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A CHARACTERIZATION OF COMPACT PARTS OF L^p SPACES APPLICATION TO SOBOLEV EMBEDDINGS

Abstract

To characterize compact parts of spaces $L^p(\Omega)$, we introduce a concept of equi-integrability based on the approximation of elements of $L^p(\Omega)$ by simple functions. The resulting theorem will be used to develop a new methodology to prove and extend results about the compactness of Sobolev embeddings.

Introduction

This paper is divided into two parts. In sections 1 to 4, we develop a characterization of compact subsets of $L^p(\Omega)$. Results and proofs are simple but appear to be unknown until now. More precisely, for a metric locally compact space Ω, we define a notion of equi-integrability which allows us to state an Ascoli theorem for $L^p(\Omega)$. This approach is a continuation of some work in generalized Riemann theory of integration framework [4]. In the second part, we develop a methodology to retrieve and improve standard results about Sobolev embeddings and compact embeddings $W^{1p}(\Omega) \to L^q(\Omega)$. In the classical approach (Cf [1, 2, 3, 7] for instance), the Sobolev-Gagliardo-Nirenberg inequality is proved on $\mathbb{R}^N$ and is extended to some extension domains (i.e. with a bounded extension operator $W^{1p}(\Omega) \to W^{1p}(\mathbb{R}^N)$). This provides the continuity of Sobolev embeddings. Obtaining the Rellich-Kondrachov theorem requires the use of a theorem characterizing the compact parts of $L^p(\Omega)$ using the approximations of functions f by the translated functions

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$\tau_h(f)(x) = f(x+h)$. To prove those results, we must extend functions from $W^{1p}(\Omega)$ to $W^{1p}(\mathbb{R}^N)$: this is an "external" approach limited to extension domains. Our approach is "internal". Results on Sobolev spaces will arise from our characterization of compact part of $L^p(\Omega)$ and from the Meyers-Serrin theorem, which is available for every open set Ω. We always stay in Ω and, for this reason, we will be able to extend Sobolev and Rellich-Kondrachov results to more general domains than extension domains. Precisely, the compacity of the embedding $W^{1p} \to L^p(\Omega)$ is proved very easily in section 6 for bounded convex open subsets of $\mathbb{R}^N$. We need some additional estimation to prove the continuity of embedding $W^{1p} \to L^{p^*}(\Omega)$ with compacity for $W^{1p} \to L^{p^*}(\Omega)$, $1 \leqslant q < p^*$ (see section 8). Finally, we prove in section 10 that compact embedding $W^{1p} \to L^p(\Omega)$ can be achieved for a wide class of domains which overshoot the Lipschitz condition, the cone condition (see [1]) or the domain extention condition.

1 Definitions and Notations

In section 2 and 3 (Ω, d) denotes a relatively compact part of a metric locally compact space $\tilde{\Omega}$, $\mathfrak{M}$ is a σ-algebra of $\tilde{\Omega}$ including all borelian sets, and μ is a measure over $\mathfrak{M}$ satisfying the following conditions (Cf. [6]):

(i) $\mu(K) < +\infty$ for every compact subset $K \subset \Omega$.

(ii) If $E \in \mathfrak{M}$, then $\mu(E) = \inf\{\mu(V), E \subset V \text{ and } V \text{ open }\}$.

(iii) If E is open with a finite measure, then $\mu(E) = \sup\{\mu(K), K \subset E \text{ and } K \text{ compact }\}$.

(iv) If $E \in \mathfrak{M}$, $A \subset E$ and $\mu(E) = 0$, then $A \in \mathfrak{M}$.

Briefly, μ is a Radon measure. We also assume that $\mu(\Omega) > 0$.

In section 5, (Ω, d) denotes a metric locally compact space and μ is a Radon measure on Ω.

In particular, in sections 6, 7, 8, 9 and 10 Ω will be a bounded open subset of $\mathbb{R}^N$ and μ will be the usual Lebesgue measure on $\mathbb{R}^N$. We denote $|\ |$ as the euclidian norm of $\mathbb{R}^N$.

We write $diam(E)$ for the diameter of $E \subset \Omega$ and E^c for the complement of E in Ω.

For a normed vector space $(X, \|\ \|)$, we denote by $B_X(y, \alpha)$ the closed ball of center y and radius α.

For $p \in [1, +\infty[$ and every measurable function $f : \Omega \to \mathbf{C}$, we set $\|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p}$. We denote $\mathcal{L}^p(\Omega)$ as the set of functions satisfying $\|f\|_p < \infty$.

As usual, $L^p(\Omega)$ is the quotient of $\mathcal{L}^p(\Omega)$ modulo the negligibility relation and $\| \ \|_p$ is the usual norm of $L^p(\Omega)$.

