RESEARCH

**Real Analysis Exchange**
Vol. (), , pp. 123–130

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# ON A CONCAVE DIFFERENTIABLE MAJORANT OF A MODULUS OF CONTINUITY

**Abstract**

In this paper we prove that for any modulus of continuity on $[0, \infty)$ there exists a concave majorant that is infinitely differentiable on $(0, \infty)$ and satisfies an additional inequality. This extends the results of Stechkin and Korneychuk obtained previously without the requirement that majorants be differentiable.

Recall that a real function $\omega$ on $[0, \infty)$ or on $[0, l]$, $0 < l < \infty$, is called a modulus of continuity if $\omega$ is continuous, semiadditive, nondecreasing and $\omega(0) = 0$. The concavity of $\omega$ is sometimes a desirable property but in general $\omega$ fails to be concave. In certain cases this difficulty can be surmounted by using a concave majorant of $\omega$. Throughout this paper we assume that $\omega$ differs from the zero function. The following lemma is due to S. B. Stechkin. It was published and applied for the first time in [1].

**Lemma A.** *Let $\omega$ be a modulus of continuity on $[0, \pi]$. Then there exists a concave modulus of continuity $\bar{\omega}$ such that $\omega(t) \le \bar{\omega}(t) < 2\omega(t)$ for $t \in (0, \pi]$. Moreover, the constant 2 cannot be reduced.*

The proof in [1] remains valid if $\pi$ is replaced by any positive number $l$. Later N. P. Korneychuk [2] proved the following lemma.

**Lemma B.** *Let $\omega$ be a modulus of continuity on $[0, \infty)$ and $\bar{\omega}$ be the minimal concave majorant of $\omega$. Then $\bar{\omega}(\mu t) < (1 + \mu)\omega(t)$ for any $t > 0$, $\mu > 0$. This inequality is best possible for each $t > 0$ and each natural $\mu$.*

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Key Words: modulus of continuity, majorant, concave
Mathematical Reviews subject classification: 26A15
Received by the editors October 2, 2000
*This research was carried out with the partial financial support of the Russian Foundation for Basic Research (grant no.99-01-00062) and the program "Leading Scientific Schools" (grant no.00-15-96143).

The construction of $\bar{\omega}$ in [1, 2] proves that any modulus of continuity has a minimal concave majorant which is a modulus of continuity as well. As is known any concave function of one variable is differentiable at each point in its domain except at most a countable set. The question that we are dealing with was suggested by P. L. Ul'janov. Let $\omega$ be a modulus of continuity on $[0,\infty)$. The question is whether there exist a constant $c$ and a concave modulus of continuity $\omega_0$ on $[0,\infty)$ so that the restriction of $\omega_0$ to $(0,\infty)$ has a given order of smoothness and satisfies $\omega(t) \le \omega_0(t) < c\,\omega(t)$. It is natural in view of Lemma B to consider the problem with the inequality of the form $\omega_0(\mu t) < c(\mu)\omega(t)$. The next theorem yields an answer to Ul'janov's problem.

**Theorem.** *Let $\omega$ be a modulus of continuity on $[0,\infty)$ and $I$ be a closed interval in $(0,\infty)$. Then there exists a concave modulus of continuity $\omega_0$ on $[0,\infty)$ such that the restriction of $\omega_0$ to $(0,\infty)$ is infinitely differentiable and satisfies $\omega(\mu t) \le \omega_0(\mu t) < (1+\mu)\omega(t)$ for $t > 0$ and $\mu \in I$. Moreover, if $\omega'(0) < \infty$, then $\omega_0(t) = \omega'(0)t$ on some neighborhood of zero.*

To prove the theorem we need two more lemmas. In Lemma 1 it is possible that $\bar{\omega}'(0) = \infty$.

