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ON SOME PROPERTIES OF THE CLASS OF REAL FUNCTIONS WITH λ AND λ' GRAPHS

Abstract

We show that (under CH) there exists a CIVP function with a λ' graph. We examine some properties of $\mathcal{M}_a(\lambda)$ and $\mathcal{M}_a(\lambda')$ class of real valued functions.

1 General Notation

First, let us recall a couple of definitions:

1. A set L is said to be a λ*-set* if every countable subset of L is G_δ relative to L.

2. A subset L of a space X is said to be a λ' (rel X) if for every countable subset D of X, $L \cup D$ has property λ.

We shall need also the following well-known fact, basic in our investigations.

Fact 1.1. *If $A \subseteq \mathbb{R}$ has property λ' (rel $\mathbb{R}$) and is the one-to-one projection of the subset H of $\mathbb{R}^2$ (i.e., H is the graph of an arbitrary real valued function with domain A), then H has property λ' (rel $\mathbb{R}^2$). The corresponding assertion for the λ property instead λ' holds.*

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(See for example [1] or [8]). Note also that property λ' is countably additive. On the other hand, property λ is not preserved even under taking finite unions.

For $A \subseteq \mathbb{R}^2$ we denote by A_x and $A^{(y)}$ the *x-section* and *y-section* of A, respectively (i.e., $A_x = \{y \colon \langle x, y\rangle \in A\}$, $A^{(y)} = \{x \colon \langle x, y\rangle \in A\}$). Throughout this paper no distinction is made between a function and its graph. Therefore, to shorten notation we write $f \in \lambda$ (λ', resp.) in the case when f has a λ (λ', resp.) graph or we say that f is a λ (λ', resp.) function. Let $\mathcal{C}$ denote as usual the ternary Cantor set.

We say that a function $f \in \mathbb{R}^{\mathbb{R}}$ has the CIVP property (*Cantor Intermediate Value Property*) if for all different $x, y \in \mathbb{R}$ such that $f(x) \neq f(y)$ and for every Cantor set P between $f(x)$ and $f(y)$, there exists a Cantor set Q between x and y such that $f(Q) \subseteq P$.

Let Mkb denote the Mokobodzki σ-ideal:

$$\mathrm{Mkb} = \{X \subseteq \mathbb{R}^2 \colon \forall_{\epsilon>0} \exists_{U \supseteq X} \left[U \textit{ open } \wedge \forall_x \mu(U_x) < \epsilon\right]\}.$$

Let us denote by Mkb^{-1} the "inverse" Mokobodzki σ-ideal, namely:

$$\mathrm{Mkb}^{-1} = \{i(X) \colon X \in \mathrm{Mkb}\},$$

where $i(\langle x, y\rangle) = \langle y, x\rangle$. It is well known that the σ-ideals Mkb, Mkb^{-1} are generated by G_δ sets; i.e.,

$$\forall_{X \in \mathcal{I}} \exists_{G \in G_\delta \cap \mathcal{I}} X \subseteq G.$$

where $\mathcal{I} = \mathrm{Mkb}$ or $\mathcal{I} = \mathrm{Mkb}^{-1}$.

In the sequel, $\mathit{Perf}_{\mathcal{C}}$ denotes the collection of all perfect subsets of $\mathbb{R}$ homeomorphic to 2^ω. By *Intr* we denote the family of all open intervals (a, b), $a < b$. We define ω_1^{Ev} (ω_1^{Od}) to be the set of all even (odd, respectively) ordinals from ω_1.

2 Introduction

As it was observed by T. Natkaniec and I. Recław, under CH there exists a function $f \in \lambda'$ such that f is almost continuous. Indeed, suppose that $S \subseteq \mathbb{R}$ is a $\mathbf{c}$-dense λ' subset of $\mathbb{R}$ (for example, take any $\mathbf{c}$-dense Sierpiński set). There exists a function $f_1 \colon S \to \mathbb{R}$ such that for each $f^* \supseteq f_1$, f^* is almost continuous. (see for example [6] or [3]). Let $\tilde{f} \colon \mathbb{R} \to S$ be any one-to-one function. Note that λ' subsets of $\mathbb{R}^2$ forms a σ-ideal, thus if we define $f = \tilde{f} \restriction (\mathbb{R} \setminus S) \cup f_1$, then the function f will be almost continuous and $f \in \lambda'$.

