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THE ALMOST DISJOINTNESS CARDINAL INVARIANT IN THE QUOTIENT ALGEBRA OF THE RATIONALS MODULO THE NOWHERE DENSE SUBSETS

Abstract

The almost disjointness number is extended to arbitrary Boolean algebras and it is shown that this number is consistently less than $\mathfrak{a}$ for the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathcal{N}$ where $\mathcal{N}$ is the ideal of nowhere dense subsets of $\mathbb{Q}$.

1 Introduction

Many of the cardinal invariants of the continuum studied in the literature have obvious generalizations obtained by replacing the ideal of finite sets by some other ideal. On the other hand, generalizations can also be obtained by defining the invariants in the context of a Boolean algebra. In either case the motivation is to obtain tools for distinguishing the objects of study; Boolean algebras in one case and ideals in the other. Applications to the study of quotients of permutation groups and their maximal abelian subgroups have been established in [5]. The present paper will restrict to the common ground of the two possible generalizations, the Boolean algebras which result by considering $\mathcal{P}(\mathbb{N})$ modulo some ideal. Several assertions will be proved which, in turn, will point the way toward some interesting open questions which are collected in the last section.

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Definition 1.1. For any Boolean algebra $\mathbb{B}$ which does not have the countable chain condition, define $\mathfrak{a}(\mathbb{B})$ to be the least cardinal of an uncountable, maximal anti-chain. In other words, $\mathfrak{a}(\mathbb{B})$, is the least cardinal such that there is an uncountable set $A \subseteq \mathbb{B} \setminus 0_{\mathbb{B}}$ of cardinality $\mathfrak{a}(\mathbb{B})$ and such that $a \wedge a' = 0_{\mathbb{B}}$ for every $\{a, a'\} \in [A]^2$ and A is maximal in the sense that for every $b \in \mathbb{B} \setminus 0_{\mathbb{B}}$ there is $a \in A$ such that $a \wedge b \neq 0_{\mathbb{B}}$. For any ideal $\mathcal{I}$ on $\mathbb{N}$, which is not countably saturated, define $\mathfrak{a}(\mathcal{I}) = \mathfrak{a}(\mathcal{P}(\mathbb{N})/\mathcal{I})$.

In[3] Monk defines the cardinal invariant $c_{\mathrm{mm}}(\mathbb{B})$ of the Boolean algebra $\mathbb{B}$ to be the least infinite cardinal of a maximal anti-chain. However Definition 1.1 restricts the minimum to be taken only over uncountable anti-chains rather than just infinite ones. In many cases this is irrelevant but the difference will be crucial for the Boolean algebras considered in this paper. The assumption in Definition 1.1 that $\mathcal{I}$ is not countably saturated is required simply to guarantee that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ does not have the countable chain condition and, hence, $\mathfrak{a}(\mathcal{P}(\mathbb{N})/\mathcal{I})$ is defined.

Notation 1.1. Let $\mathcal{N}$ denote the ideal of all nowhere dense subsets of $\mathbb{Q}$ and let $\mathcal{F}in$ denote the ideal of all finite subsets of $\mathbb{N}$. Unless otherwise specified, all ideals in this paper will be considered to be ideals on $\mathbb{N}$.

2 The Ideal of Nowhere Dense Subsets of $\mathbb{Q}$

This paper is concerned with $\mathfrak{a}(\mathcal{N})$. Therefore, it is worth pointing out that $c_{\mathrm{mm}}(\mathcal{P}(\mathbb{Q})/\mathcal{N}) = \aleph_0$. The family of intervals $\{[n, n+1]\}_{n=-\infty}^{\infty}$ witnesses this. However, $\mathfrak{a}(\mathcal{N})$ is a more interesting invariant, although one might initially harbor doubts that it is the same as $\mathfrak{a}$. This will be shown not to be the case.