**Lemma 1.** *Let $\omega$ be a modulus of continuity on $[0,\infty)$ and $\bar{\omega}$ be the minimal concave majorant of $\omega$. Then $\lim\limits_{t\to\infty} \frac{\omega(t)}{t} = \lim\limits_{t\to\infty} \frac{\bar{\omega}(t)}{t} < \infty$ and $\omega'(0) = \bar{\omega}'(0)$.*

PROOF. Note that $\frac{\bar{\omega}(t)}{t}$ is a nonincreasing function on $(0,\infty)$ since $\bar{\omega}$ is concave. This ensures the existence of $\lim\limits_{t\to\infty} \frac{\bar{\omega}(t)}{t}$. The proof of Lemma B in [2] contains the inequality

$$\bar{\omega}(t) < \frac{\omega(t_0)}{t_0} t + \omega(t_0), \text{ for } t > 0 \text{ and } t_0 > 0. \tag{1}$$

Hence $\lim\limits_{t\to\infty} \frac{\bar{\omega}(t)}{t} \le \frac{\omega(t_0)}{t_0} \le \frac{\bar{\omega}(t_0)}{t_0}$ and the first statement of the lemma follows.

Since $\bar{\omega}$ is concave, it has a finite or infinite derivative from the right $\bar{\omega}'(0)$. By passing to the limit as $t_0 \to 0^+$ in (1) we get $\bar{\omega}(t) \le t \liminf\limits_{t\to 0} \frac{\omega(t)}{t}$; whence $\bar{\omega}'(0) \le \liminf\limits_{t\to 0} \frac{\omega(t)}{t}$. On the other hand, $\limsup\limits_{t\to 0} \frac{\omega(t)}{t} \le \bar{\omega}'(0)$. So $\omega'(0) = \bar{\omega}'(0)$. □

Below we make use of the following smoothing method [3]. For every $\delta > 0$ let $\varphi(\delta,t)$ be an even infinitely differentiable function on $(-\infty,\infty)$ with $\varphi(\delta,t) > 0$ on $(-\delta,\delta)$ and $\varphi(\delta,t) = 0$ elsewhere. Assume also $\int_{-\delta}^{\delta} \varphi(\delta,t)\,dt = 1$.

Then given a continuous function $f$ on $(-\infty, \infty)$ the function of $t$ defined by

$$f(\delta, t) = \int_{-\infty}^{\infty} f(x)\,\varphi(\delta, x - t)\,dx = \int_{-\delta}^{\delta} f(x + t)\,\varphi(\delta, x)\,dx$$

is infinitely differentiable and $\lim_{\delta \to 0} f(\delta, t) = f(t)$ uniformly over any compact set in $(-\infty, \infty)$ [3, p. 46]. It is easy to show that if $f$ is concave, then so is $f(\delta, t)$.

Actually, this method will be applied to special functions defined on a finite interval. Specifically, let $f$ be a continuous function defined on $[p, q]$ so that $f$ is affine on $[p, p_1]$ and $[q_1, q]$ with different slopes, $p < p_1 < q_1 < q$. Further without special mention $f$ is extended to $(-\infty, \infty)$ so that the resulting function is affine on $(-\infty, p_1]$ and $[q_1, \infty)$ respectively. For convenience the extension of $f$ is also denoted by $f$. It is easy to check that such a function satisfies $f(\delta, t) = f(t)$ outside $(p_1 - \delta, q_1 + \delta)$. We will adhere to the following convention. If $L$ is the tangent at a point $(x, y)$ to the graph of some function, then we say simply that $L$ is the tangent to the graph at the point $x$. The topology on any interval is induced, as usual, by that on $(-\infty, \infty)$.

**Lemma 2.** *Let $y = k_1 t + d_1$ and $y = k_2 t + d_2$ be the tangents to the graph of a concave function $v$ at points $a < b$ respectively. Suppose that these tangents meet outside the graph of $v$. Then given any $\varepsilon > 0$ there exists an infinitely differentiable concave function $u$ on $[a, b]$ with the following properties:*

1. $0 \le u(t) - v(t) < \varepsilon$ *for all* $t \in [a, b]$*;*
2. $u(t) = k_1 t + d_1$ *on some neighborhood of* $a$ *and*
   $u(t) = k_2 t + d_2$ *on some neighborhood of* $b$.