On the other hand, it is obvious that there is no λ function with the Ext property (for definition of Ext and almost continuous property see for example [7]).

Since $AC \not\to CIVP$, it is a natural question whether (under CH) there exists a λ' function with the CIVP property. In the sequel we shall answer this question in the affirmative. In fact, we will show something more.

Theorem 2.1. *Assume CH. There exists a function $f \in \mathbb{R}^{\mathbb{R}}$ such that*

1. *f is a λ' set.*
2. *for each interval (a,b) and for each $P \in Perf$ there is $Q \subseteq (a,b)$, $Q \in Perf$ such that $Q \subseteq f^{-1}[P]$. (in particular, f has the* CIVP *property)*

We shall use two (folklore?) lemmas some of which might be well known. We give the proofs for completeness.

Lemma 2.2. *Suppose that $X \in \mathrm{Mkb}^{-1}$. For each $P \in Perf_{\mathcal{C}}$ and for each interval $(a,b) \subseteq \mathbb{R}$ there exists $Q, R \in Perf$ such that $Q \subseteq (a,b)$, $R \subseteq P$ and $(Q \times R) \cap X = \emptyset$.*

PROOF. Since $X \in \mathrm{Mkb}^{-1}$, there exists $G \in G_\delta$ such that $G \in \mathrm{Mkb}^{-1}$ and $X \subseteq G$. Then $\forall_{y \in P} G^{(y)} \in \mathcal{N}$, thus $G \cap ((a,b) \times P)$ is of measure zero in the space $(a,b) \times P$. (homeomorphic to $(a,b) \times 2^\omega$). Next, by the classical Mycielski theorem (see [5]), the conclusion follows.

Throughout this proof, $\{C_\alpha\}_{\alpha \in \omega_1^{Ev}}$ is a fixed enumeration of all countable subsets of $\mathbb{R}^2$.

Lemma 2.3. *Assume CH. Let $\langle G_\alpha \colon \alpha \in \omega_1 \rangle$, $\langle l_\alpha \colon \alpha \in \omega_1 \rangle$ be a sequences of subsets of $\mathbb{R}^2$ such that*

1. *$\forall_{\alpha \in \omega_1} G_\alpha \in G_\delta$ and $l_\alpha \in \lambda'$,*
2. *$\bigcup_{\alpha<\theta} G_\alpha \cup l_\theta \subseteq G_\theta$ and $\bigcup_{\alpha<\theta} G_\alpha \cap l_\theta = \emptyset$ for each $\theta < \omega_1$.*
3. *For each $\theta \in \omega_1^{Ev}$, $C_\theta \setminus \bigcup_{\alpha<\theta} G_\alpha \neq \emptyset \Rightarrow l_\theta \cap C_\theta \neq \emptyset$.*

Then the set $l = \bigcup_{\alpha<\omega_1} l_\alpha$ is a λ' set.

PROOF. Let $D \subseteq l$ be a countable set. There exists $\theta \in \omega_1$ such that $D \subseteq \bigcup_{\alpha \le \theta} l_\alpha$. Since sets with a λ' property forms a σ-ideal, we have $\bigcup_{\alpha\le\theta} l_\alpha \in \lambda'$. Hence there exists a G_δ set H such that $H \cap \bigcup_{\alpha\le\theta} l_\alpha = D$. Thus

$$(G_\theta \cap H) \cap l = \left[(G_\theta \cap H) \cap \bigcup_{\alpha\le\theta} l_\alpha\right] \cup \left[(G_\theta \cap H) \cap \bigcup_{\theta<\alpha<\omega_1} l_\alpha\right] = D.$$

Therefore $l \in \lambda$.

