INROADS

Real Analysis Exchange
Vol. 27(2), 2001/2002, pp. 739–748

Alan Macdonald, Department of Mathematics, Luther College, Decorah, IA 52101. e-mail: macdonal@luther.edu

STOKES' THEOREM

Abstract

Jean Mawhin has proved a version of Stokes' theorem on a cube using a generalized Riemann integral. We give a new, much simpler, and intuitive proof of his theorem using the integral definition of the exterior derivative.

1 Introduction

Stokes' theorem on a manifold is a central theorem of mathematics. Special cases of the theorem include the fundamental theorem of calculus, the integral theorems of vector analysis, and the Cauchy-Goursat theorem (as we shall see). Jean Mawhin has proved a version of the theorem on a cube using his RP generalized Riemann integral [12, Theorem 2]:

Stokes' Theorem on a Cube. *Let ω be a differential $(n-1)$-form defined on an open set $U \supseteq \left[0,1\right]^n$. If $d\omega$ exists on $\left[0,1\right]^n$, then $d\omega$ is RP-Integrable on $\left[0,1\right]^n$ and*

$$\text{(RP)} \int_{\left[0,1\right]^n} d\omega = \int_{\partial\left[0,1\right]^n} \omega \,. \tag{1}$$

My main purpose here is to give a new, much simpler, and intuitive proof of this theorem. I then prove an easy consequence.

Corollary. *Let ω be a continuous differential $(n-1)$-form on $\left[0,1\right]^n$. Suppose that $d\omega$ exists on $(0,1)^n$ and is Lebesgue integrable there. Then*

$$\int_{\left[0,1\right]^n} d\omega = \int_{\partial\left[0,1\right]^n} \omega. \tag{2}$$

Key Words: Generalized Riemann Integral, Gauge Integral, Stokes' theorem
Mathematical Reviews subject classification: 28A99, 58C35
Received by the editors August 1, 2001

Using standard techniques [15, pp. 303, 353], [24, p. 124], [25, p. 354], [2], the corollary can then be lifted to a Lebesgue integral version of Stokes' theorem on a manifold:

Stokes' Theorem on a Manifold. *Let ω be a continuous differential $(n-1)$-form on a compact oriented n-manifold M with boundary ∂M. Suppose that ω is differentiable on $M - \partial M$ and $d\omega$ is Lebesgue integrable there. Then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{3}$$

Traditional statements of Stokes' theorem, from those of Green's theorem on a rectangle to those of Stokes' theorem on a manifold, elementary and sophisticated alike, require that $d\omega$ be continuous. See for example de Rham [5, p. 27], Grunsky [6, p. 97], Nevanlinna [16, p. 131], and Rudin [22, p. 272].

Standard versions of Green's theorem imply Cauchy's integral theorem. As Mawhin observes [12], Stokes' theorem above specializes to a version of Green's theorem which implies the Cauchy-Goursat theorem. Acker [1] points out that this counters the usual view that the Cauchy-Goursat theorem is not a corollary of Green's theorem and so requires a special proof.

Traditional statements of Stokes' theorem also require $d\omega$ to exist on all of M. But $d\omega$ need not exist on ∂M in Stokes' theorem above. Thus the Cauchy-Goursat theorem can be extended to the case where f is continuous on a simple closed curve and its interior, and analytic in its interior. This result can also be found in [11, Th. 3.10], where it is called "important".

The key to our proof of Mawhin's theorem is the integral definition of $d\omega$. The integral definition gives $d\omega$ a simple geometric meaning. The definition makes possible a simple and intuitive one line heuristic demonstration of Stokes' theorem, which shows us the *reason* for the theorem. This is the topic of §2. Mawhin's RP generalized Riemann integral is discussed in §3. In §4, I show how the integral definition of $d\omega$ and the RP integral fit hand in glove to turn the heuristic demonstration of Stokes' theorem on a cube into a new, simple, and intuitive proof of Mawhin's theorem.

The techniques of this paper were used recently to give the first - to my knowledge - rigorous proof of the fundamental theorem of geometric calculus [10].

2 The Integral Definition of $d\omega$.

Let ω be an $(n-1)$-form on $\mathbf{R}^n$. Fix $x \in \mathbf{R}^n$. Let c denote an n-cube (of arbitrary orientation) with $x \in c$. Define

$$d\omega(x) \;=\; \lim_{\substack{x\in c \\ \operatorname{diam}(c)\to 0}} \frac{1}{|c|} \int\limits_{\partial c} \omega \;. \tag{4}$$

(By a slight abuse of notation we identify the n-form $d\omega$ with its (single) coefficient function $d\omega(x)$: $d\omega = d\omega(x)\, dx_1 \wedge \ldots \wedge dx_n$.)

