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QUASI SELF-SIMILARITY AND MULTIFRACTAL ANALYSIS OF CANTOR MEASURES*

Abstract

We examine quasi self-similarity for Cantor measures on Cantor sets. A characterization is obtained in terms of the ratios of dissection of the Cantor set. The multifractal theory of Cantor measures is studied, extending the analysis for quasi self-similar measures.

1 Introduction

The main problem in multifractal theory is to estimate the size, typically the Hausdorff or packing dimension, of

$$X(\alpha) \equiv \{x \in \operatorname{supp} \mu : \lim_{r \to 0} \frac{\log \mu\left(B(x,r)\right)}{\log r} = \alpha\},$$

where μ is a probability measure. The sets $X(\alpha)$ can be viewed as a decomposition of $\operatorname{supp} \mu$ into a family of subfractals, the dimensions of these subfractals $X(\alpha)$ being known as the multifractal spectrum of μ. The heuristic arguments developed by physicists and mathematicians (for an extensive bibliographic list see [4] and [6]) suggest that these dimensions should be equal to the Legendre transform of a suitable function. Olsen in [6] developed mathematical theories to put these arguments on a firm foundation and detailed analysis of the multifractal spectrum has been carried out for particular examples, such as

self-affine measures (see [5], [7]), self-similar measures ([2], [3]), and measures on cookie cutters ([9]).

Recently, the multifractal spectrum was determined for quasi self-similar measures in metric spaces by O'Neil [8]. These are measures for which there exists a map from small parts of the space to itself and vice versa, which distorts distances and the measure in a bi-Lipschitz fashion.

In this paper the quasi self-similarity properties of Cantor sets and Cantor measures are investigated. We introduce a variation of quasi self-similarity for Cantor sets and characterize this property in terms of the ratios of dissection of the Cantor set. Sets which have this quasi self-similarity property are seen to have positive Hausdorff dimension. We also show that this variation coincides with the more general definition for most Cantor sets.

The fine multifractal structure of Cantor measures is then analyzed. We extend the work of [8] by showing that the multifractal spectrum of many Cantor measures is the Legendre transform of a function of the ratios of dissection of the Cantor set. The class of measures to which our work applies is larger than the class of quasi self-similar Cantor measures.

2 Notation and Definitions

2.1 Cantor Sets

By a Cantor set we mean a compact, totally disconnected, perfect subset of $[0,1]$ which can be constructed in a similar fashion to the classical middle third Cantor set. We remove from $[0,1]$ a non-empty, centered, open interval, leaving two closed intervals of equal, positive length called the Cantor intervals of step one. The ratio between the length of the Cantor intervals of step one and $[0,1]$ is called the ratio of dissection of step one. A similar operation is performed on each Cantor interval of step one, producing the (four) closed equal-length Cantor intervals of step 2 and the ratio of dissection at step two. Repeating this construction yields a decreasing sequence of closed sets whose intersection is a Cantor set.

It is convenient to label the elements of the construction of the Cantor set by W, the set of binary words of finite length,

$$W = \{w = w_1 w_2 \cdots w_n : w_i \in \{0,1\}, \ n \in \mathbb{N}\} \bigcup \{e\},$$

where e is the empty word. We let $I_e = [0,1]$. More generally, if the length of the word $w \in W$ (written $|w|$) is k, then we denote by I_{w0} and I_{w1} the left and right Cantor intervals of step $k+1$ obtained after removing the open interval (or gap) G_w of step k, from I_w, a Cantor interval of step k. We let T denote

the truncation operator, $I_{T(wi)} = I_w$, for $i = 0, 1$. The ratio of dissection at step k will be denoted r_k and is equal to $|I_{wi}| / |I_w|$ for any word w of length $|w| = k - 1$, and $i = 0, 1$. The Cantor set $C_{\{r_k\}}$ is the set $\bigcap_{k \in \mathbb{N}} \bigcup_{|w|=k} I_w$, and is uniquely determined by the ratios of dissection $\{r_k\}$. The classical middle third Cantor set is the case when the ratios of dissection are all equal to $1/3$.

We will also write I_k and G_k for Cantor intervals or gaps of step k when only the step is relevant. Lastly, we note that sometimes it is convenient to write I_w for the Cantor interval I_w intersected with the Cantor set; the meaning should be clear from the context.

2.2   Quasi Self-Similar Measures

The measure which arises as the invariant measure from an iterated function system of similarity transforms is referred to as a self-similar measure; the attractor is a self-similar set. These sets and measures have been extensively studied (c.f. [4] and the references cited therein). Generalizing this idea, O'Neil in [8] defined the notion of a quasi self-similar measure on a compact metric space X.

Throughout the paper we let $B(x, d)$ denote the closed ball in X with center $x \in X$ and radius d.

Definition 1. ([8]) Let μ be a Borel probability measure supported on X. We say that μ is m**-quasi self-similar by balls from below on the space** X if for all $x \in X$ and balls $B(x, d)$ there exist mappings $\Phi : B(x, d) \to X$ (called the **quasi self-similarity maps**) such that whenever $B(y, s) \subset B(x, d)$ we have

$$B\left(\Phi(y), m^{-1}\frac{s}{d}\right) \subset \Phi(B(y, s)) \subset B\left(\Phi(y), m\frac{s}{d}\right),$$

and whenever $A \subset B(x, d)$ is a Borel set, then

$$m^{-1}\frac{\mu(A)}{\mu(B(x,d))} \leq \mu(\Phi(A)) \leq m\frac{\mu(A)}{\mu(B(x,d))}.$$

We say that μ is m**-quasi self-similar by balls from above on** X if for all $x \in X$ and balls $B(x, d)$ there exist mappings $\Phi : X \to B(x, d)$ such that if $\rho \leq$ diameter X, then for all $z \in X$ one can find $\zeta \in B(x, d)$ such that

$$B(\zeta, m^{-1}\rho d) \subset \Phi\left(B(z, \rho)\right) \subset B(\zeta, m\rho d),$$

and if $A \subset X$ is Borel, then

$$m^{-1}\mu(A) \leq \frac{\mu\left(\Phi(A)\right)}{\mu\left(B(x,d)\right)} \leq m\mu(A).$$

The measure μ is **quasi self-similar by balls** if it is both quasi self-similar from above and below on X.

For Cantor sets it is natural to to study a variant of quasi self-similarity where balls (contained in the Cantor set) are replaced in the definition of quasi self-similarity by Cantor intervals.

