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SETWISE QUASICONTINUITY AND Π-RELATED TOPOLOGIES

Abstract

A function is *quasicontinuous* if inverse images of open sets are semi-open. We generalize this definition: a collection of functions is **setwise quasicontinuous** if finite intersections of inverse images of open sets by functions in the collection are semi-open (so a function is quasicontinuous if and only if its singleton is a setwise quasicontinuous set). Two topologies on the same space are Π-related if each nonempty open set (in each) has non-empty interior with respect to the other. This paper demonstrates that a dynamical system is setwise quasicontinuous if and only if the original topology can be strengthened to one which is Π-related to it, and with respect to which each of the functions is continuous to the range space.

Further, the set of iterates $\{1_X, f, f\circ f, \dots\}$ of a self-map $f : X \to X$, is setwise quasicontinuous if and only if the topology can be extended to a Π-related one, so that each iterate is continuous from the new space to the new space.

We present a quasicontinuous function on the unit interval which is discontinuous on a dense subset of the interval; and show that conjugacies of dynamical systems via quasicontinuous bijections preserve much of the desired structure of the systems.

1 Introduction

This paper is largely motivated by two which appeared far apart in time, and apparently far apart in subject matter. One of them, [1], appeared in 1992, and showed that for continuous functions, the requirement of sensitive dependence

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on initial conditions in Devaney's definition of chaotic behavior was redundant. But many of the functions studied in dynamical systems are not continuous. In particular, two well known dynamical systems reviewed later in this paper, the baker's transformation and the tent map, are conjugate to each other, and the tent map is continuous, but the baker's transformation is not.

The study of topological dynamics has largely focused on the study of continuous systems, with occasional discontinuous functions included as "special cases". But in [3], the authors investigated a generalization of continuity (quasicontinuity) for which many of the standard theorems of continuous topological dynamics hold, including the result in [1]. The idea of quasicontinuity, was popularized by Levine in [12], in 1963 (where it was called semicontinuity), and forms the second motivation for this paper. But this idea goes at least as far back as 1932, when S. Kempisty called it quasi-continuity, in [11]. Both names are still used; in this article we will use "quasicontinuity" in order to avoid confusion between Levine's semicontinuity and the more familiar upper- and lower- semicontinuities. We are indebted to Marc Frantz and David Rose for bringing the history of quasicontinuity to our attention, and to them as well as Mel Henriksen, Mario Martelli, and Aaron Todd, for useful discussion of what follows and related material.

Although we will concern ourselves with compositions of quasicontinuous functions and with quasicontinuous dynamical systems, these functions have been studied in their own right. A frequent topic of study, for example, involves the sums and products of quasicontinuous or strongly-quasicontinuous functions, and the resulting weaker cliquish functions (see [2], [7], [8], and [13]). Ewert [5] examined sets of points which can be quasicontinuity points for some function.

Mimna [14] recently investigated omega-limit sets and the dense mapping property (DMP) for quasicontinuous functions $f : \mathbb{R} \to \mathbb{R}$. We claim that the extension from continuous systems to quasicontinuous systems is a natural one; moreover it can easily build on the topological and analytical groundwork which has already been laid.

We attempt to shed light on the structure of quasicontinuous functions by describing this structure in two different languages. Firstly, we draw connections between quasicontinuity and Π-related topologies of the underlying space (two topologies on the same space are Π-related if every non empty open set of each topology contains a non empty set open in the other). This structure will allow us to describe families of functions which remain quasicontinuous under composition, addition, and multiplication. But also, we describe quasicontinuity as driven by the dynamical system notion of topological transitivity, which concerns the intersections of inverse images of open sets. The aim of this paper is to tie together these two notions: we show that a function is quasicontinuous if and only if inverse images of open sets form a topology which is Π-related to the original topology. We then use this characterization to investigate the results of combining such functions.

Section 2 provides basic definitions and examples of quasicontinuity. Π-related topologies and their role in the characterization of quasicontinuity are discussed in section 3. The purpose of that section is to relate these notions (in 3.3 and 3.4) and to demonstrate how investigating different topologies on the same space helps us to determine whether a given dynamical system is quasicontinuous. Setwise quasicontinuity is similarly characterized in section 3, and there it is shown that the standard ways of combining setwise quasicontinuous functions result in quasicontinuous functions.

The last section gives an example of a function which is quasicontinuous everywhere on $[0, 1]$, but discontinuous on a dense subset. This function acts as a conjugacy map between the baker's transformation and the tent map, and so we present some results on systems which are equivalent via quasicontinuous conjugacies.

2 Quasicontinuous Dynamical Systems

We always assume that we have a main topology τ_X on X, and one τ_Y, on Y. But we will use help from other topologies on X and Y, which are introduced as needed. We denote the closure of A by $\mathrm{cl}(A)$, and its interior by $\mathrm{int}(A)$.

