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Pamela B. Pierce, Department of Mathematical Sciences, The College of Wooster, Wooster, OH 44691, USA. e-mail: ppierce@acs.wooster.edu

Daniel Waterman, Department of Mathematics, Florida Atlantic University, Home Address: 7739 Majestic Palm Dr., Boynton Beach, FL 33437, USA. e-mail: fourier@earthlink.net

A Δ_2-EQUIVALENT CONDITION

Abstract

We give a condition which is shown to be equivalent to the Δ_2 condition and use it to prove a well-known result of Musielak and Orlicz.

Let φ be a continuous increasing function defined on $[0,\infty)$ with $\varphi(0) = 0, \varphi(x) > 0$ for $x > 0$. Let f be any function defined on the interval $I = [a,b]$, and let $P = \{I_n\}$ be a partition of I. For any interval $[\alpha,\beta]$, let $f([\alpha,\beta]) = f(\beta) - f(\alpha)$. The quantity

$$V_\varphi(f) = V_\varphi(f;I) = \sup_P \sum_{i=1}^{m} \varphi\left(|f(I_i)|\right),$$

where the supremum is taken over all partitions P of I, is called the total φ-variation of f on I. If $V_\varphi(f)$ is finite, then f is said to be of bounded φ-variation on I. It is easy to see that this is equivalent to the requirement that the infinite sum $\sum_{n=1}^{\infty} \varphi(|f(I_n)|)$ be finite whenever $\{I_n\}_{n=1}^{\infty}$ is a collection of *nonoverlapping* intervals in $[a,b]$. The class ΦBV is defined to be the set of all functions f of bounded φ-variation. This class was first considered in less generality by L.C. Young [Y]. Wiener introduced the notion for $\varphi(x) = x^p, p > 1$, and this was developed further by L.C. Young and E.R. Love [LY]. An interesting application of φ-variation to Fourier series, which generalizes the earlier results for p-variation, is due to Salem [S].

We begin by proving the equivalence of the definitions of ΦBV given above. Suppose there exists a sequence $\{I_n\}$ of non-overlapping intervals such that $\sum \varphi(|f(I_n)|)$ diverges. Then $\{I_n\}_{n=1}^{N}$ and the intervals complementary to $\bigcup_1^N I_n$ form a partition J_n such that $\sum \varphi(|f(J_n)|) \geq \sum_{i=1}^{N} \varphi(|f(I_n)|)$, and this last sum can be made as large as we please.

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The opposite implication is less obvious. Clearly, each definition implies that f is bounded; it is almost as clear that f is regulated, i.e., has only simple discontinuities, but we shall not need this fact.

Using ΦBV in the original sense, we note that if $a < b < c$ and f is in ΦBV on $[a,b]$ and $[b,c]$, then f is in ΦBV on $[a,c]$. Suppose $\{I_n\}$ is a partition of $[a,c]$ and b is an endpoint of some $\{I_n\}$. Then

$$\sum \varphi(|f(I_n)|) \leq V_\varphi(f;[a,b]) + V_\varphi(f;[b,c]).$$

Otherwise, if b is in the interior of an interval I_n,

$$\sum \varphi(|f(I_n)|) \leq V_\varphi(f;[a,b]) + V_\varphi(f;[b,c]) + \varphi(2\sup|f(x)|).$$

If we now use the standard bisection argument, we may show that if $f \notin \Phi BV$ on I, then there is an $x_0 \in I$ such that, on one side of x_0, $f \notin \Phi BV$ on any interval terminating at x_0. Let $\{I_n^1\}_1^{N_1}$ be a partition of such an interval, $[\alpha, x_0]$, such that $\sum_1^{N_1} \varphi(|f(I_n^1)|) > 2\sup|f(x)| + 1$. Then $\sum_1^{N_1-1} \varphi(|f(I_n^1)|) > 1$. Repeat this process on the interval I_{N_1}. We may find a partition of I_{N_1}, $\{I_n^2\}_1^{N_2}$ such that $\sum_1^{N_2-1} \varphi(|f(I_n^2)|) > 1$. Continuing inductively and enumerating the intervals $\{\{I_n^k\}_{n=1}^{N_k-1}\}_{k=1}^\infty$ from left to right to form $\{I_n\}_1^\infty$, we see that $\sum \varphi(|f(I_n)|)$ diverges.