Let $D \subseteq \mathbb{R}^2$ be a countable set such that $D \cap l = \emptyset$. Then there exists $\theta \in \omega_1^{Ev}$ such that $D = C_\theta$. We have $C_\theta \setminus \bigcup_{\alpha<\theta} G_\alpha = \emptyset$: if not we would have $l_\theta \cap C_\theta \neq \emptyset$, a contradiction. This means that $D = C_\theta \subseteq \bigcup_{\alpha<\theta} G_\alpha$. Since $\bigcup_{\alpha\leq\theta} l_\alpha \in \lambda'$, there exists $H \in G_\delta$ such that $H \cap \left[\bigcup_{\alpha\leq\theta} l_\alpha \cup D\right] = D$. Define $H^* = G_\theta \cap H$, obviously $H^* \in G_\delta$. Next, we have

$$\begin{aligned} H^* \cap (l \cup D) &= (G_\theta \cap H) \cap \left[\bigcup_{\alpha\leq\theta} l_\alpha \cup \bigcup_{\alpha>\theta} l_\alpha\right] \cup D \\ &= \left[H \cap \bigcup_{\alpha\leq\theta} l_\alpha\right] \cup D = D. \end{aligned}$$

This finally proves $l \in \lambda'$ and finishes the proof of Lemma 2.3. □

PROOF OF THEOREM 2.1. Enumerate $\mathit{Intr} \times (\mathit{Perf}_{\mathcal{C}} \cap \mathcal{N})$ as $\{\langle I_\alpha; P_\alpha\rangle : \alpha \in \omega_1^{Od}\}$. We will construct inductively sequences $\langle G_\theta : \theta \in \omega_1\rangle$ and $\langle l_\theta : \theta \in \omega_1\rangle$ assuming the following induction hypothesis:

1. For each $\theta \in \omega_1$, $G_\theta \in G_\delta \cap \mathrm{Mkb} \cap \mathrm{Mkb}^{-1}$;
2. $l_\theta : \mathrm{dom}(l_\theta) \to \mathbb{R}$ is a one-to-one function such that $\mathrm{dom}(l_\theta) \in \mathcal{N}$ and $l_\theta \subseteq G_\theta$.

Let $\theta \in \omega_1$. Consider two cases:

Case 1: $\theta \in \omega_1^{Od}$.

Define $A_\theta^* = \bigcup_{\alpha<\theta} G_\alpha$ and $H_\theta^* = \left[\bigcup_{\alpha<\theta} \mathrm{dom}(l_\alpha)\right] \times \mathbb{R}$. One easily checks that $A_\theta^* \in \mathrm{Mkb} \cap \mathrm{Mkb}^{-1}$ and $H_\theta^* \in \mathrm{Mkb}^{-1}$. It follows that $A_\theta^* \cup H_\theta^* \in \mathrm{Mkb}^{-1}$, hence by Lemma 2.2 there exists $R_\theta \in \mathit{Perf}(P_\theta)$ and $Q_\theta \in \mathit{Perf}$, $Q_\theta \subseteq I_\theta$ such that $(Q_\theta \times R_\theta) \cap (A_\theta^* \cup H_\theta^*) = \emptyset$. Without loss of generality, one can assume that $Q_\theta \in \mathcal{N}$. Since $R_\theta \in \mathit{Perf}$, there is a λ' set $S_\theta \subseteq R_\theta$ of size 2^ω. Let l_θ be any bijection from Q_θ onto S_θ. Clearly $l_\theta \in \mathrm{Mkb} \cap \mathrm{Mkb}^{-1}$ (since $Q_\theta \in \mathcal{N}$ and $P_\theta \in \mathcal{N}$). Hence $A_\theta^* \cup l_\theta \in \mathrm{Mkb} \cap \mathrm{Mkb}^{-1}$; thus, there exists $G_\theta \in G_\delta \cap \mathrm{Mkb} \cap \mathrm{Mkb}^{-1}$ such that $A_\theta^* \cup l_\theta \subseteq G_\theta$.

Case 2: $\theta \in \omega_1^{Ev}$.