Definition 2.1. A subset A of X is **semi-open** if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$. Given a function $f : X \to Y$, we say that $f : (X, \tau_X) \to (Y, \tau_Y)$ is **quasicontinuous** if and only if for each $T \in \tau_Y$, $f^{-1}[T]$ is semi-open.

A set S of functions from X to Y, is **separately quasicontinuous** from (X, τ_X) to (Y, τ_Y) if and only if each $f \in S$ is quasicontinuous.

A set S of functions is **setwise quasicontinuous** from (X, τ_X) to (Y, τ_Y) if for each $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \tau_Y$, $f_1, \ldots, f_n \in S$, $\bigcap_1^n f_i^{-1}[T_i]$ is semi-open.

Example 2.2. The baker's function $b : [0, 1] \to [0, 1]$ (with the usual topology) defined by $b(x) = 2x$ for $0 \le x < \frac{1}{2}$ and $2x - 1$ for $\frac{1}{2} \le x \le 1$ is quasicontinuous. If we denote by b^n the n-th iterate of b—so that $b^3(x) = b(b(b(x)))$—then the set $\{b^n\}_{n=0}^{\infty}$ is setwise quasicontinuous. These assertions are easier to show after we prove some general results, and are further discussed in 4.2 and throughout section 5.

Example 2.3. The baker's transformation sends the point of discontinuity ($x = \frac{1}{2}$) to 0, but $\tilde{b}$, the baker's transformation except that $\tilde{b}(\frac{1}{2}) = 1$ is also quasicontinuous. The set of functions $\{b, \tilde{b}\}$, defined from $[0,1]$ to itself with the usual topology, are separately quasicontinuous, but not setwise quasicontinuous. Note for example that $b^{-1}[0, \frac{1}{8}) \cap \tilde{b}^{-1}(\frac{7}{8}, 1] = \{\frac{1}{2}\}$, which is nonempty but has empty interior (and so is not semi-open).

In [3], the authors define the quasicontinuity of $f : X \to X$ by the property that, for all non-empty, open sets $U, V \subset X$, the intersection $f^{-1}(U) \cap V$ is either empty or has nonempty interior. As we state in 2.6 (c), this definition is equivalent to the one found in this paper. They also say (f, X) is a **quasicontinuous system** if $f^n : X \to X$ is quasicontinuous for all $n > 0$. It is equivalent to say that $\{f^n\}_{n=1}^{\infty}$ is separately quasicontinuous (as in 2.1). This alternative definition was motivated by **topological transitivity**: $f : X \to X$ is topologically transitive if for all non-empty, open sets $U, V \subset X$, there is some positive integer n such that $f^{-n}(U) \cap V \neq \emptyset$.

The purpose of [3] is to show that many nice theorems about continuous discrete dynamical systems hold for quasicontinuous systems. For example, they prove the following two theorems, which are well-known for continuous systems:

Theorem 2.4. *If (f, X) is a quasicontinuous system and X is a compact topological space, then (f, X) is topologically transitive if and only if it has a dense orbit—that is, if and only if there is some $x \in X$ with $\{f^n(x)\}_{n=1}^{\infty}$ dense in X.*

Theorem 2.5. *Suppose (f, X) is a quasicontinuous system and X is a compact metrizable topological space; if (f, X) is topologically transitive and has dense periodic points, then the system has sensitive dependence on initial conditions—that is, there is some $\delta > 0$ so that for every non-empty open $U \subset X$, there is some $n > 0$ and some $x, y \in U$ with $d(f^n(x), f^n(y)) > \delta$.*

In the continuous case, the first of these is a folk-theorem and the second is due independently to Banks *et al* [1] and Glasner and Weiss [6].

The next lemma gives alternate ways of determining whether a given function—or system of functions—is quasicontinuous. For its proof, note that if A is any set and T is open, then $\mathrm{int}(A) \cap T = \mathrm{int}(A) \cap \mathrm{int}(T) = \mathrm{int}(A \cap T)$; also, if $\mathrm{cl}(A) \cap T \neq \emptyset$, then $A \cap T \neq \emptyset$. Certainly, the empty set is semi-open.

Lemma 2.6. *(a) A is semi-open if and only if whenever T is open and $A \cap T \neq \emptyset$, then $\mathrm{int}(A) \cap T \neq \emptyset$.*

(b) If A is semi-open and T is open, then $A \cap T$ is semi-open.

(c) Let $f : (X, \tau_X) \to (Y, \tau_Y)$. Then the following are equivalent:

(i) f is quasicontinuous,

(ii) $\{f\}$ *is setwise quasicontinuous.*

(iii) for each $T \in \tau_Y, U \in \tau_X$, *if* $f^{-1}[T] \cap U \neq \emptyset$, *then* $\operatorname{int} f^{-1}[T] \cap U \neq \emptyset$.