The class ΦBV is not, in general, a vector space, and hence one often considers the vector space ΦBV^*, which we define to be

$$\Phi BV^* = \{f | kf \in \Phi BV \text{ for some constant } k \neq 0\}.$$

Clearly $\Phi BV \subseteq \Phi BV^*$. The following conditions are usually placed on the function φ:

1. φ is convex
2. $\varphi(x)/x \to 0$ as $x \to 0$
3. $\varphi(x)/x \to \infty$ as $x \to \infty$.

The latter two conditions ensure that ΦBV is a generalization of BV while the first makes computation more amenable.

A function φ is said to satisfy the condition Δ_2 (often called Δ_2 for small values) if there exists an $a > 0$ and a $\delta > 0$ such that $\dfrac{\varphi(2x)}{\varphi(x)} \leq \delta \quad \text{for } x \in (0,a]$. The following result is due to Musielak and Orlicz [MO]:

Theorem (Musielak and Orlicz)**.** *The class ΦBV is linear (i.e. $\Phi BV = \Phi BV^*$) if and only if Δ_2 is satisfied.*

Hence it is usual to assume that φ satisfies condition Δ_2 and it is useful to have equivalent conditions which may, at times, be more obviously applicable. One such condition is:

A Known Δ_2-Equivalent Condition. *For any $c > 1$, there exist $\delta > 0$ and $a > 0$ such that $\dfrac{\varphi(cx)}{\varphi(x)} \leq \delta$ for $x \in (0, a]$.*

For $c = 2$ this reduces to Δ_2. Given Δ_2, choose n so that $c < 2^n$ and apply the Δ_2 condition n times, yielding the desired result on the interval $[0, a/2^{n-1}]$. A suitable δ in this condition is then the n^{th} power of the constant in the condition Δ_2.

Another equivalent formulation which we have found useful is given in the following result. After proving the equivalence of the two formulations, we will use the new one to prove the theorem of Musielak and Orlicz.

Theorem. *A function φ satisfies the condition Δ_2 iff*

$$\Delta_\Sigma : \quad \textit{for any } k \in (0,1) \textit{ and } x_n \searrow 0, \quad \sum_{n=1}^{\infty} \frac{\varphi(kx_n)}{\varphi(x_n)} = \infty.$$

PROOF. We first show that $\Delta_2 \Longrightarrow \Delta_\Sigma$.

Choose $k \in (0,1)$ and an arbitrary sequence $\{x_n\}$ such that $x_n \searrow 0$. The Δ_2-equivalent condition above implies that there is a $\delta > 0$ and an $\alpha > 0$ such that $\dfrac{\varphi(x/k)}{\varphi(x)} \leq \delta$ for $x \in (0, \alpha]$ or $\dfrac{\varphi(kx)}{\varphi(x)} \geq \delta^{-1}$ for $x \in (0, \alpha]$. Thus there is an $N > 0$ such that $\sum_{n=1}^{\infty} \dfrac{\varphi(kx_n)}{\varphi(x_n)} \geq \sum_{n=N}^{\infty} \delta^{-1} = \infty$, which is condition Δ_Σ.

