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A DIAGONALIZATION PROPERTY BETWEEN HUREWICZ AND MENGER

Abstract

In classical works, Hurewicz and Menger introduced two diagonalization properties for sequences of open covers. Hurewicz found a combinatorial characterization of these notions in terms of continuous images. Recently, Scheepers has shown that these notions are particular cases in a large family of diagonalization schemas. One of the members of this family is weaker than the Hurewicz property and stronger than the Menger property, and it was left open whether it can be characterized combinatorially in terms of continuous images. We give a positive answer. This paper can serve as an exposition of this fascinating subject.

1 Introduction

The following property was introduced by Menger [5].

Definition 1.1. A set of reals X has the *Menger* property if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X there exist finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$, $n\in\mathbb{N}$, such that the collection $\{\cup\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a cover of X.

For example, every compact, or even σ-compact, set of reals satisfies the Menger property.

Following Hurewicz [2], Scheepers introduced the following diagonalization procedure of a sequence of covers [7] (see Figure 1):

Definition 1.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be collections of open covers of a space X. X has *property* $\mathsf{U}_{fin}(\mathfrak{A},\mathfrak{B})$ if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{A}$ there exist finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$, $n\in\mathbb{N}$, such that either $\cup\mathcal{F}_n = X$ for some n, or else the collection $\{\cup\mathcal{F}_n\}_{n\in\mathbb{N}}\in\mathfrak{B}$.