(d) If S *is setwise quasicontinuous from* (X, τ_X) *to* (Y, τ_Y), $T_1, \ldots, T_n \in \tau_X$, $f_1, \ldots, f_n \in S$, $n \in \mathbb{N}$, *and* $\bigcap_1^n f_i^{-1}[T_i] \neq \emptyset$, *then* $\operatorname{int} \bigcap_1^n f_i^{-1}[T_i] \neq \emptyset$. *If* $X = Y$ *and the identity* $1_X \in S$, *then the converse holds as well.*

Proof. (a) First assume A is semi-open and let T be open, $A \cap T \neq \emptyset$. Then $\operatorname{cl}(\operatorname{int}(A)) \cap T \neq \emptyset$, so $\operatorname{int}(A) \cap T \neq \emptyset$. Conversely, if whenever $A \cap T \neq \emptyset$, then $\operatorname{int}(A) \cap T \neq \emptyset$, let $x \in A$. If T is an open set such that $x \in T$, then $x \in A \cap T$, so $\operatorname{int}(A) \cap T \neq \emptyset$. But then by the arbitrary nature of T, $x \in \operatorname{cl}(\operatorname{int}(A))$, therefore by the arbitrary nature of x, $A \subseteq \operatorname{cl}(\operatorname{int}(A))$.

(b) Using (a), it will do to show that if U is open, and $(A \cap T) \cap U \neq \emptyset$, then $\operatorname{int}(A \cap T) \cap U \neq \emptyset$. But if $(A \cap T) \cap U \neq \emptyset$, then $A \cap (T \cap U) \neq \emptyset$, so since A is semi-open, $\emptyset \neq \operatorname{int}(A) \cap (T \cap U) = (\operatorname{int}(A) \cap T) \cap U = \operatorname{int}(A \cap T) \cap U$, as required.

Both (c) and (d) are immediate from (a), (b), and 2.1. □

The case, treated in (d) above, in which $X = Y$, is of course centrally important, and in that case it will often be useful to consider the simplifying assumption that $1_X \in S$ – for example, see 3.3.

3 Π-Related Topologies

The following definition introduces the Π-relation, and also a kindred N-relation, which allow us to compare two topologies on the same space. Since we look at several topologies at once below, it's useful to decorate the notation to indicate which we are referring to at the moment. For example, the closure of A with respect to the topology σ is denoted $\operatorname{cl}_\sigma(A)$.

Definition 3.1. For any collection σ of subsets of X, let $\sigma^+ = \sigma - \{\emptyset\}$. For topologies τ, υ on X, define $\tau \mathsf{N} \upsilon$ if for each $U \in \upsilon^+$, there is an $T \in \tau^+$ such that $T \subseteq U$ (that is, if each nonempty υ-open set has nonempty τ-interior).

We say $\tau \, \Pi \, \upsilon$ if $\tau \mathsf{N} \upsilon$ and $\upsilon \mathsf{N} \tau$; τ and υ are said to be **Π-related** if $\tau \, \Pi \, \upsilon$.

Example 3.2. The Sorgenfrey topology—the topology on $\mathbb{R}$ generated by all sets of the form $[a, b)$ — is Π-related to the usual topology on $\mathbb{R}$.

Surely the relation N is transitive, thus the Π-relation is an equivalence relation on the set of topologies on X. In addition, if $\upsilon \supseteq \theta$, then $\upsilon \mathsf{N} \theta$, and it follows that if $\tau \mathsf{N} \upsilon$ and $\upsilon \supseteq \theta$, then $\tau \mathsf{N} \theta$. Also, $\tau \mathsf{N} \upsilon$ if and only if for each $U \in \upsilon$, $U \subseteq \operatorname{cl}_\upsilon \operatorname{int}_\tau U$.

Of course, the Π-relation tells us more about the relationship between two topologies; further, the Π-relation has a history and literature. It is traditionally defined somewhat differently: a π^o**-base** is a set υ such that each nonempty open set contains a member of υ^+, and each member of υ^+ has non-empty interior. Often this relation is defined with the easily established equivalent statement: $\tau \Pi \upsilon$ if and only if τ^+ is a π^o-base for υ (see, eg. [9] Theorem 3.4).

But it is certainly simpler to verify that $\tau \,\mathsf{N}\, \upsilon$ than that $\tau \,\Pi\, \upsilon$, and many proofs are simplified, so we use the relation N more often than Π. For example, notice that if υ, θ are topologies Π-related to τ and $\upsilon \subseteq \theta$, then any topology ρ, between the two is also Π-related to τ, since $\rho \subseteq \theta$ implies $\rho \,\mathsf{N}\, \tau$ and $\upsilon \subseteq \rho$ implies $\tau \,\mathsf{N}\, \rho$.

If S is a set of functions from X to X, and τ is a topology on X, then we use $w(S,\tau)$ to denote the weakest topology θ on X such that each $f \in S$ is continuous from (X,θ) to (X,τ).

Notice that $\{\bigcap_1^n f_i^{-1}[T_i] \mid n \in \mathbb{N},\ T_1,\dots,T_n \in \tau,\ f_1,\dots,f_n \in S\}$ is a base for a topology which must necessarily be the weakest for which each $f_i \in S$ is continuous.