Definition 2. We say that a Borel probability measure μ on a Cantor set $C = C_{\{r_k\}}$ is m-**quasi self-similar by Cantor intervals from below on** C if for all Cantor intervals I_w there exist mappings $\Phi : I_w \to C$ such that for all $I_v \subset I_w$ there exists $x \in I_v$ such that

$$B\left(\Phi(x), m^{-1}\frac{|I_v|}{|I_w|}\right) \subset \Phi(I_v) \subset B\left(\Phi(x), m\frac{|I_v|}{|I_w|}\right),$$

and for all Borel sets $A \subset I_w$,

$$m^{-1}\frac{\mu(A)}{\mu(I_w)} \leq \mu(\Phi(A)) \leq m\frac{\mu(A)}{\mu(I_w)}.$$

We say that a measure μ on a Cantor set C is m-**quasi self-similar by Cantor intervals from above on** C if for all Cantor intervals I_w there exist mappings $\Phi : C \to I_w$ such that for all $z \in C$ there exists $\zeta \in I_w$ such that if $z \in I_v$, then

$$B(\zeta, m^{-1}|I_v||I_w|) \subset \Phi(I_v) \subset B(\zeta, m|I_v||I_w|),$$

and for all Borel sets $A \subset C$,

$$m^{-1}\mu(A) \leq \frac{\mu(\Phi(A))}{\mu(I_w)} \leq m\mu(A).$$

A measure on a Cantor set will be said to be **quasi self-similar by Cantor intervals** if it is both quasi self-similar by Cantor intervals from below and above.

There are a natural class of Borel probability measures on Cantor sets to study.

Definition 3. Let $C = C_{\{r_k\}}$ be a Cantor set and $0 < p < 1$. A Borel probability measure $\mu = \mu_p$, supported on C, is called a p-**Cantor measure on** C if for all finite words w we have $\mu(I_{w0}) = p\mu(I_w)$ and $\mu(I_{w1}) = (1-p)\mu(I_w)$.

If the finite word w has j zeros and k ones, then $\mu(I_w) = p^j(1-p)^k$.

One easy observation, which we will use extensively, is that any ball $B(x, d)$, with center $x \in C$ and $d < \min\{|G_k| : k \leq n\}$, cannot intersect two Cantor intervals of step $n+1$ as these are separated by more than d. Hence $B(x, d)$ must be a subset of the step $n+1$ Cantor interval containing x.

3   Quasi Self-Similarity by Cantor Intervals

The main result of this section will be a characterization of quasi self-similarity by Cantor intervals in terms of the ratios of dissection of the Cantor set. Before stating this we prove a preliminary result which is of independent interest.

Proposition 1. *If the p-Cantor measure μ on $C_{\{r_k\}}$ is quasi self-similar by Cantor intervals, then $\inf_k r_k > 0$.*

PROOF. Assume μ is m quasi self-similar by Cantor intervals and $\inf_k r_k = 0$. Pick n such that $m \max(p, 1-p)^n < \min(p, 1-p)$, and choose $\varepsilon > 0$ so that the gaps of any step $j \leq n - 1$ are greater than $m\varepsilon$ in length. Then for each $z \in C$ the ball $B(z, m\varepsilon)$ is contained in some Cantor interval I_v (depending on z) of step n.

Choose k such that $r_k < \varepsilon$ and let I_w be any Cantor interval of step k. Since μ is quasi self-similar from below by Cantor intervals there exists a map $\Phi : I_{T(w)} \to C$ such that for some $z \in I_w$,

$$\Phi\left(I_w\right) \subseteq B\left(\Phi(z), \frac{m\,|I_w|}{|I_{T(w)}|}\right) \subseteq B(\Phi(z), m\varepsilon) \subseteq I_v,$$

and

$$m^{-1} \frac{\mu(I_w)}{\mu(I_{T(w)})} \leq \mu(\Phi\left(I_w\right)) \leq \mu(I_v).$$

This is a contradiction since $\mu(I_v) \leq \max(p, 1-p)^n$ and $\mu(I_w)/\mu(I_{T(w)}) \geq \min(p, 1-p)$. □

Theorem 2. *Let $C = C_{\{r_k\}}$ be a Cantor set. There is a p-Cantor measure on C which is quasi self-similar by Cantor intervals if and only if there exists n such that for all integers i, j,*

$$n^{-1} \leq \frac{r_1 \cdots r_i}{r_{1+j} \cdots r_{i+j}} \leq n. \tag{3.1}$$

Moreover, in this case every p-Cantor measure on C is quasi self-similar by Cantor intervals.

PROOF. First, assume (3.1) fails to be satisfied for any n but that C supports a p-Cantor measure which is m-quasi self-similar by Cantor intervals. By the previous proposition we may assume $\inf r_k \equiv a > 0$. Choose $k, l \in \mathbb{N}$ such that $2^k \geq m$ and $2^{l-1} > a^{-4k}$. The choice of k ensures that $|I_k| \leq m^{-1}$, therefore if $x \in C \bigcap I_k$, then $I_k \subseteq B(x, m^{-1})$.

Case 1. The ratio is unbounded from above: Obtain i, j such that

$$\frac{r_1 \cdots r_i}{r_{1+j} \cdots r_{i+j}} > 2^l m$$

and consider any word w of length j. There is a quasi self-similarity mapping $\Phi : I_w \to C$ such that for all words v of length i, there is some $x_v \in C$ with

$$B\left(x_v, m^{-1}\frac{|I_{wv}|}{|I_w|}\right) \subseteq \Phi(I_{wv}) \subseteq B\left(x_v, m\frac{|I_{wv}|}{|I_w|}\right). \qquad (3.2)$$

But $m\dfrac{|I_{wv}|}{|I_w|} = m\,|I_v|\,\dfrac{r_{1+j} \ldots r_{i+j}}{r_1 \ldots r_i} < \dfrac{|I_v|}{2^l}$. Thus

$$\bigcup_{|v|=i} \Phi(I_{wv}) \subseteq \bigcup_{|v|=i} B\left(x_v, \frac{|I_v|}{2^l}\right).$$

For each $v \in W$ chose the word $s(v)$ of minimal length such that $B\left(x_v, \dfrac{|I_v|}{2^l}\right) \supseteq I_{s(v)}$. As $s(v)$ is minimal in length, $B\left(x_v, \dfrac{|I_v|}{2^l}\right)$ will intersect at most four intervals of step $|s(v)|$. We will write $I^*_{s(v)}$ for the union of these (at most) four intervals. The fact that $B\left(x_v, \dfrac{|I_v|}{2^l}\right) \supseteq I_{s(v)}$ means that

$$\frac{r_1 \cdots r_i}{2^l} = \frac{|I_v|}{2^l} \geq \frac{\left|I_{s(v)}\right|}{2} = \frac{r_1 \cdots r_{|s(v)|}}{2},$$

and this implies $2^{-l+1} \geq r_{i+1} \ldots r_{|s(v)|} \geq a^{|s(v)|-i}$. As $2^{l-1} > a^{-4k}$ this can only be true if $|s(v)| > 4k + i$.

Taking v to be the empty word e and I_u the step k interval containing x_e, it follows from the choice of k and (3.2) that $I_u \subseteq B(x_e, m^{-1}) \subseteq \Phi(I_w)$. Since $I_w = \bigcup_{|v|=i} I_{wv}$ this gives the inclusions

$$I_u \subseteq \bigcup_{|v|=i} \Phi(I_{wv}) \subseteq \bigcup_{|v|=i} B\left(x_v, \frac{|I_v|}{2^l}\right) \subseteq \bigcup_{|v|=i} I^*_{s(v)}.$$

But the set on the right is a union of at most 2^{i+2} intervals of step $|s(v)|$, while I_u is the union of $2^{|s(v)|-k}$ such intervals. Since $s(v) > 4k + i$ this is a contradiction.