The notation $\mathbb{L}_\alpha$ will be used to denote the countable support iteration of α Laver partial orders. In Laver's original paper [2] introducing the Laver partial order, it is shown that $\mathfrak{b} = \aleph_2$, and hence also $\mathfrak{a} = \aleph_2$, in the model obtained by iterating ω_2 Laver reals with countable support. The following result can also be found there — it follows from Lemma 5 and Lemma 6 (i) of [2].

Lemma 2.1. *Suppose that $G \subseteq \mathbb{L}_\alpha$ is generic over the model V and that $\{F_n\}_{n=0}^{\infty}$ is a sequence of finite sets which belongs to V. Then, for any $f \in \prod_{n=0}^{\infty} F_n$ belonging to $V[G]$ there is a sequence of finite sets $\{a_n\}_{n=0}^{\infty} \in V$ such that $f(n) \in a_n \in [F_n]^n$ for each $n \in \mathbb{N}$.*

The following result is due to S. Shelah [6]. It is not necessary to know what NEP is, only that Laver forcing satisfies this property. (This fact is also established in [6].)

Lemma 2.2. *Let $\{B_\alpha\}_{\alpha\in\omega_1}$ be family of Borel sets in a model of set theory V such that $V \models \bigcap_{\alpha\in\omega_1} B_\alpha = \emptyset$. Let $\mathbb{P}$ be a NEP partial order with definition in V and suppose that $\{\mathbb{P}_\alpha\}_{\alpha\in\omega_2}$ is a countable support iteration such that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{P}$ for any $\alpha \in \omega_2$. If*

$$1 \Vdash_{\mathbb{P}_{\alpha+1}} \text{“} \bigcap_{\alpha\in\omega_1} B_\alpha = \emptyset \text{”}$$

for each $\alpha \in \omega_1$, then

$$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{“} \bigcap_{\alpha\in\omega_1} B_\alpha = \emptyset \text{”}$$

Theorem 2.1. *In the iterated Laver model $\mathfrak{a} = \aleph_2$ but $\mathfrak{a}(\mathcal{N}) = \aleph_1$.*

PROOF. It is shown in [2] that $b = \aleph_2$ and it is well known that $\mathfrak{a} \geq \mathfrak{b}$; hence, $\mathfrak{a} = \aleph_2$. Assuming $2^{\aleph_0} = \aleph_1$ in the ground model, let $\{(p_\xi, X_\xi)\}_{\xi\in\omega_1}$ enumerate all pairs of $\mathbb{L}_{\mu(\xi)}$ conditions and $\mathbb{L}_{\mu(\xi)}$-Names such that $\mu(\xi) \in \omega_1$ and $p_\xi \Vdash_{\mathbb{L}_{\mu(\xi)}}$ “X_ξ is dense in (a_ξ, b_ξ)”. Construct subsets of the rationals $\{A_\xi\}_{\xi\in\omega_1}$ such that

- $A_\xi \cap A_\eta$ is nowhere dense unless $\xi = \eta$
- each A_ξ is dense
- if $p_\xi \Vdash_{\mathbb{L}_{\mu(\xi)}}$ “$X_\xi \cap A_\beta \in \mathcal{N}$” for each $\beta \in \alpha$, then $p_\xi \Vdash_{\mathbb{L}_{\mu(\xi)}}$ “$X_\xi \cap A_\xi \notin \mathcal{N}$”.

Let B_ξ be the Borel set $\{X \in \mathcal{N}^+ : X \cap A_\xi \in \mathcal{N}\}$. It will suffice to show that $\bigcap_{\xi\in\omega_1} B_\xi = \emptyset$ after forcing with $\mathbb{L}_\alpha$ for each $\alpha \in \omega_1$ because, it will then follow from Lemma 2.2 for $\mathbb{L}$, that $\{A_\xi\}_{\xi\in\omega_1}$ remains a maximal anti-chain in $\mathcal{P}(\mathbb{N})/\mathcal{N}$ after forcing with $\mathbb{L}_{\omega_2}$.