The following theorem and corollary are the main results of this paper, as they tie together the notions of Π-related topologies and quasicontinuous functions.

Theorem 3.3. *Let S be a set of functions from X to Y. The following are equivalent:*

(i) S is setwise quasicontinuous from (X,τ_X) to (Y,τ_Y),

(ii) $\tau_X \,\mathsf{N}\, (w(S,\tau_Y) \vee \tau_X)$, the join of $w(S,\tau_Y)$ and τ_X, and

(iii) there is a topology $\upsilon \supseteq \tau_X$ on X such that for each $f \in S$, $f : (X,\upsilon) \to (Y,\tau_Y)$ is continuous, and $\tau_X \,\mathsf{N}\, \upsilon$.

If $(X,\tau_X) = (Y,\tau_Y)$ and $1_X \in S$, then (i) – (iii) are also equivalent to each of the following:

(iv) for each $T_1,\dots,T_n \in \tau_Y$, $f_1,\dots,f_n \in S$, $n \in \mathbb{N}$ such that $\bigcap_1^n f_i^{-1}[T_i] \neq \emptyset$, we have $\operatorname{int} \bigcap_1^n f_i^{-1}[T_i] \neq \emptyset$, and

(v) $\tau_X \,\mathsf{N}\, w(S,\tau_Y)$.

Furthermore, replacing N by Π in (ii), (iii), and (v), results in other equivalent conditions.

PROOF. Suppose (i); if $x \in W \in w(S,\tau_Y) \vee \tau_X$, then for some $n \in \mathbb{N}$, there are $T_1,\dots,T_n \in \tau_Y$, $f_1,\dots,f_n \in S, U \in \tau_X$, such that $x \in (\bigcap_1^n f_i^{-1}[T_i]) \cap U \subseteq W$. Since $\bigcap_1^n f_i^{-1}[T_i]$ is semi-open with respect to τ_X, $(\bigcap_1^n f_i^{-1}[T_i]) \cap U$ has nonempty τ_X-interior, thus $\tau_X \,\mathsf{N}\, w(S,\tau_Y) \vee \tau_X$, showing (ii).

(ii) clearly implies (iii), and if (iii) holds, then for each $T_1,\dots,T_n \in \tau_Y$, $U \in \tau_X$ and $f_1,\dots,f_n \in S$, $(\bigcap_1^n f_i^{-1}[T_i]) \cap U \in \upsilon$ so if $\bigcap_1^n f_i^{-1}[T_i] \cap U \neq \emptyset$, then by the fact that $\tau_X \,\mathsf{N}\, \upsilon$, there is a $V \in \tau_X^+$ such that $V \subseteq (\bigcap_1^n f_i^{-1}[T_i]) \cap U$, so $\bigcap_1^n f_i^{-1}[T_i]$ is semi-open, by 2.4 (a). Thus S is setwise quasicontinuous from (X,τ_X) to (X,τ_X).

In the case that $X = Y$ and $1_X \in S$, since $1_X^{-1}[T] = T$ for each $T \in \tau_X$, this requires that τ_X is weaker than $w(S, \tau_X)$, that is, $w(S, \tau_X) = w(S, \tau_X) \vee \tau_X$; thus (iii) is equivalent to (v). Finally, (iv) is simply a restatement of (v). The final statement results from the comments immediately following 3.1 of the relations N and Π. □

The following Corollary is the result most immediately useful to topological dynamical systems:

Corollary 3.4. . *A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is quasicontinuous if and only if there is a topology υ on X such that $f : (X, \upsilon) \to (Y, \tau_Y)$ is continuous, $\tau_X \,\Pi\, \upsilon$ and $\tau_X \subseteq \upsilon$.*

Proof. This is the special case of the theorem in which $S = \{f\}$. □

4 Setwise Quasicontinuity

In section 3, we considered the setwise quasicontinuity of general sets of functions between spaces. In this section we turn to dynamical systems and the question that motivated this paper: Under what circumstances are all the iterates of a map $f : X \to X$ quasicontinuous? We here give an equivalence for a stronger condition: that the set $\{f^n\}_{n=0}^{\infty}$ is setwise quasicontinuous (with respect to the given topology on X).

Theorem 4.1. *(a) Let S be a semigroup of maps from X to X, with $1_X \in S$. S is setwise quasicontinuous from (X, τ_X) to (X, τ_X) if and only if there is a topology $\upsilon \supseteq \tau_X$ on X such that $\tau_X \,\Pi\, \upsilon$ and for each $f \in S$, $f : (X, \upsilon) \to (X, \upsilon)$ is continuous.*

(b) Let $f : X \to X$; the semigroup $S_f = \{1_X, f, f^2, \dots\}$ is setwise quasicontinuous from (X, τ_X) to (X, τ_X) if and only if there is a topology $\theta \supseteq \tau$ on X such that $\tau_X \,\Pi\, \theta$ and $f : (X, \theta) \to (X, \theta)$ is continuous.

