbb{N}\}, \ x_n \in X \setminus A, \ x_n \to x$. Since f agrees with F off A and is continuous at x, we have

$$f(x) = \lim f(x_n) = \lim F(x_n) \leq M(F, x) \leq M(f, x) = f(x).$$

Now let $x \in X \setminus A$. Since f is USC and $F(x) = f(x)$, we may write at once $f(x) \leq M(F, x) \leq M(f, x) = f(x)$ We have thus shown that $m(F, \cdot) = 0$ and $M(F, \cdot) = f$. Whence $\omega(F, \cdot) = f$, what was to be proved. □

Remark. As we already know from (b) in Examples 1.1, the Sorgenfrey line $\mathbb{R}_s$ does not have any neighborhood system satisfying (N2). Therefore Theorem 2.1 cannot be applied to $\mathbb{R}_s$. On the other hand all assumptions of Theorem 2.6 are satisfied for $X = \mathbb{R}_s$. Whence each USC function $f : \mathbb{R}_s \to [0, \infty)$ has an ω-primitive. This shows by the way that condition (N2) is but sufficient for an ω-primitive to exist.

We shall complete Section 2 by giving some observations concerning the existence of ω-primitives in the case when comparable topologies are involved. Let τ_1, τ_2 be two topologies on the same set X. Given $F : X \to \mathbb{R}$, denote by $M_i(F, \cdot), \ m_i(F, \cdot)$ the Baire functions (9) computed in the space (X, τ_i), and let $\omega_i(F, \cdot) = M_i(F, \cdot) - m_i(F, \cdot), \ i = 1, 2$.
It is easy to see that

$$\tau_1 \subset \tau_2 \Rightarrow M_2(F, \cdot) \leq M_1(F, \cdot), \ m_1(F, \cdot) \leq m_2(F, \cdot), \tag{10}$$

so that for each $F : X \to \mathbb{R}$ we have $\tau_1 \subset \tau_2 \Rightarrow \omega_2(F, \cdot) \leq \omega_1(F, \cdot)$.

Theorem 2.7. *Let (X, τ_1) be dense in itself and let $f : X \to [0, \infty)$ a τ_1-USC function (i.e. f is USC for the topology τ_1). Assume that there is a topology $\tau_2 \supset \tau_1$ on X such that (X, τ_2) is a first countable dense in itself massive T_1-space having a neighborhood system $\mathcal{N}$ satisfying conditions (N1) and (N2). Then there exists an ω_1-primitive $F : X \to [0, \infty)$ for f on (X, τ_1).*

Proof. According to Theorem 2.1 there exists an ω_2-primitive F for f on (X, τ_2) such that $0 \le F \le f,\ M_2(F, x) = f(x),\ m_2(F, x) = 0$ for each $x \in X$. By (10) this yields that for each $x \in X$ we have

$$0 \le m_1(F, x) \le m_2(F, x) = 0,\ f(x) = M_2(F, x) \le M_1(F, x) \le M_1(f, x) = f(x)$$

Thus F is an ω_1-primitive for f as well. □

Remark. It is worth noting that in Theorem 2.7 we make no assumptions on the space (X, τ_1) except that it is dense in itself.

Corollary 2.3. *Let $X = \mathbb{R} \times [0, \infty)$ and let τ_N be the Niemytzki topology on X. Let $\tau \subset \tau_N$ be any topology on X such that (X, τ) be dense in itself. Then given any τ-USC function $f : X \to [0, \infty)$ there exists an ω_τ-primitive $F : X \to [0, \infty)$ for f, $F \le f$, such that we have $\omega_\tau(F, x) = \omega_{\tau_N}(F, x) = f(x)$ for $x \in X$.*

Corollary 2.4. *Let X be a Banach space. Denote by τ, τ_w respectively the norm topology and the weak topology on X (cf. Example 1.3). Then each τ_w-USC function $f : X \to [0, \infty)$ has an ω_{τ_w}-primitive $F : X \to [0, \infty)$, $F \le f$, so that we have $\omega_{\tau_w}(F, x) = \omega_\tau(F, x) = f(x)$ for $x \in X$.*

Corollary 2.5. *Let τ, τ_L, τ_R be respectively: usual, left and right topologies on $X = \mathbb{R}$. Then given any τ_L-USC (τ_R-USC) function $f : X \to [0, \infty)$, there exists an ω_{τ_L}-primitive (an ω_{τ_R}-primitive) $F : X \to [0, \infty)$ for f, so that we have $\omega_{\tau_L}(F, x) = \omega_\tau(F, x) = f(x)$ (respectively $\omega_{\tau_R}(F, x) = \omega_\tau(F, x) = f(x)$) for all $x \in X$.*