Suppose that $\{A_\xi\}_{\xi\in\zeta}$ have been constructed and that $p_\zeta \Vdash_{\mathbb{L}_{\mu(\zeta)}}$ “$X_\zeta \cap A_\beta \in \mathcal{N}$” for each $\beta \in \zeta$. Let $\{\eta(n)\}_{n\in\omega}$ enumerate ζ and let $E_j = \bigcup_{n\in j} A_{\eta(n)}$. Let $\{I_n\}_{n\in\omega}$ enumerate all rational intervals in (a_ζ, b_ζ). Recalling that

$$p_\zeta \Vdash_{\mathbb{L}_{\mu(\zeta)}} \text{“}X_\zeta \text{ is dense in } (a_\zeta, b_\zeta)\text{”},$$

let x_n be a name such that $p_\zeta \Vdash_{\mathbb{L}_{\mu(\zeta)}}$ “$x_n \in X_\zeta \cap I_n \setminus E_n$”. Observe that x_n exists since $p_\zeta \Vdash_{\mathbb{L}_{\mu(\zeta)}}$ “$X_\zeta \cap E_n \in \mathcal{N}$”.

Now let $\mathcal{J}_n$ be a partition of I_n into n^2+1 non-degenerate rational intervals. Using Lemma 2.1 find $a_n^m \in [\mathcal{J}_n]^n$ such that $p_\zeta \Vdash_{\mathbb{L}_\alpha}$ “$x_m \in \cup a_n^m$” for each n. Choose $J_n \in \mathcal{J}_n \setminus \bigcup_{i\in n} a_n^i$ and let $D_m = (a_\zeta, b_\zeta) \setminus (\bigcup_{i=m}^\infty J_i \cup E_m)$. Observe that $p_\zeta \Vdash_{\mathbb{L}_\alpha}$ “$x_m \in D_n$” for any $n \geq m$. Also $D_m \cap A_{\eta(n)} = \emptyset$ if $n \geq m$ and $D_m \cap A_{\eta(n)} \in \mathcal{N}$ if $n < m$. Hence if $A_\zeta = \bigcup_{m=0}^\infty D_m$, then $A_\zeta \cap A_{\eta(n)} \in \mathcal{N}$ for each $n \in \mathbb{N}$. Moreover $p_\zeta \Vdash_{\mathbb{L}_\alpha}$ “$x_m \in A_\zeta$” for each m and so $p_\zeta \Vdash_{\mathbb{L}_{\mu(\zeta)}}$ “$X_\zeta \cap A_\zeta$ is dense in (a_ζ, b_ζ)”. It follows that A_ζ satisfies all the required conditions. □

It is also worth observing that $\mathfrak{a}(\mathcal{N})$ can be arbitrarily large.

Proposition 2.1. *$\mathfrak{a}(\mathcal{N}) \geq \mathfrak{p}$.*

PROOF. Suppose that $\mathcal{S}$ is an uncountable family of subsets of $\mathbb{Q}$ such that $S \cap S' \in \mathcal{N}$ for each pair $\{S, S'\} \in [\mathcal{S}]^2$ and each $S \in \mathcal{S}$ is somewhere dense.

Observe that there must be some interval $[a, b] \subseteq \mathbb{Q}$ such that $\{S \cap [a', b']\}_{S \in \mathcal{S}}$ forms a proper ideal on $[a', b']$ whenever $a \leq a' < b' \leq b$. To see this, note that if it were not the case, then for every interval I there would be a subinterval $J_I \subseteq I$ and a finite subset $\mathcal{S}_I \subseteq \mathcal{S}$ such that $J_I \subseteq \cup \mathcal{S}_I$. Choosing $S \in \mathcal{S}$ such that S does not belong to $\mathcal{S}_I$ for any rational interval I it would follow that, letting $I(S)$ be an interval witnessing that $S \notin \mathcal{N}$, then $S \cap S'$ is dense somewhere in $J_{I(S)}$ for some $S' \in \mathcal{S}_{I(S)}$. If $|\mathcal{S}| < \mathfrak{p}$, then for every subinterval $I \subseteq [a, b]$ choose an infinite set $X_I \subseteq J_I$ such that $|X_I \cap S| < \aleph_0$ for each $S \in \mathcal{S}$. Using the fact that $\mathfrak{b} \geq \mathfrak{p}$ choose $x_I \in X_I$ in such a way that for each $S \in \mathcal{S}$ there are only finitely many subintervals $I \subseteq [a, b]$ such that $x_I \in S$. Then $X = \{x_{[p,q]}\}_{a \leq p < q \leq b}$ is dense in $[a, b]$ yet $X \cap S$ is finite for each $S \in \mathcal{S}$. □