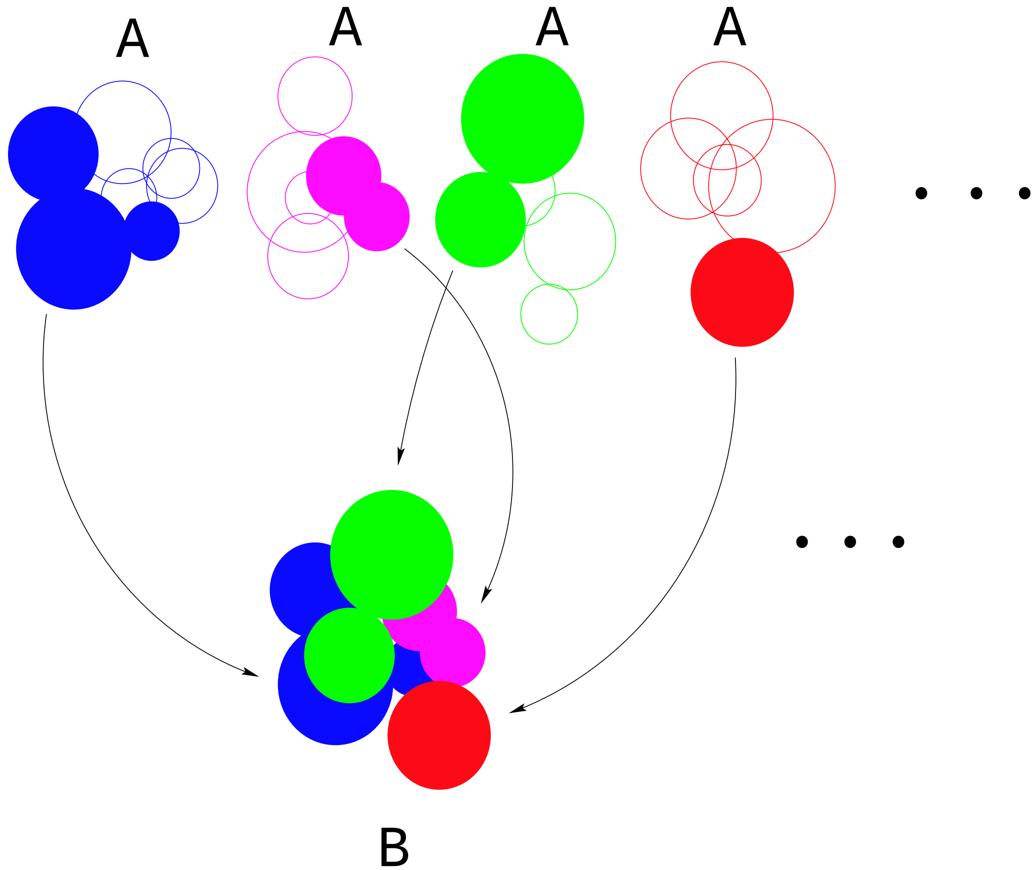

Figure 1: The selection principle $\mathsf{U}_{fin}(\mathfrak{A}, \mathfrak{B})$

We will consider sets of reals for which the usual induced topology has a subbase whose elements are *clopen* (both closed and open), that is, sets which are *zero-dimensional*. For convenience, we will also consider the Baire space ${}^{\mathbb{N}}\mathbb{N}$ of infinite sequences of natural numbers (equipped with the product topology). The Baire space, as well as any separable and zero-dimensional metric space is homeomorphic to a set of reals; thus our results about sets of reals are actually results about this more general case.

Let X be a set of reals. An ω-cover of X is a cover such that each finite subset of X is contained in some member of the cover. It is a γ-cover if it is infinite, and each element of X belongs to all but finitely many members of the cover. Let $\mathcal{O}$, Ω, and Γ denote the collections of countable[1] open covers, ω-covers, and γ-covers of X, respectively. In [3] Hurewicz studied the classes $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ (Menger property) and $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ (*Hurewicz property*). These diagonalization principles as well as several other natural diagonalization principles (see Definition 2.1) were studied by Scheepers, et. al., in a long series of papers ([7], [4], [8], etc.).

For each of the diagonalization properties, it is desirable to have a simple description of its underlying combinatorial structure. For some of the properties this involves the quasi-ordering $\leq^*$ defined on the Baire space ${}^{\mathbb{N}}\mathbb{N}$ by eventual dominance:

$$f \leq^* g \text{ if } f(n) \leq g(n) \text{ for all but finitely many } n.$$

A subset $\mathcal{F}$ of ${}^{\mathbb{N}}\mathbb{N}$ is *dominating* if for each g in ${}^{\mathbb{N}}\mathbb{N}$ there exists $f \in \mathcal{F}$ such that $g \leq^* f$.

Hurewicz ([3], see also Recław [6]) has found the following elegant characterizations of the Menger and Hurewicz properties.

Theorem 1.1 (Hurewicz). *Let X be a zero-dimensional set of reals.*

1. *X satisfies $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ if, and only if, every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is not dominating.*

2. *X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ if, and only if, every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded (with respect to $\leq^*$).*

Recław proved similar results for other important classes; see [6].

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[1] There is no loss of generality in restricting attention to countable covers here, since the spaces in question are Lindelöf.

2 The New Property $\mathsf{U}_{fin}(\mathcal{O},\Omega)$

Using finite unions, one can turn any countable cover into a γ-cover. Thus, for each collection of covers $\mathfrak{B}$, the properties $\mathsf{U}_{fin}(\mathcal{O},\mathfrak{B})$, $\mathsf{U}_{fin}(\Omega,\mathfrak{B})$, and $\mathsf{U}_{fin}(\Gamma,\mathfrak{B})$ are equivalent [4]. Therefore, the only (possibly) new property introduced by the general scheme of Definition 1.2 (when restricting attention to $\mathcal{O}$, Ω, and Γ)[2] is $\mathsf{U}_{fin}(\mathcal{O},\Omega)$. The property $\mathsf{U}_{fin}(\mathcal{O},\Omega)$ is weaker than the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ and stronger than the Menger property $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$, and according to [4] it is indeed new, that is, it is not provably equivalent to any of the classical properties of Menger and Hurewicz. Unlike these classical properties, the combinatorial counterpart of the new property was less evident. In [9] it was proved that an analogous property (involving *Borel* instead of open covers) can be characterized in terms of the combinatorial structure of Borel images. In Remark 10 of [9] it was left open whether a similar result can be obtained for $\mathsf{U}_{fin}(\mathcal{O},\Omega)$. We answer this question positively. The proof is essentially an application of Recław's arguments from [6] to the proof of the corresponding Borel result from [9].

For a finite subset F of ${}^{\mathbb{N}}\mathbb{N}$, define $\max(F) \in {}^{\mathbb{N}}\mathbb{N}$ to be the function g such that $g(n) = \max\{f(n) : f \in F\}$ for each n. In [9] the following notion was introduced. For a subset Y of ${}^{\mathbb{N}}\mathbb{N}$,

$$\operatorname{maxfin}(Y) := \{\max(F) : F \text{ is a finite subset of } Y\}.$$

We prove the following characterization of the new property $\mathsf{U}_{fin}(\mathcal{O},\Omega)$.