A subdivision of Ω is a partition $S = (E_i)_{1 \leqslant i \leqslant q}$ of Ω with measurable parts satisfying $\mu(E_i) > 0$ for all $1 \leqslant i \leqslant n$ and we set $\tau(S) = \mathrm{Max}_{1 \leqslant i \leqslant q} \operatorname{diam}(E_i)$.

A simple function $f : \Omega \to \mathbf{C}$ is a (finite) linear combination of characteristic functions of measurable sets. We say that a subdivision $S = (E_i)_{1 \leqslant i \leqslant q}$ of Ω and a simple function f are adapted to each other if f is constant over E_i, for all $1 \leqslant i \leqslant q$. We denote $E(\Omega)$ as the classes of simple functions modulo the negligibility relation. We say that a subdivision $S = (E_i)_{1 \leqslant i \leqslant q}$ of Ω and $F \in E(\Omega)$ are adapted to each other if there is a simple function $f \in F$ adapted to S.

Let $f : \Omega \to \mathbf{C}$ be an integrable function and a subdivision $S = (E_i)_{1 \leqslant i \leqslant q}$ of Ω. We denote $T(f, S)$ the simple function such that

$$\forall i \in \{1, \dots, q\},\ \forall t \in E_i, \quad T(f, S)(t) = \frac{1}{\mu(E_i)} \int_{E_i} f.$$

For every $F \in L^p(\Omega)$, $f \in F$ and S, a subdivision of Ω, we still denote by $T(F, S)$ the class of $T(f, S)$.

2 A Theorem on Approximation by Simple Functions

In this section, $1 \leqslant p < +\infty$ and Ω is bounded. Let us recall a usual approximation theorem (Cf. [6] for instance).

Theorem 2.1. *Let $f \in \mathcal{L}^p(\Omega)$. For every $\varepsilon > 0$, there exists $g \in C_c(\Omega)$ such that $\|f - g\|_p \leqslant \varepsilon$.*

Lemma 2.1. *Let $(f, g) \in \mathcal{L}^p(\Omega)^2$ and S be a subdivision of Ω, we have*

$$\|T(f, S) - T(g, S)\|_p \leqslant \|f - g\|_p.$$

PROOF. Indeed, if $S = (E_i)_{1 \leqslant i \leqslant q}$, we have

$$\begin{aligned}\int_\Omega |T(g, S) - T(f, S)|^p &\leqslant \sum_{i=1}^{q} \mu(E_i) \left[\frac{1}{\mu(E_i)} \int_{E_i} |g - f| \right]^p \\ &\leqslant \sum_{i=1}^{q} \mu(E_i) \frac{1}{\mu(E_i)} \int_{E_i} |g - f|^p \leqslant \int_\Omega |g - f|^p.\end{aligned}$$

□

We can state an approximation theorem of integrable functions by simple functions.

Theorem 2.2. *Let* $(f_k)_{0\leqslant k\leqslant n}$ *be a finite family of* $\mathcal{L}^p(\Omega)$*. For every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that for every subdivision* S *of* Ω *satisfying* $\tau(S) < \eta$ *we have*

$$\forall k \in \{0,\dots,n\},\quad \|f_k - T(f_k,S)\|_p \leqslant \varepsilon.$$

PROOF. We first extend the functions to $\overline{\Omega}$ (by 0 for instance). For $\varepsilon > 0$ and $k \in \{0,\dots,n\}$, there exists $g_k \in C(\Omega)$ such that $\|f_k - g_k\|_p \leqslant \dfrac{\varepsilon}{3}$.

There exists $\eta > 0$ such that

$$\forall k \in \{0,\dots,n\},\quad \forall (u,v)\in\Omega^2,\quad d(u,v)\leqslant \eta \Rightarrow |g_k(u)-g_k(v)| \leqslant \frac{\varepsilon}{3\mu(\Omega)^{1/p}}.$$

Let $S = (E_i)_{1\leqslant i\leqslant q}$ be a subdivision of Ω such as $\tau(S) \leqslant \eta$ (note that Ω is bounded). For every $0 \leqslant k \leqslant n$,

$$\begin{aligned}\int_\Omega |g_k - T(g_k,S)|^p &= \sum_{i=1}^{q}\int_{E_i}\left|g_k(u) - \frac{1}{\mu(E_i)}\int_{E_i} g_k(v)dv\right|^p du\\ &\leqslant \sum_{i=1}^{q}\int_{E_i}\left[\frac{1}{\mu(E_i)}\int_{E_i}|g_k(u)-g_k(v)|dv\right]^p du\\ &\leqslant \frac{\varepsilon^p}{3^p}.\end{aligned}$$

From Lemma 2.1, we also have $\|T(g_k,S) - T(f_k,S)\|_p \leqslant \dfrac{\varepsilon}{3}$ and we have,

$$\|f_k - T(f_k,S)\|_p \leqslant \|f_k - g_k\|_p + \|g_k - T(g_k,S)\|_p + \|T(g_k,S) - T(f_k,S)\|_p \leqslant \varepsilon.$$

□

3 A Characterization of Compact Sets in $L^p(\Omega)$

In this section, $1 \leqslant p < +\infty$ and Ω is bounded.