Case 2. The ratio is unbounded from below: The argument is similar to the first case. Select i, j such that $\dfrac{r_1 \cdots r_i}{r_{1+j} \cdots r_{i+j}} < \dfrac{1}{m2^l}$.

Let w be a word of length j and let $\Phi : C \to I_w$ be an m-quasi self-similarity map. For all words v of length i we have

$$\Phi(I_v) \subseteq B\left(x_v, m\,|I_v|\,|I_w|\right) \subseteq B\left(x_v, \frac{|I_{wv}|}{2^l}\right).$$

The fact that $2^k \geq m$ also ensures that for some $u \in W$ of length $k \geq 1$, the Cantor interval I_{wu} is contained in $B(x_e, m^{-1}\,|I_w|)$. Thus the quasi self-similarity property of Φ also implies that $B(x_e, m^{-1}\,|I_w|) \subseteq \Phi(I_e) \subseteq \bigcup_{|v|=i} \Phi(I_v)$, and therefore $I_{wu} \subseteq \bigcup_{|v|=i} B\left(x_v, \frac{|I_{wv}|}{2^l}\right)$.

Choose $s(v)$ of minimal length such that $B\left(x_v, \frac{|I_{wv}|}{2^l}\right) \supseteq I_{s(v)}$; then $\frac{|I_{wv}|}{2^{l-1}} \geq |I_{s(v)}|$ and $B\left(x_v, \frac{|I_{wv}|}{2^l}\right) \subseteq I^*_{s(v)}$ where $I^*_{s(v)}$ is a union of at most four step $s(v)$ intervals. Similar arguments to those used in the first case show that $|s(v)| > i + j + 4k$, which leads to the contradiction $I_{wu} \subseteq \bigcup_{|v|=i} I^*_{s(v)}$. This proves inequality (3.1) holds when C supports a p-Cantor measure quasi self-similar by Cantor intervals.

Now we will show that if (3.1) holds, then every p-Cantor measure is quasi self-similar by Cantor intervals from below. The proof for quasi self-similar from above is similar. Given a Cantor interval I_w we define a map $\Phi : I_w \to C$ by the rule $\Phi(I_{wv}) = I_v$ and extend this to elements of I_w in the natural way. First, we will show that there exists m such that for all $v \in W$ there is some $x \in I_{wv}$ with

$$B\left(\Phi(x), m^{-1}\frac{|I_{wv}|}{|I_w|}\right) \subseteq \Phi(I_{wv}) \subseteq B\left(\Phi(x), m\frac{|I_{wv}|}{|I_w|}\right).$$

If one takes x such that $\Phi(x)$ is an endpoint of G_v, then these inclusions will be satisfied if there exists some m (independent of v) with $|I_v| \leq m\frac{|I_{wv}|}{|I_w|}$ and $|I_{v0}| \geq m^{-1}\frac{|I_{wv}|}{|I_w|}$. Upon simplifying, these inequalities reduce to

$$(r_{|v|+1}m)^{-1} \leq \frac{r_1 \cdots r_{|v|}}{r_{1+|w|} \cdots r_{|v|+|w|}} \leq m.$$

Thus if there exists $m \geq n$ such that $(r_{|v|+1}m)^{-1} \leq n^{-1}$ for all $v \in W$, then we are done. However, if one takes $i = 1$, and $j = |v|$ in (3.1), the inequality $n/r_{|v|+1} \leq n^2/r_1$ is derived. Thus if $m \geq n^2/r_1$, we have the desired inclusions.

If μ is any p-Cantor measure, then since $\mu(I_{wv})/\mu(I_w) = \mu(I_v)$, the inequalities $\dfrac{m^{-1}\mu(A)}{\mu(I_w)} \le \mu(\Phi(A)) \le \dfrac{m\mu(A)}{\mu(I_w)}$ hold for all Cantor intervals $A \subseteq I_w$ and any $m \ge 1$. As these sets generate the Borel sets in C the same relationship holds for all Borel sets A. This completes the proof that μ is quasi self-similar from below. $\qquad\square$

Corollary 3. *If C supports a p-Cantor measure which is quasi self-similar by Cantor intervals, then every p-Cantor measure is quasi self-similar by Cantor intervals.*

The Hausdorff dimension of the Cantor set $C_{\{r_k\}}$ is known ([1]) to equal

$$\liminf_{k\to\infty} \frac{\log 2}{\frac{1}{k}\,|\log r_1 \cdots r_k|},$$

hence it is of interest to know how condition (3.1) compares to convergence properties of the geometric means of the ratios of dissection. We have the following relationships.

Proposition 4. *(i) If condition (3.1) is satisfied, i.e. there exists n such that for all integers i, j $n^{-1} \le \dfrac{r_1 \cdots r_i}{r_{1+j} \cdots r_{i+j}} \le n$, then $(r_1 \cdots r_k)^{1/k}$ converges (to a non-zero limit). The converse is not true.*

(ii) Condition (3.1) will be satisfied for some n if there exist $0 < a, b, \lambda < \infty$ such that $a \le \dfrac{r_1 \cdots r_k}{\lambda^k} \le b$ for all k.

PROOF. (i) Let $H_k = r_1 \cdots r_k$ and $S_k = \log(H_k)$. Condition (3.1) implies that for all i, j we have $n^{-1} \le \dfrac{H_i H_j}{H_{i+j}} \le n$. Thus if we let $\delta = \log n$ and rearrange terms we obtain

$$-\delta + S_i + S_j \le S_{i+j} \le \delta + S_i + S_j. \tag{3.3}$$

It suffices to prove that $\{S_j/j\}$ forms a Cauchy sequence. First note that an induction argument (on i) using (3.3) shows that $-i\delta + iS_j \le S_{ij} \le i\delta + iS_j$. Manipulating this inequality gives $-(i+j)\delta + iS_j \le jS_i \le (i+j)\delta + iS_j$, and thus $\left|\dfrac{S_i}{i} - \dfrac{S_j}{j}\right| \le \left(\dfrac{1}{i} + \dfrac{1}{j}\right)\delta$. Clearly this implies that $\{S_j/j\}$ is a Cauchy sequence.

To see that the converse is not true note that $(r_1 \cdots r_k)^{1/k}$ can converge, and yet $\inf r_k = 0$. From Proposition 1 we know that condition (3.1) cannot be satisfied in this case.

(ii) One can easily check that (3.1) is satisfied with $n = \max(b^2/a, b/a^2)$. $\qquad\square$

Corollary 5. *(i) If C supports a p-Cantor measure which is quasi self-similar by Cantor intervals, then C has positive Hausdorff dimension.*

(ii) There are Cantor sets of positive Hausdorff dimension which do not support any quasi self-similar p-Cantor measures.