PROOF. Denote by $(c, y_0)$ the point of intersection of the two tangents. Clearly $k_1 > k_2$, $a < c < b$ and $y_0 > v(c)$. The function $v_1$ defined by $v_1(t) = k_1 t + d_1$ on $[a, c]$ and $v_1(t) = k_2 t + d_2$ on $[c, b]$ is concave. Let $\varepsilon > 0$. Set $v_0(t) = \min\{v_1(t), v(t) + \frac{\varepsilon}{2}\}$, $a \le t \le b$. Then $0 \le v_0(t) - v(t) \le \frac{\varepsilon}{2}$. If $t = a$ or $t = b$, then $v_0(t) - v(t) = 0$, while $v_0(c) - v(c) > 0$. Consequently, there are an $\varepsilon_0 > 0$ and points $a_1 \in (a, c)$, $b_1 \in (c, b)$ such that $\varepsilon_0 < \frac{\varepsilon}{2}$ and $v_0(a_1) - v(a_1) = v_0(b_1) - v(b_1) = \varepsilon_0$. The definition of $v_0$ and the concavity of $v$ imply $v_0(t) = v_1(t)$ for $t \in [a, a_1] \cup [b_1, b]$ and $v_0(t) - v(t) \ge \varepsilon_0$ for $a_1 \le t \le b_1$. By the continuity of $v_0$ and $v$ we have also $v_0(t) - v(t) \ge \frac{1}{2}\varepsilon_0$ for $t \in [a_1 - \delta, b_1 + \delta]$ if $\delta$ is sufficiently small, $\delta > 0$, provided $a_1 - \delta > a$ and $b_1 + \delta < b$.

Let us smooth $v_0$ by means of the above method. For a sufficiently small $\delta$ we get an infinitely differentiable concave function $u(t) = v_0(\delta, t)$ with the following properties. The inequality $|u(t)-v_0(t)|<\frac{1}{2}\varepsilon_0$ holds for all $t\in[a,b]$ and hence $u(t)-v(t)<\varepsilon$. If $t\in[a,a_1-\delta]\cup[b_1+\delta,b]$, then $u(t)=v_1(t)\geq v(t)$. If $t\in[a_1-\delta,b_1+\delta]$, then $u(t)-v(t)=u(t)-v_0(t)+v_0(t)-v_1(t)>-\frac{1}{2}\varepsilon_0+\frac{1}{2}\varepsilon_0=0$, completing the proof. $\square$

Proof of the theorem. Observe that $\omega(\mu t)\leq\omega_0(\mu t)<(1+\mu)\omega(t)$ for all $t>0$ and $\mu\in I$ if and only if $\omega(t)\leq\omega_0(t)<(1+\mu)\omega(\frac{t}{\mu})$ for all $t>0$ and $\mu\in I$. Let $\bar\omega$ be the minimal concave majorant of $\omega$ and let $I=[\mu_1,\mu_2]$. First we construct $\omega_0$ sometimes disregarding the statement concerning $\omega_0(t)=\omega'(0)t$. Let us consider different cases. The case with a linear $\bar\omega$ is trivial. Assume that $\bar\omega$ is of the form: $\bar\omega(t)=k_1t$ on $[0,t_0]$ and $\bar\omega=k_2t+d$ on $[t_0,\infty)$ with $k_1>k_2$ and some $t_0>0$. The minimality of $\bar\omega$ implies $\bar\omega(t_0)=\omega(t_0)$. If $k_2=0$, then $\omega(t)=\bar\omega(t)=d$ on $[t_0,\infty)$. Therefore $(1+\mu)\omega(\frac{t}{\mu})-\bar\omega(t)=\mu d\geq\mu_1 d$ for all $\mu\in I$ and sufficiently large $t$. If $k_2>0$, then we argue as follows. By Lemma 1, $\lim\limits_{t\to\infty}\frac{\omega(t)}{t}=\lim\limits_{t\to\infty}\frac{\bar\omega(t)}{t}=k_2$. Then given any $\varepsilon>0$ there is an arbitrarily large $\tau>0$ such that $\frac{1}{t}\bar\omega(t)\leq(1+\varepsilon)k_2$ and $\frac{\mu}{t}\omega(\frac{t}{\mu})\geq(1-\varepsilon)k_2$ for $t\geq\tau$ and all $\mu\in I$. If $\varepsilon$ is sufficiently small, then a simple calculation shows that $(1+\mu)\omega(\frac{t}{\mu})-\bar\omega(t)\geq\frac{1}{2}k_2\tau\mu_2^{-1}$, where $t\geq\tau$, $\mu\in I$. Thus in both cases: $k_2=0$ and $k_2>0$, we have $\min\limits_{t\geq t_0,\,\mu\in I}\Big((1+\mu)\omega(\frac{t}{\mu})-\bar\omega(t)\Big)=2m$ with $m>0$. Denote by $(t_1,y_1)$ the point of intersection of the lines $y=k_1t$ and $y=k_2t+d+m$. Set $\omega_1(t)=k_1t$ for $0\leq t\leq t_1$ and $\omega_1(t)=k_2t+d+m$ for $t\geq t_1$. Then $\omega_1(t_1)>\bar\omega(t_1)$ and $\bar\omega(t)\leq\omega_1(t)<(1+\mu)\omega(\frac{t}{\mu})$ for all $t>0$. By smoothing we obtain a function $\omega_0(t)=\omega_1(\delta,t)$ with all the desired properties if $\delta$ is sufficiently small.