Proof. For (a) if there is such a topology, then certainly $f : (X, \upsilon) \to (X, \tau_X)$ is continuous for each $f \in S$, so S is setwise quasicontinuous from (X, τ_X) to (X, τ_X). Conversely, if S is setwise quasicontinuous; by 3.3, there is a topology υ on X so that for each $f \in S$, $f : (X, \upsilon) \to (X, \tau_X)$ is continuous, $\tau_X \subseteq \upsilon$, and $\tau_X \,\Pi\, \upsilon$. In fact, then each $f \in S$ is continuous from (X, υ) to (X, υ), for if $f(x) \in U \in \upsilon$, then for some $T_1, \dots, T_n \in \tau_X$ and $f_1, \dots, f_n \in S$, $f(x) \in \bigcap_1^n f_i^{-1}[T_i] \subseteq U$, so $x \in f^{-1}[\bigcap_1^n f_i^{-1}[T_i]] \subseteq f^{-1}[U]$, but $f^{-1}[\bigcap_1^n f_i^{-1}[T_i]] = \bigcap_1^n (f_i f)^{-1}[T_i] \in \upsilon$. This shows that $f^{-1}[U]$ is an υ-neighborhood of each of its points, thus in υ, thus that f is continuous from (X, υ) to (X, υ).

For (b), if there is such a θ, then each f^n, $n \geq 0$ (including 1_X) is continuous from $f : (X, \theta) \to (X, \theta)$, so by (a) S_f is setwise quasicontinuous; conversely, again by (a) if S_f is setwise quasicontinuous, then there is a $\theta \supseteq \tau_X$

on X such that $\tau_X \,\Pi\, \theta$ and each $f^n : (X, \theta) \to (X, \theta)$, in particular, f, is continuous. $\square$

Example 4.2. Let us consider an application of this theorem. We claimed in 2.2 that for the baker's function $b : [0,1] \to [0,1]$ with the usual topology, the set $\{b^n\}_{n=0}^{\infty}$ is setwise quasicontinuous. One way to prove this is to demonstrate that for any finite collection of open intervals $\{(\alpha_i, \beta_i)\}$, the intersection $\cap_{i=0}^{n} b^{-i}(\alpha_i, \beta_i)$ is either empty or has non-empty interior. However, by part (b) of the above theorem, it suffices to note that we can find an appropriate Π-related topology on which b^i is continuous for all $i > 0$. Define θ to be the topology generated by intervals of the form (α, β), $(\alpha, 1]$, or $[\frac{k}{2^n}, \beta)$, with $0 \leq \alpha < \beta \leq 1$ and $0 \leq k < \beta 2^n$. Then it is clear that each b^i is continuous with respect to the topology θ, and that $\theta \supseteq \tau$. Moreover, because every element of τ contains a non-empty element of θ and vice versa, we see that $\tau_X \,\Pi\, \theta$.

The first half of the next section is devoted to further discussion of this example. For now, we return to the notion of quasicontinuous system (f, X) of [3]—that is, $f : X \to X$ with $\{f^n\}$ separately quasicontinuous. In 2.3, we demonstrated a pair of functions which is separately, but not setwise, quasicontinuous. It is worth asking whether such examples can exist under the added structure of a semigroup. In particular, if a dynamical system is quasicontinuous (as a system), must the iterates of the function form a setwise quasicontinuous set? The answer is no, as can be seen by the following counterexample.

Example 4.3. *A self-map whose generated semigroup is separately, but not setwise, quasicontinuous.*

We define $f : [0,4] \to [0,4]$ by $f(x) = \begin{cases} 2, & \text{for } 0 \leq x \leq 1 \\ x+1, & \text{for } 1 \leq x \leq 2 \\ 3-x, & \text{for } 2 < x < 3 \\ 2, & \text{for } 3 \leq x \leq 4. \end{cases}$

By simple iteration, $f^2(x) = \begin{cases} 3, & \text{for } 0 \leq x \leq 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 2, & \text{for } 2 \leq x < 3 \\ 3, & \text{for } 3 \leq x \leq 4. \end{cases}$

Both of these functions are quasicontinuous, as are all the higher order iterations:

$$f^3(x) = f^5(x) \cdots = \begin{cases} 2, & \text{for } \leq x < 2 \\ 3, & \text{for } 2 \leq x < 3 \\ 2, & \text{for } 3 \leq x \leq 4 \ , \end{cases}$$

$$f^4(x) = f^6(x) \cdots = \begin{cases} 3, & \text{for } 0 \leq x < 2 \\ 2, & \text{for } 2 \leq x < 3 \\ 3, & \text{for } 3 \leq x \leq 4. \end{cases}$$

We see that f is continuous from the left at 2, and f^n is continuous from the right at 2 for $n > 1$. From this, we can see that $f^{-1}(3-\epsilon, 3+\epsilon) = (2-\epsilon, 2]$ and $f^{-2}(2-\epsilon, 2+\epsilon) = [2, 3)$. The intersection of these sets is $\{2\}$, which is neither empty nor has non-empty interior.