This *integral definition* gives $d\omega$ a clear geometric meaning. The integral definition tells us the *reason* for Stokes' theorem. To see this, partition $\big[\,0,1\big]^n$ with small cubes $\big\{c_j\big\}$ and let $x_j \in c_j$. Then if $d\omega$ is Riemann integrable,

$$\int\limits_{\partial\left[0,1\right]^n} \omega = \sum_j \int\limits_{\partial c_j} \omega \approx \sum_j d\omega\big(x_j\big)\big|c_j\big| \to \int\limits_{\left[0,1\right]^n} d\omega \;. \tag{5}$$

The integral definition is essential in turning this heuristic argument into our proof of Mawhin's version of Stokes' theorem on a cube.

There is a step-by-step parallel between the heuristic argument and a proof of the fundamental theorem of calculus which requires that $f\,'$ exist and be Riemann integrable on (a,b) (and that f be continuous on $[a,b]$): Let $a = x_0 < \ldots < x_j < \ldots < x_n = b$. Then using a telescoping series and the mean value theorem,

$$f(b) - f(a) = \sum_{j=1}^{n} \{f(x_j) - f(x_{j-1})\} = \sum_{j=1}^{n} f\,'(c_j)(x_j - x_{j-1}) \to \int_a^b f\,'. \tag{6}$$

The integral definition of $d\omega$ is invariant under a rotation of coordinates. In contrast, the usual *derivative definition* of $d\omega$ is given in terms of partial derivatives with respect to some coordinate system. It must then be proved that the derivative definition is invariant under a rotation of coordinates.

One might say that the integral definition tells us *what* $d\omega$ *is*, whereas the derivative definition tells us how to *compute* it.

In Section 5 we show that if ω is differentiable; i.e., its coefficient functions are linearly approximate, then $d\omega$ exists and the integral definition is equivalent to the derivative definition. This shows that the derivative definition of $d\omega$ is invariant under a rotation of coordinates.

The integral definition of $d\omega$ and the heuristic demonstration of Stokes' theorem are used in many physics oriented texts, e.g., [2, p. 188], [21, p. 10], [23, Section 5.8], and [26, pp. 83, 93]. They should be better known to mathematicians.

3 The RP Integral

The first generalized Riemann integral was the *Henstock-Kurzweil integral* in $\mathbf{R}^1$ [7] [9]. Bartle has given an excellent elementary account of this integral [3]. See also [14].

The HK integral solves a problem in formulating the fundamental theorem of calculus: a derivative need not be Riemann, or even Lebesgue, integrable. Among the impressive features of the HK integral is its formulation of the fundamental theorem:

If f' exists on $[a,b]$, then f' is HK-integrable on $[a,b]$ and

$$\text{(HK)}\int_a^b f'(x)\,dx = f(b) - f(a). \tag{7}$$

Equally impressive is the trivial proof of the theorem. All this even though f' need not be Lebesgue integrable. Moreover, the HK integral is *super Lebesgue*: If f is Lebesgue integrable, then it is HK integrable to the same value.

The HK integral in $\mathbf{R}^n$ does not always integrate $d\omega$ [18, Example 5.7]. Mawhin designed his RP integral to overcome this deficiency.

According to Mawhin's theorem, the RP integral always integrates $d\omega$. In addition, the integral is super Lebesgue. See [18, Prop. 4.1] for a short proof and [4] for a different proof. Why, then, don't we abandon the Lebesgue integral in favor of the RP integral? Most important for us, the change of variable theorem fails [19, p. 143], and so the integral cannot be lifted to manifolds. Fubini's theorem also fails [18, Remark 5.8]. And there are other deficiencies [18, Remark 7.3].

Unlike $\mathbf{R}^1$, where the HK integral seems to be completely satisfactory, none of the several generalized Riemann integrals in higher dimensions seems to have enough desirable properties to make it a useful general purpose integral. Thus current versions of Stokes' theorem stated in terms of a generalized Riemann integral (e.g., [8], [13], [17], [19], [20]) cannot serve as a general purpose Stokes' theorem. We consider the RP integral to be only a catalyst to compute the Lebesgue integral on the left side of Eq. (3).