Consider the case, where $\bar\omega(t)=k_1t$ on $[0,a_1]$ and $\bar\omega(t)=k_2t+d$ on $[b_1,\infty)$ with $0<a_1<b_1$ provided the lines $y=k_1t$ and $y=k_2t+d$ meet outside the graph of $\bar\omega$. In this case the conclusion of the theorem follows at once from Lemma 2.

Assume now that $\bar\omega(t)=kt$ on some interval $[0,b_1]$, but $\bar\omega$ is not affine on any infinite interval. We suppose $\bar\omega'(b_1)=k$, since otherwise $b_1$ can be replaced by a smaller positive number. Choose a $b_2>b_1+1$ so that $\bar\omega'(b_2)$ exists and the tangents to the graph of $\bar\omega$ at $b_1$ and $b_2$ meet outside the graph. Such a choice is possible, since otherwise $\bar\omega$ would be affine on some infinite interval. Again, choose a $b_3>b_2+1$ so that $\bar\omega'(b_3)$ exists and the tangents at $b_2$ and $b_3$ meet outside the graph of $\bar\omega$. Continuing this process we obtain a sequence $\{b_i\}_{i=1}^\infty$ such that $b_{i+1}>b_i+1$ and the tangents at $b_i$ and $b_{i+1}$ meet outside the graph of $\bar\omega$. Denote by $L_i$ the tangent at $b_i$, $i\in\mathbb{N}$. Set $\varepsilon_i=\min\limits_{b_i\leq t\leq b_{i+1},\mu\in I}\Big((1+\mu)\omega(\frac{t}{\mu})-\bar\omega(t)\Big)$. By Lemma 2 for each $i\in\mathbb{N}$ there is infinitely differentiable concave function $\omega_{i0}$ on $[b_i,b_{i+1}]$ such that $0\leq\omega_{i0}(t)-$

$\bar{\omega}(t) < \varepsilon_i$ and hence $\omega(t) \le \omega_{i0}(t) < (1+\mu)\omega(\frac{t}{\mu})$, $t \in [b_i, b_{i+1}]$, $\mu \in I$. Moreover, the graph of $\omega_{i0}$ coincides with $L_i$ on some neighborhood of $b_i$ and with $L_{i+1}$ on some neighborhood of $b_{i+1}$. Set $\omega_0(t) = kt$ on $[0, b_1]$ and $\omega_0(t) = \omega_{i0}(t)$ on $[b_i, b_{i+1}]$, $i \in \mathbb{N}$. The values of $\omega_{i0}$ on consecutive intervals are coordinated so that $\omega_0$ is infinitely differentiable. The concavity of $\omega_0$ is obvious. Thus the conclusion of the theorem holds.