Corollary 6. *Every p-Cantor measure on $C_{\{r_k\}}$ is quasi self-similar by Cantor intervals if there exist $0 < a, b, \lambda < \infty$ such that $a \leq \dfrac{r_1 \ldots r_k}{\lambda^k} \leq b$ for all k.*

4 Equivalence of Quasi Self-Similarity by Balls and Cantor Intervals

In this section we will show that for p-Cantor measures supported on Cantor sets with ratios of dissection bounded away from $1/2$, quasi self-similarity by balls or Cantor intervals coincide, but that they need not coincide in general. The assumption on the ratios of dissection is important because it ensures that balls and appropriate Cantor intervals are comparable. (It effectively replaces the strong separation condition.) One formulation of this is given in the lemma below.

Lemma 7. *Suppose $C = C_{\{r_k\}}$ is a Cantor set with $\sup r_k < 1/2$. There exists an integer N such that if $x \in I_w$, then $I_w \subseteq B(x, |I_w|) \subseteq I_{T^N(w)}$. Similarly, if $x \in C$ and $d > 0$, then there is some $w \in W$ such that $x \in I_w \subseteq B(x, d) \subseteq I_{T^N(w)}$.*

PROOF. The assumption that $\sup r_k < 1/2$ is equivalent to the statement that there exists $\delta > 0$ (namely, $\delta = \sup(1 - 2r_k)$) such that $|G_v| \geq \delta |I_v|$ for all $v \in W$. Choose N such that $2^{-N+1} < \delta$. As $\left|I_{T^N(w)}\right| \geq 2^N |I_w|$, this ensures that $|I_w| < |G_v|$ for any v such that $|v| \leq |w| - N$. Thus $B(x, |I_w|)$ must be contained in $I_{T^N(w)}$.

To prove the second statement, choose k such that $r_1 \cdots r_k \leq d < r_1 \cdots r_{k-1}$, let I_w be the Cantor interval of step k containing x, and argue as in the first paragraph. □

Theorem 8. *Suppose $C = C_{\{r_k\}}$ is a Cantor set with $\sup r_k < 1/2$. Any p-Cantor measure on C is quasi self-similar by balls if and only if it is quasi self-similar by Cantor intervals.*

PROOF. Throughout the proof we fix δ with $0 < \delta < \sup(1 - 2r_k)$. First, assume μ is m-quasi self-similar by balls on C. We will begin by establishing that μ is quasi self-similar by Cantor intervals from below, so let I_w be a Cantor interval and denote by x an endpoint of I_w. Let $\Phi : B(x, |I_w|) \to C$

be a quasi self-similarity map from below, suppose $I_v \subseteq I_w$ and assume z is the endpoint of I_v closest to x. Because $\delta \, |I_v|$ is less than the size of any gap of step $j \leq |v|$, $B(z, \delta \, |I_v|) \subseteq I_v$. The choice of z ensures that

$$I_v \subseteq B(z, |I_v|) \subseteq B(x, |I_w|);$$

so the definition of an m-quasi self-similar map yields the inclusions

$$B\left(\Phi(z), \frac{m^{-1}\delta \, |I_v|}{|I_w|}\right) \subseteq \Phi\left(B(z, \delta \, |I_v|)\right) \subseteq \Phi(I_v)$$

$$\subseteq \Phi\left(B(z, |I_v|)\right) \subseteq B\left(\Phi(z), \frac{m \, |I_v|}{|I_w|}\right).$$

Moreover, for all Borel sets $A \subseteq B(x, |I_w|)$ we have

$$\frac{m^{-1}\mu(A)}{\mu\left(B(x, |I_w|)\right)} \leq \mu(\Phi(A)) \leq \frac{m\mu(A)}{\mu\left(B(x, |I_w|)\right)}.$$

By the lemma there is a constant N (independent of x and w) such that $I_w \subseteq B(x, |I_w|) \subseteq I_{T^N(w)}$. Thus $\mu\left(B(x, |I_w|)\right)$ is comparable to $\mu(I_w)$. These observations show that for a suitable constant m_1 and all Borel sets $A \subseteq I_w$, we have

$$B\left(\Phi(z), \frac{m_1^{-1} \, |I_v|}{|I_w|}\right) \subseteq \Phi(I_v) \subseteq B\left(\Phi(z), \frac{m_1 \, |I_v|}{|I_w|}\right),$$

and

$$\frac{m_1^{-1}\mu(A)}{\mu(I_w)} \leq \mu(\Phi(A)) \leq \frac{m_1\mu(A)}{\mu(I_w)}.$$

Thus Φ restricted to I_w is a suitable map and therefore μ is quasi self-similar by Cantor intervals from below.

To see that μ is quasi self-similar by Cantor intervals from above we again start with a Cantor interval I_w. Choose x in the Cantor set belonging to I_w and let $\Phi : C \to B(x, \delta \, |I_w|) \subseteq I_w$ be a m-quasi self-similarity map from above. If $z \in C \bigcap I_v$, then choosing ζ as in the definition of quasi self-similarity by balls we obtain the inclusions

$$B(\zeta, m^{-1}\delta \, |I_v| \, \delta \, |I_w|) \subseteq \Phi(B(z, \delta \, |I_v|)) \subseteq \Phi(I_v)$$

$$\subseteq \Phi(B(z, |I_v|)) \subseteq B(\zeta, m \, |I_v| \, \delta \, |I_w|).$$

If N is chosen such that $2^{-N} \leq \delta$, then $B(x, \delta \, |I_w|)$ will contain the step $|w|+N$ Cantor subinterval of $I_{|w|}$ which contains x. This ensures that $\mu(B(x, \delta \, |I_w|))$ is comparable to $\mu(I_w)$, and similar arguments to those used in the earlier part of the proof imply that μ is quasi self-similar by Cantor intervals from above.

Now assume μ is quasi self-similar by Cantor intervals on C. We will show quasi self-similarity by balls from below; the proof for quasi self-similarity from above is similar. Consider a ball $B(x, d)$ with center $x \in C$ and, as in the lemma, choose the Cantor interval I_w and integer N independent of x and d such that $|I_w| \le d < |I_{T(w)}|$ and $x \in I_w \subseteq B(x, d) \subseteq I_{T^N(w)}$. Let $\Phi : I_{T^{2N}(w)} \to C$ be a quasi self-similarity map from below and suppose $B(y, s) \subseteq B(x, d)$. Choose v such that $I_v \subseteq B(y, s) \subseteq I_{T^N(v)}$. Since $I_v \subseteq I_{T^N(w)}$, we have $I_{T^N(v)} \subseteq I_{T^{2N}(w)}$, thus the quasi self-similarity properties of Φ imply that

$$B\left(\Phi(x), \frac{m^{-1}|I_v|}{|I_{T^{2N}(w)}|}\right) \subseteq \Phi(I_v) \subseteq \Phi(B(y, s))$$

$$\subseteq \Phi(I_{T^N(v)}) \subseteq B\left(\Phi(x), \frac{m|I_{T^N(v)}|}{|I_{T^{2N}(w)}|}\right).$$