We now give a series of definitions leading to the RP integral [12] [13], specialized to $\left[0,1\right]^n$. The definition of the RP integral becomes that of the Riemann integral if the function $\delta(x)$ below is replaced with a constant δ and the cubes c_j with rectangles.

A *gauge* on $\left[0,1\right]^n$ is a positive function $\delta(x)$ on $\left[0,1\right]^n$.

A *tagged regular partition* $\{c_j,\ x_j\}_{j=1}^k$ of $\left[0,1\right]^n$ is a decomposition of $\left[0,1\right]^n$ into closed subcubes $\{c_j\}$ together with points $x_j \in c_j$. The c_j are disjoint except for boundaries.

Let δ be a gauge on $\left[0,1\right]^n$. A tagged regular partition $\{c_j,\ x_j\}_{j=1}^k$ is δ-*fine* if $\operatorname{diam}(c_j) \le \delta(x_j),\ j = 1 \dots k$.

Let $d\omega$ be an n-form defined on $[0,1]^n$. A number, denoted $(\mathrm{RP})\int_{[0,1]^n} d\omega$, is the *RP integral* of $d\omega$ over $\left[0,1\right]^n$ if, given $\varepsilon > 0$, there is a gauge δ on $\left[0,1\right]^n$ so that for every δ-fine tagged regular partition $\{c_j,\ x_j\}$ of $\left[0,1\right]^n$,

$$\left| (\mathrm{M}) \int_{\left[0,1\right]^n} \mathrm{d}\omega - \sum_{\mathrm{j=1}}^{\mathrm{k}} \mathrm{d}\omega\big(\mathrm{x_j}\big)\left|\,\mathrm{c_j}\right| \right| \le \varepsilon. \tag{8}$$

If this definition is to make sense, we need to prove two things:

(i) *Given a gauge δ on $\left[0,1\right]^n$, there is a δ-fine tagged regular partition of* $\left[0,1\right]^n$ (Cousin's lemma). To see this, first note that if a cube c is partitioned into subcubes, each of which has a δ-fine regular partition, then c has a δ-fine regular partition. Thus if $\left[0,1\right]^n$ has no δ-fine regular partition, then there is a sequence $\left[0,1\right]^n \supset c_1 \supset c_2 \supset \dots$ of compact cubes with no δ-fine regular partition and $\operatorname{diam}(c_i) \to 0$. Let $\{x\} = \bigcap_i c_i$. Choose j so that $\operatorname{diam}(c_j) \le \delta(x)$. Then $\{(c_j, x)\}$ is a δ-fine regular partition of c_j, which is a contradiction.

(It is interesting to note that the standard proof of the Cauchy-Goursat theorem and Acker's proof of Stokes' theorem [1] use similar compactness arguments.)

(ii) *If the RP integral exists, then it is unique.* For if δ_1 and δ_2 are gauges and $\delta = \operatorname{Min}(\delta_1, \delta_2)$, then a δ-fine regular partition is also δ_1-fine and δ_2-fine.

4 Proof of Stokes' Theorem

PROOF OF STOKES THEOREM.

Given $\varepsilon > 0$, define a gauge $\delta(x) > 0$ on $\left[0,1\right]^n$ as follows. Choose $x \in \left[0,1\right]^n$. Then, according to the integral definition of $d\omega$, Eq. (4), there is a $\delta(x) > 0$ so that if $x \in c$, a cube with $\operatorname{diam}(c) \le \delta(x)$, then $\left|\int_{\partial c} \omega - d\omega\big(x\big)|c\,|\right| < \varepsilon|c\,|$. Now let $\left\{c_j, x_j\right\}$ be a δ-fine tagged regular partition of $\left[0,1\right]^n$. Then

$$\left| \int\limits_{\partial\left[0,1\right]^n} \omega - \sum_j d\omega\big(x_j\big)\,|c_j| \right| = \left| \sum_j \int\limits_{\partial c_j} \omega - \sum_j d\omega\big(x_j\big)\,|c_j| \right| < \sum_j \varepsilon\big|c_j\big| = \varepsilon.$$

By the definition of the RP integral, Eq. (8), $(\mathrm{RP})\int_{\left[0,1\right]^n} d\omega$ exists and is equal to $\int_{\partial\left[0,1\right]^n} \omega$. $\square$

As stated in Section 1, "The RP integral fits hand in glove with the integral definition of $d\omega$ to turn the heuristic demonstration of Stokes' on a cube [Eq. (5)] into a simple and intuitive proof."