Consider the case where $\bar{\omega}(t) = kt+d$ on some half-line $[a_1, \infty)$, $a_1 > 0$, and $\bar{\omega}$ is not linear on any neighborhood of zero. We can assume that $\bar{\omega}'(a_1) = k$. As in the preceding case we choose a decreasing sequence $\{a_i\}_{i=1}^{\infty}$ so that $a_i \to 0$ as $i \to \infty$ and the tangents at $a_i$, $a_{i+1}$ meet outside the graph of $\bar{\omega}$. Applying again Lemma 2 and setting $\omega_0(0) = 0$ we construct a function $\omega_0$ so that $\omega_0$ has the desired properties except for the last statement of the theorem.

Suppose that $\bar{\omega}$ is not affine on $[t_1, \infty)$ as well as on $[0, t_2]$ for any positive $t_1$, $t_2$. Choose an $a_1 > 0$ so that $\bar{\omega}'(a_1)$ exists and set $b_1 = a_1$. We determine two sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ in the same way as in the two previous cases. The further construction being clear we omit the details.

We have still to prove the last statement of the theorem when $\omega'(0) < \infty$ and $\bar{\omega}$ is not linear on any neighborhood of zero. To this end let us change our construction somewhat. Choose $\varepsilon > 0$ so that $(1+\mu_2^{-1})(1-\varepsilon) > 1$. By Lemma 1 $\omega(t) \sim \bar{\omega}'(0)t$ as $t \to 0$. Then for all $\mu \in I$ and $t$ sufficiently small

$$(1+\mu)\omega(\frac{t}{\mu}) \ge \frac{1+\mu}{\mu}(1-\varepsilon)\bar{\omega}'(0)t \ge (1+\mu_2^{-1})(1-\varepsilon)\bar{\omega}'(0)t > \bar{\omega}'(0)t.$$

Therefore, while constructing the sequence $\{a_i\}_{i=1}^{\infty}$ one can single out a $j$ such that $\bar{\omega}'(0)t < (1+\mu)\omega(\frac{t}{\mu})$ for $t \in (0, a_j]$ and $\mu \in I$. We define $\omega_0$ on $[a_j, \infty)$ just as above. If $0 < t < a_j$, then we construct $\omega_0$ in a different way. Let $B$ denote the point $(a_j, \bar{\omega}(a_j))$. The tangent at $B$ to the graph of $\bar{\omega}$ meets the line $y = \bar{\omega}'(0)t$ at a point $A$ with an abscissa $a \in (0, a_j)$. The union of the line segments $OA$ and $AB$ is the graph of some function $\omega_2$ on $[0, a_j]$. The segment $AB$, except for $A$, lies under the half-line $OA$. It follows that $\omega_2(t) < (1+\mu)\omega(\frac{t}{\mu})$ for $t \in (0, a_j]$ and $\mu \in I$. Observe also that $\omega_2(a) > \bar{\omega}(a)$. Smoothing $\omega_2$ with a sufficiently small $\delta > 0$, $\delta < \min\{a, a_j - a\}$, and setting $\omega_0(t) = \omega_2(\delta, t)$ on $[0, a_j]$ we finally obtain $\omega_0$ on $[0, \infty)$ with all the properties claimed in the theorem. $\square$

Clearly, the unimprovability asserted in Lemma B remains valid for $\omega_0$.

Let us remark that the condition of the theorem cannot be weakened by assuming only $\mu \in (0, \infty)$ instead of $\mu \in I$. Indeed, take $\omega$ such that $\omega(t) = t$ on $[0, 1]$ and $\omega(t) = 1$ on $[1, \infty)$. Then $\omega_0(t) = \omega(t)$ for $0 \le t \le 1$, since otherwise $\omega_0'(0) \ne \omega'(0)$. In order that $\omega_0$ be a concave differentiable majorant of $\omega$ it is necessary that $\omega_0(t) > \omega(t)$ for $t > 1$ and, in particular, $\omega_0(2) > 1$.

If $\omega_0(2) < (1+\mu)\omega(\frac{2}{\mu})$ for all $\mu > 0$, then taking the limit as $\mu \to 0$ we obtain $\omega_0(2) \le 1$ contradicting $\omega_0(2) > 1$. It remains unclear if the condition $\mu_1 \le \mu \le \mu_2$ can be weakened by assuming only $\mu \ge \mu_1 > 0$.