The assumption of quasi self-similarity by Cantor intervals ensures that the ratios of dissection of C are bounded away from zero, hence $|I_{T^{2N}(w)}|$ is comparable to d, and the lengths of I_v and $I_{T^N(v)}$ are comparable to s. Also, $\mu(I_{T^{2N}(w)})$ and $\mu(B(x, d))$ are comparable, hence there is a constant m_1 such that

$$B\left(\Phi(x), \frac{m_1^{-1}s}{d}\right) \subseteq \Phi(B(y, s)) \subseteq B\left(\Phi(x), \frac{m_1 s}{d}\right),$$

and for all Borel sets $A \subseteq B(x, d)$,

$$\frac{m_1^{-1}\mu(A)}{\mu(B(x, d))} \le \frac{m^{-1}\mu(A)}{\mu(I_{T^{2N}(w)})} \le \mu(\Phi(A)) \le \frac{m\mu(A)}{\mu(I_{T^{2N}(w)})} \le \frac{m_1\mu(A)}{\mu(B(x, d))}.$$

Thus Φ restricted to the ball $B(x, d)$ is a suitable quasi self-similarity function. $\square$

Corollary 9. *Suppose $\sup_k r_k < 1/2$. Then a p-Cantor measure on $C_{\{r_k\}}$ is quasi self-similar by balls if and only if there exists n such that (3.1) holds for all i, j. If $C_{\{r_k\}}$ supports a p-Cantor measure which is quasi self-similar by Cantor intervals (or, equivalently, balls), then every p-Cantor measure is quasi self-similar by Cantor intervals and balls.*

However, the two quasi self-similarity properties are not the same. To show this we will prove one additional property of quasi self-similarity by balls.

Proposition 10. *Suppose C is a Cantor set whose gaps decrease in length as the step increases. If there is a p-Cantor measure on C which is quasi self-similar by balls, then $\sup_k r_k < 1/2$.*

Proof. Suppose μ is m-quasi self-similar and choose n such that

$$m \max(p, 1-p)^n < \min(p, 1-p).$$

Assume to the contrary that $\sup_k r_k = 1/2$. Then we can choose $k > n$ such that $\dfrac{m\,|G_{k-1}|}{|I_k| + |G_{k-1}|} = \dfrac{m(1 - 2r_k)}{1 - r_k} < |G_n|$. Set $s = |G_{k-1}|$, $d = |I_k|$ and let $x \in I_k$ be an endpoint of G_{k-1}. Clearly, $I_k \subseteq B(x, d) \subseteq I_{k-1}$. Since μ is quasi self-similar by balls, there exists a quasi self-similarity map $\Phi : B(x, d+s) \to C$ which has the property that for any $z \in B(x, d)$,

$$\Phi(B(z, s)) \subseteq B\left(\Phi(z), \frac{ms}{r+s}\right).$$

The assumption of decreasing gap lengths ensures that any gap of the Cantor set, contained in $B(x, d)$, has length at most s. This means we can choose elements of the Cantor set, $z_1, ..., z_m$ in $B(x, d)$, such that the balls $B(z_i, s)$ cover $B(x, d)$ and for each i, $B(z_i, s) \bigcap B(z_{i+1}, s)$ is non-empty. But because $ms/(d + s)$ is less than the length of any gap of step at most n, $B\left(\Phi(z), \dfrac{ms}{d+s}\right)$ must be contained in a Cantor interval of step $n+1$. Consequently, $\Phi(B(z_i, s)) \subseteq I_{v_i}$ for some $v_i \in W$ of length $n+1$. Since $\Phi(B(z_i, s))$ and $\Phi(B(z_{i+1}, s))$ intersect non-trivially, the intervals I_{v_i} intersect, and therefore coincide. Thus $\Phi(B(x, d)) \subseteq I_v$ for some $|v| = n + 1$. The quasi self-similarity property also implies that

$$\frac{m^{-1}\mu(B(x, d))}{\mu(B(x, d+s))} \leq \mu(\Phi(B(x, d)) \leq \mu(I_v).$$

Finally, we note that $B(x, d + s) \subseteq I_{k-1}$, thus

$$\frac{m^{-1}\mu(B(x, d))}{\mu(B(x, d+s))} \geq \frac{m^{-1}\mu(I_k)}{\mu(I_{k-1})} \geq m^{-1}\min(p, 1-p).$$

But n was chosen so that $\mu(I_v) < m^{-1}\min(p, 1-p)$ which is a contradiction. $\square$

An immediate corollary of this Proposition and Theorem 8 is the following.

Corollary 11. *If C is a Cantor set with decreasing gaps, then any p-Cantor measure which is quasi self-similar by balls is quasi self-similar by Cantor intervals.*

Corollary 12. *A Cantor set with ratios of dissection increasing to* $1/2$ *does not support any p-Cantor measure quasi self-similar by balls.*

PROOF. Since the ratios are increasing, the gaps must be decreasing in size, and hence the proposition directly applies. $\square$

We can now give an example of a p-Cantor measure which is quasi self-similar by Cantor intervals but not balls. We do not know if there are p-Cantor measures which are quasi self-similar by balls but not by intervals.

Corollary 13. *If* $r_k = 2^{-1-1/k^2}$, *then every p-Cantor measure on the Cantor set* $C_{\{r_k\}}$ *is quasi self-similar by Cantor intervals but not by balls.*

PROOF. One can easily check that $\{r_k\}$ satisfies (3.1). However, the ratios increase to $1/2$ and thus the no p-Cantor measure is quasi self-similar by balls. $\square$

5 Multifractal Analysis of p-Cantor Measures

In [8] O'Neil showed that the multifractal formalism developed by Olsen [6] could be applied to quasi self-similar measures. In this section we obtain similar results for a broad class of p-Cantor measures, including those which are not necessarily quasi self-similar, but are supported on Cantor sets where the geometric means of the ratios of dissection converge.

Throughout the section we will assume that $\sup r_k < 1/2$, and μ will denote a p-Cantor measure where, without loss of generality, we will assume $p = \max(p, 1-p)$.

We begin our study of the multifractal structure of Cantor sets by recalling the definition of the pointwise local upper and lower dimensions of μ at $x \in C$:

$$\overline{\dim}_{loc}\mu(x) = \limsup_{d \to 0} \frac{\log \mu(B(x,d))}{\log d};$$
$$\underline{\dim}_{loc}\mu(x) = \liminf_{d \to 0} \frac{\log \mu(B(x,d))}{\log d}.$$

The pointwise local dimension of μ is $\dim_{loc}\mu(x) = \lim_{d \to 0} \frac{\log \mu(B(x,d))}{\log d}$ if this limit exists.