PROOF OF COROLLARY.

We can now prove the corollary from Section 1. Let $c_k = \left[k^{-1}, 1-k^{-1}\right]^n$. From the result just proved and the fact that the RP integral is super Lebesgue [18, Prop. 4.1], we have

$$\int_{c_k} d\omega = \int_{\partial c_k} \omega. \tag{9}$$

Let $k \to \infty$ in Eq. (9). The left side approaches the left side of Eq. (2) by the Lebesgue dominated convergence theorem. And the right side approaches the right side of Eq. (2) by the uniform continuity of ω on $\left[\,0,1\right]^n$. □

5 Existence of $d\omega$ and Its Coordinate Representation

Let

$$\omega = \sum_{j=1}^{n} f_j(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \tag{10}$$

be an $(n-1)$-form, where the hat indicates that dx_j is omitted. If the f_j are differentiable at 0, then $d\omega(0)$ (defined by the integral definition) exists and is given by the derivative definition:

$$d\omega(0) = \sum_{j=1}^{n} (-1)^{j-1} \partial_j f_j(0). \tag{11}$$

PROOF. By the integral definition of $d\omega$, Eq. (4), we must show that

$$\lim_{\substack{0 \in c \\ \mathrm{diam}(c) \to 0}} \frac{1}{|c|} \int_{\partial c} \sum_{j=1}^{n} f_j(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n = \sum_{j=1}^{n} (-1)^{j-1} \partial_j f_j(0). \tag{12}$$

We first prove Eq. (12) for cubes with sides parallel to the x-axes. For such cubes it suffices to show that for an arbitrary p and differentiable function f,

$$\lim_{\substack{0 \in c \\ \mathrm{diam}(c) \to 0}} \frac{1}{|c|} \int_{\partial c} f(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_p} \wedge \ldots \wedge dx_n = (-1)^{p-1} \partial_p f(0). \tag{13}$$

Let c have width ε and sides $s_j^{\pm}$, on which x_j is constant. The only sides in ∂c contributing to the integral in Eq. (13) are $s_p^{\pm}$. And by definition, the orientation of $s_p^\pm$ in ∂c is $\pm(-1)^{p-1}$ times the orientation of $s_p^\pm$ in $\mathbf{R}^n$; i.e., $\pm(-1)^{p-1}(x_1, \dots \widehat{x_p}, \dots x_n)$ [24, p. 98]. Thus Eq. (13) can be written

$$\lim_{\varepsilon\to 0} \frac{(-1)^{p-1}}{\varepsilon^n}\left[\int_{s_p^+} f(x) - \int_{s_p^-} f(x)\right] = (-1)^{p-1}\partial_p f(0). \tag{14}$$

Our hypothesis that f is differentiable at 0 means that

$$f(x) = f(0) + \sum_{k=1}^{n} \partial_k f(0)\, x_k + R\big(x\big), \tag{15}$$

where $\big|R(x)\big|/\big|x\big| \to 0$ as $\big|x\big| \to 0$.

We prove Eq. (14) by substituting separately the three terms on the right side of Eq. (15) for $f(x)$ in the left side of Eq. (14). The result will be the right side of Eq. (14).

FIRST TERM. Substitute $f(0)$ for $f(x)$ in the left side of Eq. (14). The two integrals are equal and so the result is zero.

SECOND TERM. For $x \in s_p^+$, let $\tilde{x} = (\tilde{x}_1, \dots \tilde{x}_p, \dots \tilde{x}_n)$ be the corresponding point on the opposite side s_p^-. Then $\tilde{x}_p = x_p - \varepsilon$, and for $k \neq p$, $\tilde{x}_k = x_k$. Substitute $\partial_k f(0)\, x_k$ for $f(x)$ in the left side of Eq. (14), omitting the limit:

$$\frac{\big(-1\big)^{p-1}\partial_k f(0)}{\varepsilon^n}\int_{s_p^+}(x_k - \tilde{x}_k).$$

If $k = p$, this expression is equal to the right side of Eq. (14). If $k \neq p$, the expression is zero.