In spite of examples like the above and 2.3, we claim that setwise quasicontinuity is a natural restriction to consider when studying classes of functions. These classes need not be restricted to iteration in order for setwise quasicontinuity to be useful. For example, sums of separately quasicontinuous functions $f, g : X \to \mathbb{R}$ need not be quasicontinuous (consider $b + \tilde{b}$ of Example 2.3). But sums of setwise quasicontinuous functions remain quasicontinuous:

Theorem 4.4. *Suppose S is setwise quasicontinuous from (X, τ_X) to (Y, τ_Y), $f_1, \ldots, f_n \in S$, and $\sigma : (Y, \tau_Y)^n \to (Y, \tau_Y)$ is a continuous n-ary operation. Then $S \cup \{\sigma(f_1, \ldots, f_n)\}$ is setwise quasicontinuous from (X, τ_X) to (Y, τ_Y). (Of course, by $\sigma(f_1, \ldots, f_n)$ we mean the map from X to Y, defined by $\sigma(f_1, \ldots, f_n)(x) = \sigma(f_1(x), \ldots, f_n(x))$.)*

In particular, sums, products, differences, and quotients (when the denominator is non-zero) of a setwise quasicontinuous pair of real-valued functions of a real variable are quasicontinuous real-valued functions of a real variable.

Proof. Since S is setwise quasicontinuous from (X, τ_X) to (Y, τ_Y) there is, by 3.3 (iii), a topology $\upsilon \supseteq \tau_X$, such that $\tau_X \mathsf{N} \upsilon$, and each $f_i : (X, \upsilon) \to (Y, \tau_Y)$ is continuous. Notice that for the product topologies, $\upsilon^n \supseteq \tau_X^n$ and $\tau_X^n \mathsf{N} \upsilon^n$, and the map $(f_1, \ldots, f_n) : (X, \upsilon) \to (X, \upsilon)^n$, defined by $(f_1, \ldots, f_n)(x) = (f_1(x), \ldots, f_n(x))$ is continuous, and that $\sigma(f_1, \ldots, f_n)$ is $\sigma \circ (f_1, \ldots, f_n)$, a composition of the continuous function $(f_1, \ldots, f_n) : (X, \upsilon) \to (X, \upsilon)^n$, by the continuous $\sigma : (X, \upsilon)^n \to (Y, \tau_Y)$ (since we have enlarged the topology from which σ was assumed continuous), thus which is continuous from (X, υ) to (Y, τ_Y). So by 3.3 (iii), $S \cup \{\sigma(f_1, \ldots, f_n)\}$ is quasicontinuous from (X, τ_X) to (Y, τ_Y). $\square$

But compositions don't work quite the same way. Note that by 4.1 (a), the semigroup generated by a setwise quasicontinuous set of self-maps on (X, τ_X)

is setwise quasicontinuous if and only if τ_X can be strengthened to a Π-related topology υ on X such that for each $f \in S$, $f : (X, \upsilon) \to (X, \upsilon)$ is continuous. But by 2.6 (c), for the function f of 4.3, $\{f\}$ is setwise quasicontinuous, but the semigroup it generates is not.

5 *A function which is quasicontinuous with dense discontinuities*

Throughout the following, a, b, k, n, m will represent positive integers.

We will define this function inductively. Let $h_0 = 1_{[0,1]}$. For $n > 0$ we will define $h_n : [0,1] \to [0,1]$ in such a way that it is linear with slope 1 or -1 on each subinterval $[\frac{k}{2^n}, \frac{k+1}{2^n})$ and continuous from the right at any point $\frac{k}{2^n} \in [0,1)$.

Suppose h_{n-1} maps the interval $[\frac{a}{2^{n-1}}, \frac{a+1}{2^{n-1}})$ into the interval $[\frac{b}{2^{n-1}}, \frac{b+1}{2^{n-1}}]$. It does so either with slope $+1$ or -1.

Slope $+1$: We require that h_n maps $[\frac{2a}{2^n}, \frac{2a+1}{2^n})$ into $[\frac{2b}{2^n}, \frac{2b+1}{2^n})$ with slope $+1$, and $[\frac{2a+1}{2^n}, \frac{2a+2}{2^n})$ into $(\frac{2b+1}{2^n}, \frac{2b+2}{2^n}]$ with slope -1.

Slope -1: We require that h_n maps $[\frac{2a}{2^n}, \frac{2a+1}{2^n})$ into $[\frac{2b+1}{2^n}, \frac{2b+2}{2^n})$ with slope $+1$, and $[\frac{2a+1}{2^n}, \frac{2a+2}{2^n})$ into $(\frac{2b}{2^n}, \frac{2b+1}{2^n}]$ with slope -1.