If $C_\theta \subseteq \bigcup_{\alpha<\theta} G_\alpha$, then let $l_\theta = \emptyset$. If $C_\theta \setminus \bigcup_{\alpha<\theta} G_\alpha \neq \emptyset$, then pick an arbitrary $z_\theta \in C_\theta \setminus \bigcup_{\alpha<\theta} G_\alpha$ and define $l_\theta = \{z_\theta\}$. Next, choose an arbitrary G_δ set G from $\mathrm{Mkb} \cap \mathrm{Mkb}^{-1}$ which contains $\bigcup_{\alpha<\theta} G_\alpha \cup l_\theta$ and define $G_\theta = G$. Observe that

$$\forall_{\alpha,\beta\in\omega_1^{Od}} \alpha \neq \beta \Rightarrow \mathrm{dom}(l_\alpha) \cap \mathrm{dom}(l_\beta) = \emptyset.$$

Indeed, suppose that $\alpha < \beta$. Since $(Q_\beta \times R_\beta) \cap (A^*_\beta \cup H^*_\beta) = \emptyset$, we have $Q_\beta \cap \bigcup_{\mu<\beta} \mathrm{dom}(l_\mu) = \emptyset$, thus $\mathrm{dom}(l_\alpha) \cap \mathrm{dom}(l_\beta) = \emptyset$. Let us define $k = \bigcup_{\theta\in\omega_1^{Od}} l_\theta$. It is easy to see that k is a real function a domain of which is a subset of $\mathbb{R}$. Since the sequences $\langle G_\theta \colon \theta \in \omega_1\rangle$ and $\langle l_\theta \colon \theta \in \omega_1\rangle$ fulfill the conditions (1)-(3) of Lemma 2.3, we conclude that $\bigcup_{\alpha\in\omega_1} l_\alpha \in \lambda'$, thus $k \in \lambda'$.

Suppose that $I \in Intr$, $P \in Perf_{\mathcal{C}}$. We will show that there exists a perfect set Q such that $Q \subseteq k^{-1}(P) \cap I$. Choose $\theta \in \omega_1^{Od}$ such that $I = I_\theta$ and $P = P_\theta$. Then we have

$$Q_\theta \subseteq l_\theta^{-1}(P_\theta) \subseteq k^{-1}(P_\theta).$$

Moreover, $Q_\theta \subseteq I_\theta = I$. Therefore if we extend k arbitrarily to $l \in \lambda'$ defined on whole real line we obtain the conclusion of Theorem 2.1. □

3 Compositions

Theorem 3.1. *Assume that there exists a set $X \in \lambda'$ of size 2^ω. Then every real function $h \in \mathbb{R}^\mathbb{R}$ can be expressed as the composition of two λ' functions.*

PROOF. Let $\Lambda \subseteq \mathbb{R}$ be a λ' set of size 2^ω. Let $h \in \mathbb{R}^\mathbb{R}$ be arbitrary. Let $f \colon \mathbb{R} \to \Lambda$ be an arbitrary bijection. Let us define $g \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} h(f^{-1}(x)) & \text{if } x \in \Lambda \\ f(x) & \text{if } x \notin \Lambda \end{cases}$$

Since $g \restriction \Lambda \in \lambda'$ and $f \restriction (\mathbb{R} \setminus \Lambda) \in \lambda'$ we infer that $g \in \lambda'$. It is evident that $g \circ f = h$. □

We end this chapter by an example which is useful later in this article. We use it in Example 4.4, Theorem 4.5 and Theorem 5.1.

Example 3.2.