We now show that $\Delta_\Sigma \Longrightarrow \Delta_2$.
If φ does not satisfy Δ_2 then, for any $\delta > 0$, we may choose $x > 0$, arbitrarily small, such that $\dfrac{\varphi(x/2)}{\varphi(x)} < \delta$.
Choose a sequence $\{\delta_j\}$, $\delta_j \searrow 0$ such that $\sum_{j=1}^{\infty} \delta_j < \infty$. We now choose a sequence $\{c_n\}$ in the following manner: For $j = 1$ we choose c_1 such that $\varphi(c_1/2)/\varphi(c_1) \leq \delta_1$. For $j = 2$ we choose $c_2 < \min\{c_1, 1/2\}$ and such that $\varphi(c_2/2)/\varphi(c_2) \leq \delta_2$. We proceed inductively so that at the n^{th} stage we choose $c_n < \min\{c_{n-1}, 1/2^{n-1}\}$ and such that $\varphi(c_n/2)/\varphi(c_n) \leq \delta_n$. We then have $\sum_{j=1}^{\infty} \dfrac{\varphi(c_j/2)}{\varphi(c_j)} \leq \sum_{j=1}^{\infty} \delta_j < \infty$.

Thus φ does not satisfy Δ_Σ, and we have shown the two conditions to be equivalent.

We now use this equivalent condition to give an alternative proof of the result of Musielak and Orlicz.

PROOF. We shall show first that Δ_Σ implies that $\Phi BV^* \subseteq \Phi BV$. This will be accomplished if we show that for $c \in (0,1)$ and $\overline{\varphi}(x) = \varphi(cx)$, we have $\overline{\Phi}BV \subseteq \Phi BV$. Now Δ_Σ implies that $\sum \dfrac{\overline{\varphi}(x_j)}{\varphi(x_j)} = \infty$, for $x_j \searrow 0$. If

$$\liminf_{x \searrow 0} \frac{\overline{\varphi}(x)}{\varphi(x)} = 0,$$

then there exists a sequence $\{x_j\} \searrow 0$ such that $\sum \dfrac{\overline{\varphi}(x_j)}{\varphi(x_j)} < \infty$, which contradicts Δ_Σ. Thus $1 \geq \liminf_{x\searrow 0} \dfrac{\overline{\varphi}(x)}{\varphi(x)} = \delta > 0$, implying that $(2/\delta)\overline{\varphi}(x) > \varphi(x)$ for small x. Thus, for a bounded f, there is a finite M such that, for any interval I, $M\overline{\varphi}(|f(I)|) \geq \varphi(|f(I)|)$, implying that $MV_{\overline{\varphi}}(f) \geq V_\varphi(f)$ and so $\overline{\Phi}BV \subseteq \Phi BV$.

We have noted that $f \in \Phi BV$ if and only if, for every sequence of nonoverlapping intervals, $\{I_n\}, \sum \varphi(|f(I_n)|)$ converges. We shall use this fact to show that $\Phi BV^* \subseteq \Phi BV$ implies Δ_Σ. We show, in particular, that if, for any $C > 1$, $\sum \varphi(x_n) < \infty$ implies $\sum \varphi(Cx_n) < \infty$ for sequences $\{x_n\} \searrow 0$, then Δ_Σ holds.

Suppose Δ_Σ does not hold and let $\overline{\varphi}(x)$ denote $\varphi(Cx)$. Then there is a $C > 1$ and a sequence $\{x_n\} \searrow 0$ such that $\dfrac{\varphi(x_n)}{\overline{\varphi}(x_n)} \searrow 0$. By choosing subsequences, we may determine $\{x_n\}$ so that $\varphi(x_n) < \dfrac{1}{n^2}$ and $\dfrac{\varphi(x_n)}{\overline{\varphi}(x_n)} < \dfrac{1}{n}$. Choose an integer k_n so that $\frac{1}{n^2} < k_n\varphi(x_n) \leq \dfrac{2}{n^2}$. We define a sequence $\{y_n\}$ as follows: the first k_1 terms are equal to x_1, the next k_2 terms are equal to x_2, and so on. Then we have

$$\sum \varphi(y_n) = \sum k_n \varphi(x_n) < \sum \frac{2}{n^2} < \infty$$

and

$$\sum \overline{\varphi}(y_n) = \sum k_n \overline{\varphi}(x_n) \geq \sum k_n n \varphi(x_n) \geq \sum n \frac{1}{n^2} = \infty,$$

which establishes the desired result. □

We note that the argument we have just used is patterned after that of Birnbaum and Orlicz[BO], as was the argument of Musielak and Orlicz.

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