THIRD TERM. Substitute $R\,(x)$ for $f(x)$ in the left side of Eq. (14). Since $|x| \leq \sqrt{n}\varepsilon$ on c,

$$\left|\frac{(-1)^{\,p-1}}{\varepsilon^n}\int_{s_p^\pm} R(x)\right| \;\leq\; \frac{1}{\varepsilon^n}\int_{s_p^\pm}\frac{\sqrt{n}\,\varepsilon}{|x|}\,|R(x)| \;\leq\; \sqrt{n}\sup_{|x|\,\leq\,\sqrt{n}\varepsilon}\frac{|R(x)|}{|x|} \to 0. \tag{16}$$

We have now proved Eq. (12) for cubes with sides parallel to the axes. However, the limit in Eq. (12) is taken as $\operatorname{diam}(c) \to 0$ for cubes of arbitrary orientation. Thus it remains to show that the limit is *independent of* and *uniform in*, the orientation of the cubes.

The only limit taken in proving Eq. (14) is in Eq. (16). This limit is independent of and uniform in the orientation of the cubes because $R(x)$ is invariant under a rotation of coordinates. To see this, observe that the other three terms in Eq. (15) are invariant under a rotation. ($f(x)$ is independent of the coordinates assigned to the point x, and the sum is $\nabla f\cdot\mathbf{x}$, where, since f is differentiable, ∇f is a vector.) $\square$

6 Acknowledgments

I thank Professor Felipe Acker and Professor Robert Bartle for sending me unpublished materials.

References

[1] P. Acker, *The Missing Link*, Math. Intell., **18** (1996), 4–9.

[2] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[3] R. Bartle, *Return to the Riemann Integral*, Amer. Math. Monthly, **103** (1996), 625–632.

[4] R. Davies and Z. Schuss, *A Proof that Henstock's Integral Includes Lebesgues'*, J. Lond. Math. Soc., **2** (1970), 561–562.

[5] G. de Rham, *Differentiable Manifolds*, Springer-Verlag, Berlin, 1984.

[6] H. Grunsky, *The General Stokes' Theorem*, Pitman, Boston, 1983.

[7] R. Henstock, *Definitions of Riemann Type of the Variational Integrals*, Proc. London Math. Soc., **11** (1961), 402–418.

[8] J. Jarník, J. Kurzweil, *A New and More Powerful Concept of the PU-Integral*, Czech. Math. J., **38** (1988), 8–48.

[9] J. Kurzweil, *Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter*, Czech. Math. J., **82** (1957), 418–446.

[10] A. Macdonald, *The Fundamental Theorem of Geometric Calculus via a Generalized Riemann Integral*, Adv. Appl. Cliff. Alg., **8(5)** (1998).

[11] A. Markushevich, *Theory of Functions of a ccomplex Variable*, Vol. 3, Chelsea, New York, 1977.

[12] J. Mawhin, *Generalized Riemann Integrals and the Divergence Theorem for Differentiable Vector Fields*, in E. B. Christoffel, Birkhauser, Basel-Boston, 1981, 704–714.

[13] J. Mawhin, *Generalized Multiple Perron Integrals and the Green-Goursat Theorem for Differentiable Vector Fields*, Czech. Math. J., **31** (1981), 614–632.

[14] R. McLeod, *The Generalized Riemann Integral*, Carus Math. Monographs, no. 20, Math. Assn. Amer., Washington, 1980.

[15] J. Munkres, *Analysis on Manifolds*, Addison-Wesley, Redwood, 1991.

[16] F. and R. Nevanlinna, *Absolute Analysis*, Springer-Verlag, New York, 1973.

[17] D. Nonnenmacher, *A Descriptive, Additive Modification of Mawhin's Integral and the Divergence Theorem with Singularities*, Ann. Polon. Math., **59** (1994), 85–98.

[18] W. Pfeffer, *The Divergence Theorem*, Trans. Amer. Math. Soc., **295** (1986), 665–685.

[19] W. Pfeffer, *The Multidimensional Fundamental Theorem of Calculus*, J. Austral. Math. Soc. Ser. A, **43** (1987), 143–170.

[20] W. Pfeffer, *The Gauss-Green Theorem*, Adv. Math., **87** (1991), 93–147.

[21] J. Reitz et al., *Foundations of Electromagnetic Theory*, Third Ed., Addison-Wesley, Reading MA, 1980.

[22] W. Rudin, *Principles of Mathematical Analysis*, Third Ed., McGraw-Hill, New York, 1976.

[23] M. Schwartz et al., *Vector Analysis with Applications to Physics*, Harper, 1960, New York.

[24] M. Spivak, *Calculus on Manifolds*, W. A. Benjamin, Menlo Park, CA, 1965.

[25] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 1, 2nd Ed., Publish or Perish, Houston, 1970.

[26] A. Wills, *Vector Analysis*, Prentice-Hall, New York, 1931.