Assume CH. Let P be a perfect set and A its countable, dense subset. Let $S \subseteq [\omega]^\omega$ be a scale of size ω_1. Let us denote: $D = [\omega]^{<\omega}$. It is well known (see for example Theorem 5.6 of [4]) that S is a λ-set and for each $H \in G_\delta$ such that $D \subseteq H$ we have $H \cap S \neq \emptyset$. We may assume that $D \subseteq \mathcal{C}$, $\overline{D} = \mathcal{C}$ and $S \subseteq \mathcal{C}$. Let $f_D \colon A \to D$ be a function such that $\forall_{d\in D} \overline{f_D^{-1}(\{d\})} = P$. Let $S_1 \subseteq S$ be any subset of S of size ω_1 such that $|S \setminus S_1| = \omega_1$ and for each $H \in G_\delta$ such that $D \subseteq H$ we have $H \cap S_1 \neq \emptyset$. Set S_1 can be easy constructed by transfinite induction. Let R_1 be any subset of $P \setminus A$ of size ω_1 such that R_1 is a comeager subset or P and $|P \setminus R_1| = \omega_1$. Let $\{H_\theta\}_{\theta\in\omega_1}$ be an enumeration of all G_δ subsets H of $\mathbb{R}^2$ such that $f_D \subseteq H$. We will construct a sequence $\{\langle x_\theta, y_\theta\rangle\}_{\theta\in\omega_1}$ such that $x_\theta \in R_1$ and $y_\theta \in S_1$ by transfinite induction. Suppose that we have already constructed $\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha<\theta}$. It is folklore that if H is a G_δ set, then the set $\{y\colon H^{(y)} \cap P \in \text{co} - \mathcal{MGR}(P)\}$ is a G_δ set. Since $f_D \subseteq H_\theta$ and $\forall_{d\in D}\overline{f_D^{-1}(\{d\})} = P$, the set $H_\theta^* = \{y \in \mathcal{C}\colon H_\theta^{(y)} \cap P \in \text{co} - \mathcal{MGR}(P)\}$ is a G_δ and comeager subset of $\mathcal{C}$. Moreover $D \subseteq H_\theta^*$ hence $H_\theta^* \cap S_1 \setminus \{y_\alpha\colon \alpha < \theta\} \neq \emptyset$. Let y_θ be an arbitrary element of $H_\theta^* \cap S_1 \setminus \{y_\alpha\colon \alpha < \theta\}$. Next, choose an arbitrary x_θ from the set $R_1 \setminus \{x_\alpha\colon \alpha < \theta\} \cap H_\theta^{(y_\theta)}$. Extend $\{\langle x_\theta, y_\theta\rangle\colon \theta \in \omega_1\}$ to a one-to-one function $f^*\colon (\mathbb{R} \setminus A) \to S$ and then define: $f^{(P,A)} = f^* \cup f_D$.

Suppose that H is a G_δ set such that $f_D \subseteq H$. Then there exists $\theta \in \omega_1$ such that $H = H_\theta$. On the other hand, $\langle x_\theta, y_\theta\rangle \in H_\theta$. Thus $H_\theta \cap (f^{(P,A)} \setminus f_D) \neq \emptyset$. This witnesses $f^{(P,A)} \notin \lambda$. Furthermore, suppose that $\gamma \in \mathbb{R}$, $|\gamma| \geq 1$. Define

$$f_\gamma^{(P,A)}(x) = \begin{cases} f^{(P,A)}(x) & \text{if } x \in A \\ f^{(P,A)}(x) + \gamma & \text{if } x \notin A \end{cases}$$

If $\gamma \geq 1$, then $f_\gamma^{(P,A)} = (f^{(P,A)} \restriction A) \cap \big[\mathbb{R} \times (-\infty; 1\rangle\big] \cup \big(f^{(P,A)} \restriction (\mathbb{R} \setminus A) + \gamma\big) \cap \big[\mathbb{R} \times \langle 1; \infty)\big]$. Since $f^{(P,A)} \restriction A$ is countable, $f^{(P,A)} \restriction A$ is a λ set. Moreover, $f^{(P,A)} \restriction (\mathbb{R} \setminus A)\colon (\mathbb{R} \setminus A) \to S$ is one-to-one, hence $f^{(P,A)} \restriction (\mathbb{R} \setminus A) + \gamma \in \lambda$. As $\mathbb{R} \times (-\infty; 1\rangle$ and $\mathbb{R} \times \langle 1; \infty)$ are G_δ sets we obtain that $f_\gamma^{(P,A)} \in \lambda$. In a similar fashion we can prove that $f_\gamma^{(P,A)}$ is a λ set for $\gamma \leq -1$. □

4 Additive Families

Let $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ be a family of real functions. The following notion was first defined and examined by T. Natkaniec in [6].