3 Oscillation and Quasi-Uniform Convergence

Let X be a topological space and (Y, d) a metric space. By $\mathcal{F}(X, Y)$ we denote the set of all mappings from X to Y. A net $\{g_j : j \in J\}$ of mappings $g_j : X \to Y$ is said to be quasi-uniformly convergent to $g : X \to Y$ if (cf. [10])

$$\forall x \in X\ \forall \varepsilon > 0\ \exists j_0 \in J\ \forall j \ge j_0\ \exists U(x)\ \forall x' \in U(x) :\ d(f(x'), f_j(x')) < \varepsilon \quad (11)$$

where, as before, $U(x)$ stands for an open neighborhood of x. This type of convergence preserves continuity. It is known that there exists a uniformizable topology τ_{qu} on $\mathcal{F}(X,Y)$ compatible with quasi-uniform convergence. Thus a net $\{g_j : j \in J\}$ is τ_{qu}-convergent to g if and only if g is the quasi-uniform limit of $\{g_j : j \in J\}$ [10]. Moreover, on the class $C(X,Y)$ of all continuous mappings from X to Y the topology of the pointwise convergence coincides with τ_{qu}.

Theorem 3.1. *Let X be a topological space and (Y,d) a metric space. If a net $\{g_j : j \in J\}$ of mappings $g_j : X \to Y$ is quasi-uniformly convergent to a mapping $g : X \to Y$, then the net of oscillations $\{\omega(g_j,\cdot) : j \in J\}$ is quasi-uniformly convergent to $\omega(g,\cdot)$.*

PROOF. Let $x_0 \in X$ and $\varepsilon > 0$ be fixed; then by (11) there exists a corresponding j_0. Fix a $j \geq j_0$ and a neighborhood $U_j = U_j(x_0)$ such that

$$d(g(x), g_j(x)) < \varepsilon/4 \text{ for all } x \in U_j. \tag{12}$$

Let $x \in U_j$ and $\omega(g,x) < \infty$. Then we can choose a neighborhood $U(x)$ of x, $U(x) \subset U_j$, such that $d(g(x'), g(x'')) < \omega(g,x) + \varepsilon/4$ for all $x', x'' \in U(x)$. This implies $d(g_j(x'), g_j(x'')) < \omega(g,x) + 3\varepsilon/4$ for all $x', x'' \in U(x)$, so we have $\omega(g_j,x) < \omega(g,x) + \varepsilon$. Whence $\omega(g_j,x) < \infty$. Then in a similar way, using (12), we obtain $\omega(g,x) < \omega(g_j,x) + \varepsilon$. Thus we have shown that $|\omega(g,x) - \omega(g_j,x)| < \varepsilon$ for each $x \in U_j$ whenever $\omega(g,x) < \infty$.

Now let $x \in U_j$ and $\omega(g,x) = \infty$. Then

$$\sup_{x',x'' \in W(x)} d(g(x'), g(x'')) > n + \varepsilon$$

for each $n \in \mathbb{N}$ and each neighborhood $W(x)$ of x. Hence for each $W(x) \subset U_j$ and for each n we have

$$\sup_{x',x'' \in W(x)} d(g_j(x'), g_j(x'')) > n$$

which yields $\omega(g_j,x) = \infty$. We have thus shown that $|\omega(g,x) - \omega(g_j,x)| < \varepsilon$ for each $x \in U_j$. □

As an immediate consequence we get

Corollary 3.1. *Let X be a topological space and (Y,d) a metric space. Then the classes*

$$C_\omega := \{g \in \mathcal{F}(X,Y) : \ \omega(g,\cdot) \text{ is continuous}\}$$

and

$$[f] := \{g \in \mathcal{F}(X,Y) : \ \omega(g,\cdot) = f\},$$

where $f : X \to [0, \infty)$ *is any USC-function, are closed subsets in the space* $(\mathcal{F}(X, Y), \tau_{qu})$.

Corollary 3.2. *Let* X *be a topological space. Then*

$$\mathcal{F}_\omega := \{f \in \mathcal{F}(X, [0, \infty)) : \ \omega(f, \cdot) = f\}$$

is a closed subset of $(\mathcal{F}(X, [0, \infty)), \tau_{qu})$.

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