We define $h_n(1) = \frac{1}{2}$ for all $n > 0$.

The functions are easier to see if one draws the first few iterates.

We define $h : [0,1] \to [0,1]$ pointwise by $h(x) = \lim_{n\to\infty} h_n(x)$. (It is fairly easy to see that $h(0) = 0$; it is only slightly more difficult to check that $h(\frac{2}{3}) = 1$.) It is also easy to check by induction that if $x, y \in [\frac{a}{2^m}, \frac{a+1}{2^m})$, and $m \le n$, then $|f_n(x) - f_n(y)| \le \frac{1}{2^m}$.

Theorem 5.1. *For every $x \in [0,1]$, $h(x)$ exists. Unless $x = \frac{k}{2^n}$ for some positive integers n, k, then h is continuous at x. Otherwise, h is continuous from the right at x.*

Corollary 5.2. *h is quasicontinuous (with respect to the standard topology on $[0,1]$) and the discontinuities of h are dense in $[0,1]$.*

Proof of Theorem. We will give two proofs of the theorem; the first, using standard arguments and the second using the methods of this paper.

Suppose $x \ne \frac{k}{2^n}$ for each $k, n \in \mathbb{N} \cup \{0\}$. Let $\epsilon > 0$ be given. Then we can find some a, m such that the interval $I_{a,m} = [\frac{a}{2^m}, \frac{a+1}{2^m}) \subseteq (x - \epsilon, x + \epsilon)$. Then if $y \in I_{a,m}$, we have $|h_m(x) - h_m(y)| < \frac{1}{2^{m+1}} < \epsilon$ and so for all $n > m$,

$|h_n(x) - h_n(y)| < \epsilon$. Accordingly, $h(x)$ exists and h is continuous at x. The case $x = \frac{k}{2^n}$ is handled similarly.

Alternatively, let υ be the topology generated by sets of the form (c, d), $(c, 1]$, and $[\frac{k}{2^n}, d)$, with $0 \leq c < d \leq 1$ and $0 \leq k < d2^n$, and τ be the usual topology on the unit interval. Then it is clear that each h_n is continuous with respect to the topology υ and that these converge uniformly to h. Therefore, h is continuous on $[0, 1]$ with respect to υ, $\tau \subset \upsilon$, and $\upsilon \, \Pi \, \tau$. This completes the proof. □

To further appreciate the importance of this example, it helps to look at two examples of dynamical systems. The first is the baker's transformation $b(x)$ described in 2.2 and 4.2. The second of these is known as the tent map: it is defined by

$$t(x) = \begin{cases} 2x, & \text{for } 0 \leq x < 0.5 \\ 2 - 2x, & \text{for } 0.5 \leq x \leq 1. \end{cases}$$

Each of these functions is standard to dynamical systems (and each indeed appears in most introductory textbooks in the field), but the baker's transformation must be treated as a "special case" because the usual topological theorems assume continuity.

Both of these functions are defined on the interval $[0, 1]$, and both of the systems are chaotic in the sense that periodic points of each system are dense, each has a dense orbit, and each system has sensitive dependence on initial conditions. Indeed, dynamicists commonly take advantage of the fact that these two systems display exactly the same behaviors—in a sense described below—with the following exception. Both b and t have a pair of fixed points ($\{0, 1\}$ and $\{0, \frac{2}{3}\}$, respectively). Each of the fixed points of t has a pair of distinct preimages: $t^{-1}(0) = \{0, 1\}$ and $t^{-1}(\frac{2}{3}) = \{\frac{1}{3}, \frac{2}{3}\}$. However, only one of the fixed points of b has a distinct preimage: $b^{-1}(0) = \{0, \frac{1}{2}\}$, but $b^{-1}(1) = \{1\}$.

On the other hand, there is a sense in which the tent map and the baker's map are equivalent—there is a one-to-one correspondence between the kind of behaviors of one and the other. We would say that these systems are *conjugate* if there were a continuous bijective function $g : [0, 1] \to [0, 1]$ with a continuous inverse which satisfied $g(t(x)) = b(g(x))$. There is no such continuous function g, obviously. But there is a quasicontinuous function which acts as a conjugacy—it is the h that is described above.