To analyze these local dimensions we will make significant use of the fact that balls centered in C and appropriate Cantor intervals have comparable measure. For instance, if k is chosen such that $r_1 \cdots r_k \leq d < r_1 \cdots r_{k-1}$ and I_w is the step k Cantor interval containing $x \in C$, then we know from

Lemma 7 that $I_w \subseteq B(x, d) \subseteq I_{T^N(w)}$, and hence $p_{w_1} \cdots p_{w_k} \leq \mu(B(x, d)) \leq p_{w_1} \cdots p_{w_{k-N}}$. Thus

$$\frac{\log p_{w_1} \cdots p_{w_{k-N}}}{\log r_1 \cdots r_k} \leq \frac{\log \mu(B(x, d))}{\log d} \leq \frac{\log p_{w_1} \cdots p_{w_k}}{\log r_1 \cdots r_{k-1}}.$$

Since N can be chosen independently of x and d, the limiting behavior (as $k \to \infty$) of $\dfrac{\log p_{w_1} \ldots p_{w_{k-N}}}{\log r_1 \ldots r_k}$ and $\dfrac{\log p_{w_1} \cdots p_{w_k}}{\log r_1 \cdots r_{k-1}}$ are the same, and coincide with the limits of $\dfrac{\log \mu(I_w)}{\log |I_w|} = \dfrac{\log p_{w_1} \ldots p_{w_k}}{\log r_1 \ldots r_k}$. This means that in the definition of pointwise local dimension we may replace balls by Cantor intervals, in other words, $\dim_{loc} \mu(x) = \lim_{x \in I_w ; |I_w| \to 0} \dfrac{\log \mu(I_w)}{\log |I_w|}$.

Motivated by the fact that box dimensions are often easier to calculate than Hausdorff dimension, Olsen ([6] Sec 2.7) introduced the multifractal box dimensions. To define these we need the following terminology. A countable family of closed balls $(B(x_i, d))_i$ is called a centered packing of C if $x_i \in C$ and the balls are pairwise disjoint. For $q \in \mathbb{R}$, and $d > 0$ set

$$S_d^q = \sup\{\sum_i \mu(B(x_i, d))^q : (B(x_i, d))_i \text{ is a centred packing of } C\}.$$

The upper and lower multifractal q-box dimensions, C^q and C_q, are given by

$$C^q = \limsup_{d \to 0} \frac{\log S_d^q}{|\log d|} \text{ and } C_q = \liminf_{d \to 0} \frac{\log S_d^q}{|\log d|}.$$

The comparison between measures of balls and Cantor intervals enables us to easily calculate these as we demonstrate below.

Proposition 14. *There are constants $a = a(q), b = b(q)$ such that if $r_1 \cdots r_k \leq d < r_1 \cdots r_{k-1}$, then $a(p^q + (1-p)^q)^k \leq S_d^q \leq b(p^q + (1-p)^q)^k$.*

PROOF. Let $(B(x_i, d))_i$ be a centered packing of C. Let $I_w(x_i)$ be the Cantor interval of step k containing x_i, so $I_w(x_i) \subseteq B(x_i, d) \subseteq I_{T^N(w)}(x_i)$. The balls $B(x_i, d)$ are disjoint by definition, thus the intervals $I_w(x_i)$ are distinct. Since each step $k - N$ interval contains 2^N step k subintervals, a step $k - N$ interval can be the interval $I_{T^N(w)}(x_i)$ for at most 2^N choices of x_i. Hence

$$\sum_i \mu(B(x_i, d))^q \leq 2^N \sum_{|w|=k-N} \left(p_{w_1} \ldots p_{w_{k-N}}\right)^q$$
$$= 2^N (p^q + (1-p)^q)^{k-N} \equiv b(p^q + (1-p)^q)^k.$$

To obtain the reverse inequality choose N such that $\left|G_{T^j(w)}\right| \geq \left|I_{T(w)}\right|$ for all $j \geq N$ and for all $w \in W$. Let $x_1, ..., x_{2^{k-N}}$ denote the left hand endpoints of the step $k - N$ Cantor intervals. Since these step $k - N$ intervals are separated by gaps of at least d, the balls $B(x_i, d)$, $i = 1, ..., 2^{k-N}$, must be disjoint and hence form a centered packing of C. Because $d \geq r_1 \cdots r_k$, $B(x_i, d)$ contains the step k interval having x_i as its left endpoint. Thus

$$\sum \mu(B(x_i, d))^q \geq \sum_{|w|=k-N} p^{Nq} \left(p_{w_1} \cdots p_{w_{k-N}}\right)^q$$
$$= p^{Nq}(p^q + (1-p)^q)^{k-N} = a(p^q + (1-p)^q)^k.$$

$\square$

Corollary 15. *The upper and lower multifractal q-box dimensions have the values*

$$C_q = \liminf_{k \to \infty} \frac{\log(p^q + (1-p)^q)}{\left|\frac{1}{k} \log r_1 \cdots r_k\right|}; \ C^q = \limsup_{k \to \infty} \frac{\log(p^q + (1-p)^q)}{\left|\frac{1}{k} \log r_1 \cdots r_k\right|}.$$

PROOF. This is a routine consequence of the proposition and the definition of the multifractal dimensions. $\square$

For $\alpha \in \mathbb{R}$ set

$$X^\alpha = \{x \in C : \overline{\dim}_{loc}\mu(x) = \alpha\},$$
$$X_\alpha = \{x \in C : \underline{\dim}_{loc}\mu(x) = \alpha\},$$
$$X(\alpha) = \{x \in C : \dim_{loc}\mu(x) = \alpha\} = X_\alpha \bigcap X^\alpha.$$

These sets can be viewed as giving a multifractal decomposition of the Cantor set C relative to the measure μ, and the aim of the fine approach to multifractal analysis is to determine their dimensions. We will follow the method of [4], Sec. 11.2 by showing the existence of probability measures concentrated on these sets.

For this, define $\beta_i = \beta_i(q)$ by $(p^q + (1-p)^q)r_i^{\beta_i} = 1$ and define a set function $\upsilon = \upsilon_q$ on the Cantor intervals by the rule $\upsilon(I_w) = (p_{w_1} \cdots p_{w_k})^q r_1^{\beta_1} \cdots r_k^{\beta_k}$ if $|w| = k$. Notice that $\upsilon(I_{w0}) + \upsilon(I_{w1}) = \upsilon(I_w)$ and consequently υ extends to a probability measure supported on C. Since $\log \prod_{i=1}^k r_i^{\beta_i} = -k \log(p^q + (1-p)^q)$, the reader can readily use the definition of β_i to verify that when $|w| = k$,

$$\frac{\log \upsilon(I_w)}{\log |I_w|} = q \frac{\log \mu(I_w)}{\log |I_w|} + \frac{\log(p^q + (1-p)^q)}{\left|\frac{1}{k} \log r_1 ... r_k\right|}.$$