Definiton 4.1 (T. Natkaniec)**.**

$$\mathcal{M}_a(\mathcal{F}) = \{f \in \mathbb{R}^\mathbb{R}\colon \forall_{h\in\mathcal{F}} f + h \in \mathcal{F}\}$$
$$\mathcal{M}_m(\mathcal{F}) = \{f \in \mathbb{R}^\mathbb{R}\colon \forall_{h\in\mathcal{F}} f \cdot h \in \mathcal{F}\}$$

The goal of this section is to provide a detailed investigation of the families $\mathcal{M}_a(\lambda), \mathcal{M}_a(\lambda')$. We start with a straightforward observation:

Theorem 4.2. *Every continuous function belongs to $\mathcal{M}_a(\lambda)$ and to $\mathcal{M}_a(\lambda')$.*

PROOF. Suppose that f is a continuous real function. Define a function $\Phi_f\colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$\Phi_f(x, y) = \langle x, y + f(x)\rangle.$$

The following lists some easy properties of the function defined above:

1. Φ_f is a bijection.
2. If f is a continuous function, then Φ_f is an automorphism of R^2.
3. For each $g \in \mathbb{R}^{\mathbb{R}}$, $\Phi_f[g] = f + g$.

From this the theorem follows. □

Theorem 4.3. *Suppose that a function $f \in \mathbb{R}^{\mathbb{R}}$ is such that there exist a sequence of functions $\{f_n\}_{n\in\omega}$ from $\mathcal{M}_a(\lambda')$ and a partition $\{X_n\}_{n\in\omega}$ of $\mathbb{R}$ such that $f = \bigcup_{n\in\omega} f_n^* \restriction X_n$. Then f belongs to $\mathcal{M}_a(\lambda')$.*

PROOF. Suppose that $l \in \mathbb{R}^{\mathbb{R}}$ has the λ' property. Then for each $n \in \omega$ we have $f_n^* + l \in \lambda'$, therefore $f + l = \bigcup_{n\in\omega}(f_n^* + l) \restriction X_n \in \lambda'$, since λ' is a σ–ideal. □

The next example shows that our previous theorem is no longer valid for the functions with a λ graph.

Example 4.4. *Assume CH. The function $2D\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$2D(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 2 & \text{if } x \in \mathbb{Q} \end{cases}$$

does not belong to $\mathcal{M}_a(\lambda)$.

PROOF. We will use the function $f_\gamma^{\mathbb{R},\mathbb{Q})}$ from Example 3.2. We have

$$f_2^{(\mathbb{R},\mathbb{Q})}(x) + 2D(x) = f^{(\mathbb{R},\mathbb{Q})}(x) + 2 \text{ for } x \in \mathbb{Q}, \text{ and}$$
$$f_2^{(\mathbb{R},\mathbb{Q})}(x) + 2D(x) = f^{(\mathbb{R},\mathbb{Q})}(x) + 2 \text{ for } x \in \mathbb{R} \setminus \mathbb{Q}.$$

Therefore $(f_2^{\mathbb{R},\mathbb{Q})} + 2D) = f^{(\mathbb{R},\mathbb{Q})} + 2 \notin \lambda$. Since $f_2^{(\mathbb{R},\mathbb{Q})} \in \lambda$ we obtain that $2D \notin \mathcal{M}_a(\lambda)$. □

This example can be generalized to the following theorem.

Theorem 4.5. *Assume CH. Suppose that $A \subseteq \mathbb{R}$ is a countable set.*

1. *If $|\overline{A}| \leq \omega$, then $\chi_A \in \mathcal{M}_a(\lambda)$.*
2. *If $\overline{A}$ is a perfect set, then $\chi_A \notin \mathcal{M}_a(\lambda)$.*

PROOF. 1. Let $l \in \mathbb{R}^{\mathbb{R}}$ be a function such that $l \in \lambda$ and suppose that $D \subseteq \mathbb{R}$ is a countable set. Without loss of generality we may assume that $\overline{A} \subseteq D$. Since l is a λ set, there exists a G_δ set $G \subseteq \mathbb{R}^2$ such that $G \cap l = l \restriction D$. Then we have

$$\begin{aligned}(\chi_A + l) \restriction D &= (\chi_A + l) \restriction \overline{A} \cup (\chi_A + l) \restriction (D \setminus \overline{A}) \\ &= (\chi_A + l) \restriction \overline{A} \cup l \restriction D \setminus \overline{A} \\ &= \left[(\overline{A} \times \mathbb{R}) \cup [G \cap (\overline{A}^c \times \mathbb{R})]\right] \cap (\chi_A + l).\end{aligned}$$