Theorem 5.3. *Let $h, t, b : [0, 1] \to [0, 1]$ be as given above. Then h is one-to-one except that $h(\frac{1}{3}) = h(1) = \frac{1}{2}$. Moreover, $h(t(x)) = b(h(x))$ for all $x \in [0, 1] \setminus \{\frac{1}{3}\}$.*

PROOF. That h is one-to-one follows from the construction—we can construct h^{-1} in a method analogous to our construction of h. To see that h acts as a conjugacy, we can look at the "kneading sequence". For any point $x \in [0,1]$, we associate a sequence $\sigma_t(x) = .s_0 s_1 s_2 \dots$, in the following manner. If $t^n(x) < \frac{1}{2}$, then $s_n = 0$; if $t^n(x) \geq \frac{1}{2}$, then $s_n = 1$. It is easy to see that the kneading sequence associated with the baker's transformation is exactly the binary expansion of x. Pick a point $x \in [0,1]$ and compute $\sigma_t(x)$. If we let y be the number whose associated binary sequence is $\sigma_t(x)$, then $y = h(x)$. For more on kneading sequences, see for example [4], p. 141. $\square$

Notice that $\{h, b, b^2, b^3, \dots\}$ is setwise quasicontinuous, and that therefore there is a topology (described in 4.2) for which $b \circ h$ is continuous. This observation leads us to several more general theorems for conjugacy maps of quasicontinuous systems. We begin with a simple lemma:

Lemma 5.4. *Let τ, θ, υ be three topologies on a set X. (a) If $\upsilon \,\Pi\, \tau$ and $\upsilon \supseteq \tau$, where υ and τ are topologies on X, then $1_X : (X,\tau) \to (X,\upsilon)$ is quasicontinuous and $1_X : (X,\upsilon) \to (X,\tau)$ is continuous.*

(b) $\tau \,\mathsf{N}\, \upsilon$ if and only if each τ-dense subset of X is υ-dense. Thus $\tau \,\Pi\, \upsilon$ if and only if the τ-dense subsets of X and the υ-dense subsets of X are the same.

PROOF. The proof of (a) is straightforward. For (b), if $\tau \,\mathsf{N}\, \upsilon$, and P is τ-dense, pick $V \in \upsilon^+$; then we know that there is a $U \in \tau^+$ with $U \subset V$. Since P is τ-dense $\emptyset \neq P \cap U \subset P \cap V$. By the arbitrary nature of V, P is υ-dense in X.

Conversely, if it is not true that $\tau \,\mathsf{N}\, \upsilon$, there is some $U \in \upsilon^+$ so that no $T \in \tau^+$ is a subset of U. Let $P = X \setminus U$; then $P \neq X$ is υ-closed, so it is not υ-dense. But if $T \in \tau^+$, then T is not a subset of U, so $\emptyset \neq T \cap P$; and this shows that P is τ-dense.

The statement about Π-related topologies is then immediate. $\square$

Lemma 5.4 (b) extends one of the assertions of [10], proposition 3.2. For the theorems which follow, we will refer to the following hypothesis:

$$H1. f^n : (X,\theta) \to (X,\theta) \text{ is continuous for every } n > 0, \tau \subseteq \theta, \text{ and } \tau \,\Pi\, \theta.$$

Notice that this hypothesis implies by Theorem 4.1 that $\{f^n : (X,\tau) \to (X,\tau)\}$ is setwise quasicontinuous.

Theorem 5.5. *If f satisfies (H1), then f is topologically transitive with respect to τ if and only if f is topologically transitive with respect to θ.*

PROOF. Let us suppose that f is topologically transitive with respect to τ. Pick $V, V' \in \theta^+ = \theta \setminus \{\emptyset\}$. We can use the Π-relation to find $U, U' \in \tau^+$ with

$U \subset V$ and $U' \subset V'$. By the definition of topological transitivity, there is some $n > 0$ with $f^{-n}(U) \cap U' \neq \emptyset$. It follows that $f^{-n}(V) \cap V' \neq \emptyset$.

The reverse direction is proved similarly—notice that we do not need $\tau \subseteq \theta$ for this proof. □

Consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

Theorem 5.6. *If the diagram above commutes, f satisfies (H1), $h : (X, \theta) \to (Y, \upsilon)$ is onto and continuous, and $g : (Y, \upsilon) \to (Y, \upsilon)$ is quasicontinuous, then the following hold:*

(a) If X contains a dense orbit of f, then Y contains a dense orbit of g.

(b) If periodic points of f are dense in X, then periodic points of g are dense in Y.

(c) If f is topologically transitive, then g is topologically transitive.

Lemma 5.4 allows us to suppress the notation for the topology in the statement of the above theorem; it should be understood that the topology on Y is υ and the topology on X is either τ or θ.

Proof. (a) Let $x_0 \in X$ such that $\{f^n(x_0)\}_{n=0}^{\infty}$ is dense, and pick a nonempty open $V \subset Y$. Then $h^{-1}(V) \in \theta^+$ so that there is some $n > 0$ with $f^n(x_0) \in h^{-1}(V)$—that is, $g^n(h(x_0)) = h(f^n(x_0) \in V$. From this we see that the orbit of $h(x_0)$ is dense in Y.

(b) Pick an open set $V \in \upsilon^+$; then $h^{-1}(V) \in \theta^+$ so that there is some $x \in X$ and $n > 0$ with $f^n(x) = x$. Let $y = h(x)$; by commutativity of the diagram we have $g^n(y) = g^n(h(x)) = h(f^n(x)) = h(x) = y$; this shows that periodic points are dense in Y. The proof of (c) is straightforward. □

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