Theorem 2.1. *For a zero-dimensional set X of reals, the following are equivalent:*

1. *X satisfies $\mathsf{U}_{fin}(\mathcal{O},\Omega)$;*
2. *For each continuous function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$, $\operatorname{maxfin}(\Psi[X])$ is not dominating.*

Proof. 2 $\Rightarrow$ 1: Assume that $\mathcal{U}_n$, $n \in \mathbb{N}$, are open covers of X. For each n, replacing each open member of $\mathcal{U}_n$ with all of its clopen subsets we may assume that all elements of $\mathcal{U}_n$ are clopen, and thus we may assume further that they are disjoint. For each n enumerate $\mathcal{U}_n = \{U^n_m\}_{m\in\mathbb{N}}$. As we assume that the elements U^n_m, $m \in \mathbb{N}$, are disjoint, we can define a function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$ by

$$\Psi(x)(n) = m \Leftrightarrow x \in U^n_m.$$

[2]See [10] for another type of covers which can be smoothly incorporated into this framework.

Then Ψ is continuous. Therefore, $\mathrm{maxfin}(\Psi[X])$ is not dominating. Let g be a witness for that. Then for each finite $F \subseteq X$, $\max(\Psi[F])(n) < g(n)$; i.e., $F \subseteq \bigcup_{k<g(n)} U^n_k$, for infinitely many n. Thus, $\{\bigcup_{k<g(n)} U^n_k\}_{n\in\mathbb{N}}$ is an ω-cover of X.

$1 \Rightarrow 2$: This was proved in [9]. For completeness, we give a minor variation of the original proof. Since Ψ is continuous, $Y = \Psi[X]$ also satisfies $\mathsf{U}_{fin}(\mathcal{O},\Omega)$ [4]. Consider the basic open covers $\mathcal{U}_n = \{U^n_m\}_{m\in\mathbb{N}}$ defined by $U^n_m = \{f : f(n) = m\}$. Then there exist finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that either $Y = \cup\mathcal{F}_n$ for some n, or else $\mathcal{V} = \{\cup\mathcal{F}_n : n \in \mathbb{N}\}$ is an ω-cover of Y.

The first case can be split into two sub-cases: If $Y = \cup\mathcal{F}_n$ for infinitely many n, then for these infinitely many n, the set $\{f(n) : f \in Y\}$ is finite. Thus $\mathrm{maxfin}(Y)$ cannot be dominating. Otherwise $Y = \cup\mathcal{F}_n$ for only finitely many n, therefore we may replace each $\mathcal{F}_n$ satisfying $Y = \cup\mathcal{F}_n$ with $\mathcal{F}_n = \emptyset$, so we are in the second case. In the second case, since $Y \notin \mathcal{V}$ and $\mathcal{V}$ is an ω-cover of Y, we have that each finite subset of Y is contained in infinitely many elements of $\mathcal{V}$. Define $g \in {}^{\mathbb{N}}\mathbb{N}$ by $g(n) = \max\{m : U^n_m \in \mathcal{F}_n\} + 1$. For each finite $F \subseteq Y$, we have that $F \subseteq \cup\mathcal{F}_n$ and thus $\max(F)(n) < g(n)$ for infinitely many n. Then g witnesses that $\mathrm{maxfin}(Y)$ is not dominating. $\square$

The following diagonalization properties, which generalize some other classical notions, were introduced by Scheepers [7].

Definition 2.1. For a set X of reals, define the following diagonalization properties:

$\mathsf{S}_1(\mathfrak{A},\mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{A}$, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{B}$.

$\mathsf{S}_{fin}(\mathfrak{A},\mathfrak{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{A}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that each $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n \in \mathfrak{B}$.

Let $\mathcal{B}$, $\mathcal{B}_\Omega$, and $\mathcal{B}_\Gamma$ denote the collections of countable Borel covers, ω-covers, and γ-covers of X, respectively. Using Theorem 2.1 and results from [9], we get the following corollary, which answers another question from [9]. (Recall that $\mathsf{U}_{fin}(\mathcal{O},\Omega) = \mathsf{U}_{fin}(\Gamma,\Omega)$.)