Definition 3.1. Let Γ be a subset of $L^p(\Omega)$.We say that Γ is *uniformly equi-p-integrable* if one of those equivalent properties is satisfied.

(a) For every $\varepsilon > 0$ there exists $\eta > 0$ such that for every subdivision S of Ω satisfying $\tau(S) \leqslant \eta$ we have

$$\forall F \in \Gamma,\quad \|F - T(F,S)\|_p \leqslant \varepsilon.$$

(b) For every $\varepsilon > 0$ there exists $\eta > 0$ such that for every subdivision S of Ω satisfying $\tau(S) \leqslant \eta$, and for every $F \in \Gamma$, we can find $F_{s\varepsilon} \in E(\Omega)$ adapted to S such that $\|F - F_{s\varepsilon}\|_p \leqslant \varepsilon$.

Definition 3.2. Let Γ be a subset of $L^p(\Omega)$. The set Γ is equi-p-integrable if and only if one of those equivalent properties is satisfied.

(a) For every $\varepsilon > 0$ there exists a subdivision S of Ω such that

$$\forall F \in \Gamma, \quad \|F - T(F,S)\|_p \leqslant \varepsilon.$$

(b) For every $\varepsilon > 0$ there exists a subdivision S of Ω such that for all $F \in \Gamma$ we can find $F_{s\varepsilon} \in E(\Omega)$ adapted to S and such that $\|F - F_{s\varepsilon}\|_p \leqslant \varepsilon$.

Theorem 3.1. *In the above definitions the pairs of properties are equivalent*

PROOF. The equivalence of those two pairs of properties is easy to show. For the equi-p-integrability, one implication is obvious (we choose $F_{s\varepsilon} = T(F,S)$).

Conversely, we suppose that for every $\varepsilon > 0$ there exists a subdivision S of Ω such that for all $F \in \Gamma$ we can find $F_{s\varepsilon} \in E(\Omega)$ adapted to S and such that $\|F - F_{s\varepsilon}\|_p < \varepsilon$. From lemma 2.1,

$$\|F - T(F,S)\|_p \leqslant \|F - F_{s\varepsilon}\|_p + \|T(F_{s\varepsilon}, S) - T(F,S)\|_p \leqslant 2\|F - F_{s\varepsilon}\|_p \leqslant 2\varepsilon$$

because $F_{s\varepsilon} = T(F_{s\varepsilon}, S)$, and the result follows. The proof for the uniform equi-p-integrability is similar. □

Remark 3.1. *Equi-p-integrability and uniform equi-p-integrability are different notions.*

For instance, if we define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } 0 \leqslant x \leqslant \dfrac{1}{2}, \\ 1 & \text{if } \dfrac{1}{2} < x \leqslant 1, \end{cases}$$

the set of functions $\Gamma = \{\lambda f, \ \lambda \in \mathbb{R}\} \subset L^p(\Omega)$ is obviously equi-p-integrable - consider the subdivision $(0, \frac{1}{2}, 1)$, but Γ is not uniformly equi-p-integrable by considering the subdivisions $\left(\dfrac{k}{2p+1}\right)_{0 \leqslant k \leqslant 2p+1}$, $p \in \mathbb{N}$. Nevertheless, as a consequence of theorem 3.1, those concepts are equivalent for a bounded subset of $\mathcal{L}^p(\Omega)$.

Lemma 3.1. *Let (F_n) be a sequence of $E(\Omega)$ adapted to a same subdivision of Ω. If $(\|F_n\|_p)$ is bounded, we can extract a subsequence $(F_{\varphi(n)})$ converging in $(E(\Omega), \| \ \|_p)$.*

PROOF. Let $S = (E_i)_{1\leqslant i\leqslant q}$ be a subdivision adapted to (F_n). The subspace of simple functions adapted to S is of finite dimension and the result follows. □

We are now able to state a theorem characterizing the compacts subsets of $L^p(\Omega)$.

Theorem 3.2. *Let Γ be a subset of $L^p(\Omega)$. The following assertions are equivalent:*

(i) Γ *is relatively compact ;*

(ii) Γ *is bounded and uniformly equi-p-integrable ;*

(iii) Γ *is bounded and equi-p-integrable.*

PROOF. We consider three implications.