Lemma A does not answer the question if the factor 2 can be replaced by a smaller value depending on $\omega$. Lemma B gives rise to a similar question with any $\mu > 0$. It is also noteworthy that $\bar{\omega}(\mu t) < (1+\mu)\omega(t)$ if $\omega$ is defined on a finite interval $[0, l]$, since $\omega$ can be extended to $[0,\infty)$ by setting $\omega(t) = \omega(l)$ for $t > l$. However, in this case the unimprovability of the indicated inequality requires a complementary study. Indeed, taking $\mu = 1$ and $t = l$ we can write $\bar{\omega}(l) = \omega(l)$ instead of $\bar{\omega}(l) < 2\,\omega(l)$. Below we construct an example which shows the unimprovability of $\bar{\omega}(\mu t) < (1+\mu)\omega(t)$ in a sense different from that in Lemma B and thereby we complement our theorem. Incidentally, the same example enables us to remove in Lemma B the restriction that $\mu$ be a natural number.

Take a sequence of positive numbers $\{q_m\}_{m=-\infty}^{\infty}$ so that $q_m \to \infty$ as $|m| \to \infty$. Let $a_0 = c_0 = 1$. For each integer $m$ we determine inductively a triple $a_m, b_m, a_{m+1}$ that forms a geometric progression with $q_m$ as the ratio. Applying again induction on $m$ we define a continuous nondecreasing function $\omega$ on $[0,\infty)$ as follows. $\omega(0) = 0$, $\omega(t) = c_m$ for $a_m \le t \le b_m$ and $\omega(t) = \frac{c_m}{b_m} t$ for $b_m \le t \le a_{m+1}$ with suitable constants $c_m$. Note the following property of the graph of $\omega$. Given any $x > 0$, the points of the chord joining $(0,0)$ and $(x, \omega(x))$ lie under or on the graph. It follows that $\omega$ is semiadditive. Thus, $\omega$ is a modulus of continuity.

Let $\omega_l$ denote the restriction of $\omega$ to the finite interval $[0, l]$ and let $\varphi_l$ be a concave majorant of $\omega_l$. Fix a $\mu > 0$. Consider only those integers $m$ that satisfy $a_{m+1} \le l$ and $\mu b_m \in [a_m, a_{m+1}]$. The points $(a_m, c_m)$ and $(a_{m+1}, c_{m+1})$ belong to the graph of $\omega_l$. It is not hard to check that the line segment joining these points is described by $y = \dfrac{c_m}{q_m+1}(\dfrac{t}{a_m} + q_m)$, $a_m \le t \le a_{m+1}$. If $t = \mu b_m$, then $y = \dfrac{(1+\mu)q_m}{q_m+1}\omega_l(b_m)$. It follows that $\varphi_l(\mu b_m) \ge \dfrac{(1+\mu)q_m}{q_m+1}\omega_l(b_m)$ since $\varphi_l$ is a concave majorant of $\omega_l$. The same argument is valid for any concave majorant $\varphi$ of $\omega$ on $[0,\infty)$. Clearly $\dfrac{(1+\mu)q_m}{q_m+1} \to 1+\mu$ as $|m| \to \infty$. Thus, there is a single modulus of continuity $\omega$ such that any factor $1+\mu$ in $\bar{\omega}(\mu t) < (1+\mu)\omega(t)$ cannot be reduced, no matter if we consider functions on a finite or infinite interval.

The function $\omega$ constructed above can be used in another way. Let $\mu > 0$ and $t_0 > 0$ be given. For any integer $m$ set $\omega(t,m) = \omega(\frac{b_m}{t_0} t)$, $t \ge 0$. If $\psi(t)$ is concave majorant of $\omega(t,m)$, then $\varphi(t) = \psi(\frac{t_0}{b_m} t)$ is a concave majorant of

$\omega(t)$. Therefore $\psi(\mu t_0) \geq \frac{(1+\mu)q_m}{q_m+1}\omega(t_0, m)$. Since $q_m$ is arbitrarily large, it follows that the last statement of Lemma B remains valid with any $\mu > 0$.

**Acknowledgment**

The author is grateful to P. L. Ul'janov for the setting of the problem and his advice.

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