3 Product Ideals

Many ideals can be obtained by taking the Fubini product of simpler ideals. Recall that this product is defined by

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \{n \in \mathbb{N} : \{m \in \mathbb{N} : (n, m) \in A)\} \in \mathcal{J}\} \in \mathcal{I}\}.$$

Proposition 3.1. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals, then $\mathfrak{a}(\mathcal{I} \times \mathcal{J}) \leq \mathfrak{a}(\mathcal{I})$.*

PROOF. If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a maximal anti-chain of size $\mathfrak{a}(\mathcal{I})$, then

$$\{A \times \mathbb{N} : [A]_{\mathcal{I}} \in \mathcal{A}\}$$

produces a maximal anti-chain in $\mathcal{P}(\mathbb{N})/(\mathcal{I} \times \mathcal{J})$. □

An upper bound is also available but it is not clear if it is the best possible. The model of [4] where $\mathfrak{b} < \mathfrak{a}$ is worth examining in this context,

Proposition 3.2. *If $\mathcal{I}$ is any ideal, then $\mathfrak{b} \leq \mathfrak{a}(\mathcal{F}in \times \mathcal{I})$.*

PROOF. Suppose that $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be such that $\mathcal{A} \cap \mathcal{F}in \times \mathcal{I} = \emptyset$ and if A and B are distinct elements of $\mathcal{A}$, then $A \cap B \in \mathcal{F}in \times \mathcal{I}$. For $A \in \mathcal{A}$ let $\langle A \rangle_n = \{m \in \mathbb{N} : (n, m) \in A\}$.

Let $\{A_n\}_{n=0}^{\infty}$ enumerate distinct elements of $\mathcal{A}$ and for each $A \in \mathcal{A} \setminus \{A_n\}_{n=0}^{\infty}$ choose a function $H_A : \mathbb{N} \to \mathbb{N}$ such that $\langle A_i \cap A \rangle_m \in \mathcal{I}$ for all

$m > H(i)$. If $|\mathcal{A}| < \mathfrak{b}$, then it is possible to find H such that $H \geq^* H_A$ for each $A \in \mathcal{A} \setminus \{A_n\}_{n=0}^{\infty}$. Now choose an increasing sequence of integers $\{k_i\}_{i=0}^{\infty}$ such that

$$\langle A_i \setminus (\bigcup_{j \in i} A_j) \rangle_{k_i} \notin \mathcal{I}$$

for each i. Note that it is possible to do this because $A_i \cap (\bigcup_{j\in i} A_j) \in \mathcal{F}in \times \mathcal{I}$. Now let $X = \bigcup_{i=0}^{\infty}\{k_i\} \times \langle A_i \setminus (\bigcup_{j\in i} A_j)\rangle_{k_i}$ and note that $X \notin \mathcal{F}in \times \mathcal{I}$ yet $X \cap A \in \mathcal{F}in \times \mathcal{I}$ for each $A \in \mathcal{A}$. □

The question of an inequality involving the second factor of the product of the two ideals is not so clear but the cardinal invariant can be calculated in at least one simple case.

Proposition 3.3. $\mathfrak{a}(\mathcal{F}in \times \{\emptyset\}) = \mathfrak{a}(\mathcal{F}in) = \mathfrak{a}$.