As with μ, the local behavior of υ is the same whether one considers balls or Cantor intervals, thus we obtain

$$\underline{\dim}_{loc}\upsilon(x) = \liminf_{x \in I_w; |I_w| \to 0} \frac{\log \upsilon(I_w)}{\log |I_w|}$$

$$= \liminf \left(q \frac{\log \mu(I_w)}{\log |I_w|} + \frac{\log(p^q + (1-p)^q)}{\left|\frac{1}{k} \log r_1 ... r_k\right|} \right)$$

$$\geq \liminf q \frac{\log \mu(I_w)}{\log |I_w|} + C_q.$$

Notice the inequality is an equality if $\lim_{x \in I_w; |I_w| \to 0} \dfrac{\log \mu(I_w)}{\log |I_w|}$ exists, and in this case we have $\underline{\dim}_{loc}\upsilon(x) = q \lim_{x \in I_w; |I_w| \to 0} \dfrac{\log \mu(I_w)}{\log |I_w|} + C_q$. Similarly, $\overline{\dim}_{loc}\upsilon(x) \leq \limsup q \dfrac{\log \mu(I_w)}{\log |I_w|} + C^q$, again with equality if $\lim \dfrac{\log \mu(I_w)}{\log |I_w|}$ exists. Set

$$\alpha^* \equiv \alpha^*(q) = \frac{-(p^q \log p + (1-p)^q \log(1-p))}{(p^q + (1-p)^q) \limsup_k \left|\frac{1}{k} \log r_1 \cdots r_k\right|}$$

and

$$\alpha_* \equiv \alpha_*(q) = \frac{-(p^q \log p + (1-p)^q \log(1-p))}{(p^q + (1-p)^q) \liminf_k \left|\frac{1}{k} \log r_1 \cdots r_k\right|}.$$

We are now ready to state the technical result from which we can determine bounds on the Hausdorff and packing dimensions of $X_{\alpha^*} \bigcap X^{\alpha_*}$.

Theorem 16. *There exists a probability measure υ concentrated on $X_{\alpha^*} \bigcap X^{\alpha_*}$ which has the property that for υ a.e. x in $X_{\alpha^*} \bigcap X^{\alpha_*}$:*

(i) If $q \geq 0$, then $\underline{\dim}_{loc}\upsilon(x) \geq q\alpha^* + C_q$ and $\overline{\dim}_{loc}\upsilon(x) \leq q\alpha_* + C^q$;

(ii) If $q \leq 0$, then $\underline{\dim}_{loc}\upsilon(x) \geq q\alpha_* + C_q$ and $\overline{\dim}_{loc}\upsilon(x) \leq q\alpha^* + C^q$.

The proof of the theorem can be obtained in a similar fashion to [4], Prop. 11.4 once we have established the following lemma.

Lemma 17. *Let $\varepsilon > 0$, $s \in \mathbb{R}$ and let $\delta > 0$ be small. Let*

$$\alpha_s \equiv \alpha_s(q) = \frac{-(p^q \log p + (1-p)^q \log(1-p))}{(p^q + (1-p)^q) s},$$

and for $k \in \mathbb{N}$ set $\Phi_{\pm}(\alpha_s) = \log \prod_{i=1}^{k} \left(p^{q \pm \delta} + (1-p)^{q \pm \delta} \right) r_i^{\beta_i + (\varepsilon \mp \alpha_s)\delta}$. Then

$$\Phi_{\pm}(\alpha_s) \leq k \left(\mp \alpha_s s\delta + O(\delta^2) - \varepsilon\delta \log 2 \pm \frac{\alpha_s}{k}\delta \sum_{i=1}^{k} |\log r_i| \right).$$

If $t = \limsup \frac{1}{k} \sum_{i=1}^{k} |\log r_i|$, then there is a constant c such that $\Phi_{+}(\alpha_t) \leq -k\delta c\varepsilon$ for all but finitely many k, and $\Phi_{-}(\alpha_t) \leq -k\delta c\varepsilon$ for infinitely many k.

If $t = \liminf \frac{1}{k} \sum_{i=1}^{k} |\log r_i|$, then there is a constant c such that $\Phi_{+}(\alpha_t) \leq -k\delta c\varepsilon$ for infinitely many k, and $\Phi_{-}(\alpha_t) \leq -k\delta c\varepsilon$ for all but finitely many k.

PROOF. This is essentially a calculus exercise. Using the first order Taylor expansion of $(p^x + (1-p)^x)$ at $x = q$ and the fact that $-\log (p^q + (1-p)^q) = \beta_i \log r_i$, we readily obtain

$$\Phi_{\pm}(\alpha_s) = \sum_{1}^{k} \left(\log \left(p^{q \pm \delta} + (1-p)^{q \pm \delta} \right) - \log \left(p^q + (1-p)^q \right) \right)$$

$$+ \delta(\varepsilon \mp \alpha_s) \sum_{i=1}^{k} \log r_i$$

$$= \pm \delta k \left(\frac{p^q \log p + (1-p)^q \log(1-p)}{p^q + (1-p)^q} \right) + k O(\delta^2)$$

$$- \delta \sum_{i=1}^{k} (\varepsilon \mp \alpha_s) |\log r_i|.$$

Upon recalling the definition of α_s and noting that $\frac{1}{k} \sum_{i=1}^{k} |\log r_i| \geq \log 2$, the first claim is proved.

If $t = \limsup \frac{1}{k} \sum_{i=1}^{k} |\log r_i|$, then given any $\varepsilon_1 > 0$, for all but finitely many k we have $\frac{1}{k} \sum_{i=1}^{k} |\log r_i| \leq t + \varepsilon_1$, and for infinitely many k, $\frac{1}{k} \sum_{i=1}^{k} |\log r_i| \geq t - \varepsilon_1$. Because $t \geq 0$ we have $\alpha_t \geq 0$, and thus for suitably small ε_1 and δ,

$$\Phi_{\pm}(\alpha_t) \leq k \left(\mp \alpha_t t\delta + O(\delta^2) - \varepsilon\delta \log 2 \pm \alpha_t\delta(t \pm \varepsilon_1) \right) \leq -k\delta c\varepsilon$$

for all but finitely many k in the case of Φ_{+} and for infinitely many k for Φ_{-}.

The argument is similar if $t = \liminf \frac{1}{k} \sum_{i=1}^{k} |\log r_i|$. $\qquad \square$

PROOF OF THEOREM 16. Let $\alpha \in \mathbb{R}$, fix $\varepsilon > 0$ and let $\delta > 0$ be small. Temporarily fix k and let $I_w(x)$ denote the interval of step k containing x.

Then

$$\upsilon\{x : \mu(I_w(x)) \geq |I_w(x)|^{\alpha - \varepsilon}\} \leq \int \mu(I_w(x))^\delta |I_w(x)|^{-(\alpha - \varepsilon)\delta} \, d\upsilon(x)$$

$$= \sum_{|w|=k} \mu(I_w)^\delta |I_w|^{-(\alpha - \varepsilon)\delta} \upsilon(I_w)$$

$$= \prod_{i=1}^{k} \left(p^{q+\delta} + (1-p)^{q+\delta} \right) r_i^{\beta_i + (\varepsilon - \alpha)\delta} = \exp \Phi_+(\alpha).$$

Similarly, $\upsilon\{x : \mu(I_w(x)) \leq |I_w(x)|^{\alpha + \varepsilon}\} \leq \exp \Phi_-(\alpha)$.