Since $(\overline{A} \times \mathbb{R})$ and $(\overline{A}^c \times \mathbb{R})$ are G_δ sets we conclude that $\chi_A + l$ is a λ-set.

2. Let us assume that $|A| \leq \omega$ and $\overline{A}$ is a perfect set. We will use the function $f_\gamma^{(\overline{A},A)}$ from Example 3.2. We have

$$\begin{aligned}f_2^{(\overline{A},A)}(x) + 2\chi_A(x) &= f^{(\overline{A},A)}(x) + 2 \text{ for } x \in A, \text{ and} \\ f_2^{(\overline{A},A)}(x) + 2\chi_A(x) &= f^{(\overline{A},A)}(x) + 2 \text{ for } x \in \mathbb{R} \setminus A.\end{aligned}$$

Therefore $(f_2^{(\overline{A},A)} + 2\chi_A) = f^{(\overline{A},A)} + 2 \notin \lambda$. Since $f_2^{(\overline{A},A)} \in \lambda$ we obtain that $2\chi_A \notin \mathcal{M}_a(\lambda)$, hence $\chi_A \notin \mathcal{M}_a(\lambda)$. □

Theorem 4.6. *Assume CH. Suppose that $\mathcal{I}$ is a σ-ideal generated by G_δ sets containing all singletons. Let $f \in \mathbb{R}^{\mathbb{R}}$ be a function from $\mathcal{M}_a(\lambda')$. Then f has the following property:*

$$\forall_{P \in Perf} \exists_{P \supseteq P_1 \in Perf} \exists_{E \in \text{ co } -\mathcal{I}} f(P_1) + E \in \mathcal{MGR}.$$

PROOF. By way of contradiction, assume that there exists a perfect set P such that for every perfect $P_1 \subseteq P$ and for every $E \in$ co $-\mathcal{I}$ we have $f(P_1) + E \notin \mathcal{MGR}$. Let Q_P be any countable, dense subset of P. Let $\langle G_\theta \colon \theta \in \omega_1^{Od} \rangle$ be an enumeration of all G_δ sets containing $Q_P \times \mathbb{Q}$. Let $\langle C_\theta \colon \theta \in \omega_1^{Ev} \rangle$ be an enumeration of all countable subsets of $\mathbb{R}$. We will use the following result:

Fact 4.7 ([2], Exercise 19.3)**.** *If $R \subseteq 2^\omega \times \mathbb{R}$ is a comeager subset, then there exist a perfect set Q and a dense G_δ set $G \subseteq \mathbb{R}$ such that $Q \times G \subseteq \mathbb{R}$.*

We will construct by induction on $\theta \in \omega_1$ a sequences $\{\langle x_\theta, y_\theta \rangle \colon \theta \in \omega_1\}$ and $\{H_\theta \colon \theta \in \omega_1\}$ such that H_θ are G_δ sets from $\mathcal{I}$. Assume that $\langle x_\mu, y_\mu \rangle$ and H_μ have been chosen for $\mu < \theta$. Let us consider two cases.

Case 1. $\theta \in \omega_1^{Ev}$

Choose $x_\theta \in P \setminus \left[\{x_\mu \colon \mu < \theta\} \cup Q_P\right]$. There are two possible cases. If $C_\theta \setminus \bigcup_{\mu<\theta} H_\mu \neq \emptyset$, then we pick any $y_\theta \in C_\theta \setminus \bigcup_{\mu<\theta} H_\mu$. In the other case choose an arbitrary $y_\theta \in \mathbb{R} \setminus \bigcup_{\mu<\theta} H_\mu$.