Proof. Suppose that $\mathcal{A} \subseteq (\mathcal{F}in \times \{\emptyset\})^+$ and pairwise intersections of elements of $\mathcal{A}$ belong to $\mathcal{F}in \times \{\emptyset\}$. Observe that if $G : \mathbb{N} \to \mathbb{N}$ and $G^* = \{(n, m) \in \mathbb{N} \times \mathbb{N} : m \leq G(n)\}$, then $\{A \cap G^*\}_{A\in\mathcal{A}}$ is a maximal almost disjoint family on G^*. Hence, in order to show that $|\mathcal{A}| \geq \mathfrak{a}$ it suffices to find G such that $\{A \cap G^*\}_{A\in\mathcal{A}}$ is infinite. Choosing a countable subfamily of $\mathcal{A}$, picking an infinite, partial function in each member of this family and then finding G which dominates all of these works. □

Although Proposition 3.1 does not apply to $\{\emptyset\} \times \mathcal{F}in$ because $\mathfrak{a}(\{\emptyset\})$ is not defined, the following proposition is still easy to see.

Proposition 3.4. $\mathfrak{a}(\{\emptyset\} \times \mathcal{F}in) = \mathfrak{a}$.

4 Questions

Question 4.1. What is the relationship between $\mathfrak{a}(\mathcal{I} \times \mathcal{J})$ and $\mathfrak{a}(\mathcal{J})$?

Question 4.2. Can it be shown that $\mathfrak{a}(\mathcal{F}in \times \mathcal{I}) = \mathfrak{a}(\mathcal{I})$ for any arbitrary ideal $\mathcal{I}$?

If the answer to this question is negative the following question becomes of interest.

Question 4.3. Can the equality $\mathfrak{a}(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{a}$ be proved? What about $\mathfrak{a}(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{a}(\mathcal{F}in \times \mathcal{F}in \times \mathcal{F}in)$ and the other obvious questions?

Theorem 2.1 only establishes a consistent strict inequality. The following question remains open.

Question 4.4. Can the inequality $\mathfrak{a}(\mathcal{N}) \leq \mathfrak{a}$ be proved?

Finally, there remains the general question of how much information about the quotient Boolean algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ can be derived from knowledge of the cardinal invariant $\mathfrak{a}(\mathcal{I})$. Under one interpretation of this question the answer is, "None at all." The reason is that, under MA it will be the case that $\mathfrak{a}(\mathcal{I}) = 2^{\aleph_0}$ for any analytic ideal while it has been shown in [1] that the algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ tend to be non-isomorphic assuming MA and OCA. However, one might ask the following.

Question 4.5. If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals such that it is consistent that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not isomorphic to $\mathcal{P}(\mathbb{N})/\mathcal{J}$ does it follow that it is consistent that $\mathfrak{a}(\mathcal{I}) \neq \mathfrak{a}(\mathcal{J})$?

There is no reason to conjecture a positive answer to this question. Nevertheless, an investigation of $\mathfrak{a}(\mathcal{I})$ for various analytic ideals $\mathcal{I}$ may lead to some interesting results.

References

[1] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc., **148(702)** (2000), xvi+177.

[2] R. Laver, *On the consistency of Borel's conjecture*, Acta Math., **137(3–4)** (1976), 151–169.

[3] J. Monk, *Cardinal invariants on Boolean algebras*, Birkhäuser Verlag, Basel, 1996.

[4] S. Shelah, *On cardinal invariants of the continuum*, In D. A. Martin J. Baumgartner and S. Shelah, editors, Axiomatic Set Theory, Vol. 31 of *Contemporary Mathematics*, Amer. Math. Soc., Providence, (1984), 183–207.

[5] S. Shelah and J. Steprāns, *Possible cardinalities of maximal abelian subgroups of quotients of permutation groups on the integers*, http://hausdorff.math.yorku.ca/Research/menu.html, (2001).

[6] S. Shelah, *Non-elementary proper forcing notions*, unpublished, **630** in Shelah's personal classification.