Corollary 2.2. *For a zero-dimensional set X of reals, the following are equivalent:*

1. *X satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Omega)$,*
2. *Every Borel image of X satisfies $\mathsf{S}_1(\Gamma,\Omega)$,*
3. *Every Borel image of X satisfies $\mathsf{S}_{fin}(\Gamma,\Omega)$;*

4. *Every Borel image of X satisfies $\mathsf{U}_{fin}(\Gamma, \Omega)$.*

Proof. This proof follows from Theorem 2.1, the equivalences $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) = \mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) = \mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$, and the fact that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$ is closed under taking Borel images, see [9]. □

3 Finite Dominance

For a subset Y of ${}^{\mathbb{N}}\mathbb{N}$, the property that $\mathrm{maxfin}(Y)$ is not dominating can be stated in terms of finite dominance, a notion defined by Blass in a recent study [1].

Definition 3.1. Let Y be a subset of ${}^{\mathbb{N}}\mathbb{N}$, and $k \in \mathbb{N}$. Y is *k-dominating* if for each $f \in {}^{\mathbb{N}}\mathbb{N}$ there exist $g_1, \dots, g_k \in Y$, such that for all but finitely many n, $f(n) \le \max\{g_1(n), \dots, g_k(n)\}$.

In other words, Y is k-dominating when the collection $\{\max(F) : F \subseteq Y \text{ and } |F| = k\}$ is dominating.

Proposition 3.1. *For $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$, $\mathrm{maxfin}(Y)$ is dominating if, and only if, there exists a natural number k such that Y is k-dominating.*

Proof. We will prove the less trivial implication. Assume that for each k, Y is not k-dominating. For each k, let $g_k \in {}^{\mathbb{N}}\mathbb{N}$ witness that Y is not k-dominating. Since the collection $\{g_k : k \in \mathbb{N}\}$ is countable, there exists $f \in {}^{\mathbb{N}}\mathbb{N}$ bounding it with respect to eventual dominance $\le^*$. Then f witnesses that $\mathrm{maxfin}(Y)$ is not dominating. □

Corollary 3.2. *For a zero-dimensional set X of reals, the following are equivalent:*

1. *X satisfies $\mathsf{U}_{fin}(\mathcal{O}, \Omega)$;*
2. *For each continuous function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$ and for each natural number k, $\Psi[X]$ is not k-dominating.*

4 Reduced Products

The characterization in Theorem 2.1 has an elegant statement in the language of filters. Let $\mathcal{F}$ be a filter over $\mathbb{N}$. An equivalence relation $\sim_{\mathcal{F}}$ is defined on ${}^{\mathbb{N}}\mathbb{N}$ by

$$f \sim_{\mathcal{F}} g \Leftrightarrow \{n : f(n) = g(n)\} \in \mathcal{F}.$$

The equivalence class of f is denoted $[f]_{\mathcal{F}}$, and the set of these equivalence classes is denoted ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$. Using this terminology, $[f]_{\mathcal{F}} < [g]_{\mathcal{F}}$ means

$$\{n : f(n) < g(n)\} \in \mathcal{F}.$$

Lemma 4.1 ([9]). *Let $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$ be such that for each n the set $\{h(n) : h \in Y\}$ is infinite. Then the following are equivalent:*

1. $\mathrm{maxfin}(Y)$ *is not a dominating family.*
2. *There is a non-principal filter $\mathcal{F}$ on $\mathbb{N}$ such that the subset $\{[f]_{\mathcal{F}} : f \in Y\}$ of the reduced product ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$ is bounded.*

Corollary 4.2. *For a zero-dimensional set X of reals, the following are equivalent:*

1. *X satisfies $\mathsf{U}_{fin}(\mathcal{O}, \Omega)$;*
2. *For each continuous function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$, either there is a principal filter $\mathcal{G}$ for which $\Psi[X]/\mathcal{G}$ is finite, or else there is a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$ such that the subset $\Psi[X]/\mathcal{F}$ of the reduced product ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$ is bounded.*

PROOF. This follows from Theorem 2.1 and Lemma 4.1, see [9] for the proof of the corresponding Borel version. □

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