Case 2. $\theta \in \omega_1^{Od}$
Since $G_\theta \cap (P \times \mathbb{R})$ is a comeager set in $P \times \mathbb{R}$, by Fact 4.7 there exists a perfect set $Q_\theta \subseteq P$ and a comeager set K_θ such that $Q_\theta \times K_\theta \subseteq G_\theta$.
Without loss of generality we may assume that $Q_\theta \cap \left[Q_P \cup \{x_\mu : \mu < \theta\}\right] = \emptyset$.
By the assumption, $f(Q_\theta) + \left[\bigcup_{\mu<\theta} H_\mu\right]^c$ is not meager. Hence
$\left(f(Q_\theta) + \left[\bigcup_{\mu<\theta} H_\mu\right]^c\right) \cap K_\theta \neq \emptyset$. Choose $x_\theta \in Q_\theta$ and $y_\theta \in \mathbb{R} \setminus \bigcup_{\mu<\theta} H_\mu$ such that $f(x_\theta) + y_\theta \in K_\theta$.
In both those cases we define H_θ in the following way. Since
$\bigcup_{\mu<\theta} H_\mu \cup \{y_\theta\} \in \mathcal{I}$, so we can choose a G_δ set $H_\theta \in \mathcal{I}$ such that
$\bigcup_{\mu<\theta} H_\mu \cup \{y_\theta\} \subseteq H_\theta$. The construction is complete.
Let Y be defined by $Y = \{y_\theta : \theta \in \omega_1\}$. It is easy to see that such defined set Y is a λ'-set. Thus the set l^* defined by $l^* = \{\langle x_\theta, y_\theta\rangle : \theta \in \omega_1\}$ is a λ'-set, too.
Next, let l be any λ' extension of the function l^* onto $\mathbb{R}$. We have

$$f + l = \{\langle x, f(x) + l(x)\rangle : x \in \mathbb{R}\} \supseteq$$

$$\supseteq \{\langle x_\theta, f(x_\theta) + y_\theta\rangle : \theta \in \omega_1^{Od}\}.$$

For each $\theta \in \omega_1^{Od}$ we have: $f(x_\theta) + y_\theta \in K_\theta$, thus
$\langle x_\theta, f(x_\theta) + y_\theta\rangle \in Q_\theta \times K_\theta \subseteq G_\theta$. Therefore $\left[(f + l) \cap G_\theta\right] \setminus \left[Q_P \times Q\right] \neq \emptyset$.
This proves that $f + l \notin \lambda'$, which is a contradiction. This ends the proof of Theorem 4.6. □

Problem 4.8. *Characterize the classes*

$$\mathcal{M}_a(\lambda) \text{ and } \mathcal{M}_a(\lambda').$$

5 Minima and Maxima

It is obvious that for every two functions f_1, f_2 with a λ' graph we have $\min\{f_1, f_2\} \in \lambda'$. The next example shows that the analogous result does not hold for functions with a λ graph.

Theorem 5.1. *Assume CH. There exist two functions* $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ *such that* $g_1, g_2 \in \lambda$, *but* $\min\{g_1, g_2\} \notin \lambda$.

Proof. We will use the function $f_\gamma^{(\mathbb{R},\mathbb{Q})}$ from Example 3.2. Define $g_1 = f_{-2}^{(\mathbb{R},\mathbb{Q})} + 2$ and $g_2 = f_2^{(\mathbb{R},\mathbb{Q})}$. Note that

$$g_1(x) = \begin{cases} f^{(\mathbb{R},\mathbb{Q})}(x) + 2 & \text{if } x \in \mathbb{Q} \\ f^{(\mathbb{R},\mathbb{Q})}(x) & \text{if } x \notin \mathbb{Q} \end{cases}$$

and

$$g_2(x) = \begin{cases} f^{(\mathbb{R},\mathbb{Q})}(x) & \text{if } x \in \mathbb{Q} \\ f^{(\mathbb{R},\mathbb{Q})}(x) + 2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is easy to see that $\min\{g_1, g_2\}(x) = f^{(\mathbb{R},\mathbb{Q})}(x)$. Hence, $\min\{g_1, g_2\} \notin \lambda$. □

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