First, let $t = \limsup \frac{1}{k} \sum_{i=1}^{k} |\log r_i|$. Then $\alpha_t = \alpha^*$ and hence from the lemma $\upsilon\{x : \mu(I_w(x)) \geq |I_w(x)|^{\alpha^* - \varepsilon}\} \leq \exp(-k\delta c\varepsilon)$ for all but finitely many k, say for $k \geq k_1$. This means that

$$\sum_{k \geq k_1} \upsilon\{x : \mu(I_w(x)) \geq |I_w(x)|^{\alpha^* - \varepsilon}\} \leq \sum_{k \geq k_1} \exp(-k\delta c\varepsilon) < \infty,$$

and therefore the probability that $\mu(I_w(x)) \geq |I_w(x)|^{\alpha^* - \varepsilon}$ infinitely often is zero. In other words, for υ a.e. x, $\mu(I_w(x)) \leq |I_w(x)|^{\alpha^* - \varepsilon}$ eventually. Since $\varepsilon > 0$ was arbitrary $\liminf \dfrac{\log \mu(I_w(x))}{\log |I_w(x)|} \geq \alpha^*$ for υ a.e. x.

Similarly, for infinitely many k, say $k \in \{k_n\}$, the lemma shows that

$$\upsilon\{x : \mu(I_w(x)) \leq |I_w(x)|^{\alpha^* + \varepsilon}\} \leq \exp(-k\delta c\varepsilon) \ .$$

Thus

$$\sum_{k_n} \upsilon\{x : \mu(I_w(x)) \leq |I_w(x)|^{\alpha^* + \varepsilon}\} \leq \sum_{k_n} \exp(-k_n \delta c\varepsilon) < \infty,$$

and so for infinitely many k and υ a.e. x we have $\mu(I_w(x)) \geq |I_w(x)|^{\alpha^* + \varepsilon}$. Consequently,

$$\underline{\dim}_{loc}\mu(x) = \liminf \frac{\log \mu(I_w(x))}{\log |I_w(x)|} = \alpha^* \text{ for } \upsilon \text{ a.e. } x.$$

As $X_{\alpha^*} = \{x : \underline{\dim}_{loc}\mu(x) = \alpha^*\}$ it follows that $\upsilon(X_{\alpha^*}) = 1$.

Using $t = \liminf \frac{1}{k} \sum_{i=1}^{k} |\log r_i|$ and $\alpha_t = \alpha_*$, we similarly obtain that

$$\limsup \frac{\log \mu(I_w(x))}{\log |I_w(x)|} = \alpha_* \text{ for } \upsilon \text{ a.e. } x.$$

Thus we also have $\upsilon(X^{\alpha_*}) = 1$ and hence $\upsilon(X_{\alpha^*} \bigcap X^{\alpha_*}) = 1$.

Recall that

$$\underline{\dim}_{loc}\upsilon(x) \geq \liminf q \frac{\log \mu(I_w(x))}{\log |I_w(x)|} + C_q$$

and

$$\overline{\dim}_{loc}\upsilon(x) \leq \limsup q \frac{\log \mu(I_w(x))}{\log |I_w(x)|} + C^q.$$

Thus if $q > 0$,, then $\underline{\dim}_{loc}\upsilon(x) \geq q\alpha^+ + C_q$ and $\overline{\dim}_{loc}\upsilon(x) \leq q\alpha^- + C^q$ for υ a.e. $x \in X_{\alpha^*} \bigcap X^{\alpha_*}$. If $q < 0$ the roles of α^* and α_* are interchanged. $\qquad\square$

Corollary 18. *If $q \geq 0$, then*

$$\dim_H \left(X_{\alpha^*} \bigcap X^{\alpha_*} \right) \geq q\alpha^* + C_q,$$

$$\dim_P \left(X_{\alpha^*} \bigcap X^{\alpha_*} \right) \leq q\alpha_* + C^q.$$

If $q < 0$, then

$$\dim_H \left(X_{\alpha^*} \bigcap X^{\alpha_*} \right) \geq q\alpha_* + C_q$$

$$\dim_P \left(X_{\alpha^*} \bigcap X^{\alpha_*} \right) \leq q\alpha^* + C^q.$$

PROOF. This is immediate from Theorem 16 upon applying the mass distribution principle (see [4], 2.3). $\qquad\square$

If, in addition, $\lim_k \left| \frac{1}{k} \log r_1 \cdots r_k \right| = \tau$, then $\tau = \log 2 / \dim_H C$,

$$C^q = C_q = \log \left(p^q + (1-p)^q \right) / \tau$$

and

$$\alpha^* = \alpha_* = \frac{-\left(p^q \log p + (1-p)^q \log(1-p) \right)}{\left(p^q + (1-p)^q \right) \tau} \equiv \alpha(q).$$

Consequently, for all $x \in X(\alpha(q)) = \{\dim_{loc} \mu(x) = \alpha(q)\}$,

$$\dim_{loc} \upsilon(x) = q\alpha + \frac{\log \left(p^q + (1-p)^q \right)}{\tau}.$$

This gives the formulas

Corollary 19. *If $\lim_k \left| \frac{1}{k} \log r_1 \cdots r_k \right|$ exists, then*

$$\dim_H X(\alpha(q)) = \dim_P X(\alpha(q)) = q\alpha + \frac{\log \left(p^q + (1-p)^q \right) \dim_H C}{\log 2}.$$

Finally, consider the Legendre transform of the function C_q (under the assumption that $\lim_k \left| \frac{1}{k} \log r_1 \cdots r_k \right|$ exists):

$$f(\alpha) = \inf_q (q\alpha + C_q) = \inf_q \left(q\alpha + \frac{\log \left(p^q + (1-p)^q \right)}{\tau} \right).$$

A simple calculus argument shows that the infimum is attained when q is such that $\alpha = \alpha(q)$. Thus $\dim_H X(\alpha(q)) = f(\alpha(q))$. Furthermore, observe that every value in the interval $[-\log p/\tau, -\log(1-p)/\tau]$ is of the form $\alpha(q)$ for some $q \in \mathbb{R}$. Thus the multifractal spectrum of μ (i.e., the dimension of $X(\alpha)$) is given by the Legendre transform of the multifractal q-box dimensions.

Corollary 20. *Let C be a Cantor set and suppose μ is the p-Cantor measure. Assume $\tau = \lim_k \left| \frac{1}{k} \log r_1 \cdots r_k \right|$ exists. Then for every*

$$\alpha \in \left[\frac{-\log p}{\tau}, \frac{-\log(1-p)}{\tau} \right],$$

we have

$$\dim_H X(\alpha) = \dim_P X(\alpha) = f(\alpha)$$

where f is the Legendre transform of the function which gives the multifractal q-box dimensions: $q \longmapsto \log \left(p^q + (1-p)^q \right) / \tau$.

Remark 1. This improves upon results in [8] since we have seen in Prop. 4 that such Cantor measures need not be quasi self-similar.

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