• *(i)* $\Rightarrow$ *(ii):*
A relatively compact subset Γ of $L^p(\Omega)$ is bounded. We have to show that Γ is uniformly equi-p-integrable. For $\varepsilon > 0$, there exists $G_0, \dots, G_n$ in $L^p(\Omega)$ such that $\Gamma \subset \bigcup_{k=0}^{n} B\Big(G_k, \frac{\varepsilon}{3}\Big)$. From Theorem 2.2, there exists $\eta > 0$ such that

$$\forall k \in \{0, \dots, n\}, \quad \|G_k - T(G_k, S)\|_p \leqslant \frac{\varepsilon}{3}$$

for every subdivision $S = (E_i)_{1\leqslant i\leqslant q}$ of Ω satisfying $\tau(S) \leqslant \eta$. For $F \in \Gamma$, there exists $k \in \{0, \dots, n\}$ such that $\|F - G_k\|_p \leqslant \frac{\varepsilon}{3}$. Now, from Lemma 2.1, we have $\|T(G_k, S) - T(F, S)\|_p \leqslant \|G_k - F\|_p$ and we deduce

$$\|F - T(F, S)\|_p \leqslant \|F - G_k\|_p + \|G_k - T(G_k, S)\|_p + \|T(G_k, S) - T(F, S)\|_p \leqslant \varepsilon.$$

• *(ii)* $\Rightarrow$ *(iii):*
For $\varepsilon > 0$, we choose $\eta > 0$ from the equi-p-integrability hypothesis. From the compacity of $\overline{\Omega}$, there exists a subdivision S of Ω such that $\tau(S) \leqslant \eta$, and the result is proved.

• *(iii)* $\Rightarrow$ *(i):*
Let Γ be an equi-p-integrable bounded part of $L^p(\Omega)$ and $M = \mathrm{Sup}_{F\in\Gamma} \|F\|_p$. Let (F_n) be a sequence of Γ. For every $q \in \mathbb{N}$, there exists a subdivision S^q

such that for every $n \in \mathbb{N}$ we have $\|F_n - F_n^q\|_p \leqslant 2^{-q}$. Just as in Ascoli's theorem, the end of the proof is an application of the Cantor diagonal process. From lemma 3.1 there exists a strictly increasing application $\varphi_0 : \mathbb{N} \to \mathbb{N}$ such that

$$\forall (n,m) \in \mathbb{N}^2, \quad \left\| F^0_{\varphi_0(n)} - F^0_{\varphi_0(m)} \right\|_p \leqslant 1.$$

Then, for every integer q, we build a strictly increasing application $\varphi_q : \mathbb{N} \to \mathbb{N}$ such that

$$\forall (n,m) \in \mathbb{N}^2, \quad \left\| F^q_{\varphi_q(n)} - F^q_{\varphi_q(m)} \right\|_p \leqslant 2^{-q},$$

where indices $(\varphi_q(n))_{n\in\mathbb{N}}$ are selected from the previously selected indices $(\varphi_{q-1}(n))_{n\in\mathbb{N}}$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be the strictly increasing application defined by $\varphi(r) = \varphi_r(r)$. For $r < s$, we have

$$\begin{aligned} \left\| F_{\varphi(r)} - F_{\varphi(s)} \right\|_p \leqslant & \left\| F_{\varphi(r)} - F^r_{\varphi(r)} \right\|_p + \left\| F^r_{\varphi(r)} - F^r_{\varphi(s)} \right\|_p + \left\| F^r_{\varphi(s)} - F_{\varphi(s)} \right\|_p \\ \leqslant & 3 \cdot 2^{-r}. \end{aligned}$$

The sequence $(F_{\varphi(r)})_{r\in\mathbb{N}}$ satisfies the Cauchy property and the compacity of $\overline{\Gamma}$ follows from the completeness of $L^p(\Omega)$. □

We emphasize the simplicity of the above equivalences. Statement and proof are analogous to Ascoli's theorem, with the definition "à la Riemann" for the equi-p-integrability of classes of functions.

In the following, the property of equi-p-integrability will be used to establish the compacity of some parts of $L^p(\Omega)$. In fact, the characterization using $T(f, S)$ gives a precise direction to follow in order to verify the compacity of a given subset of $L^p(\Omega)$.

4 The Case $p = +\infty$

In this part,unless otherwise stated, Ω is a metric locally compact space. We are going to study how to modify the previous results in the special case $p = +\infty$. In this context, we extend the definition of subdivision and simple functions to metric locally compact spaces.

Theorem 4.1. *Let $(f_k)_{0\leqslant k\leqslant n}$ be a finite family of $\mathcal{L}^\infty(\Omega)$. For every $\varepsilon > 0$, there exists a subdivision S of Ω and simple functions $(g_k)_{0\leqslant k\leqslant n}$ adapted to S such that $\|f_k - g_k\|_\infty \leqslant \varepsilon$. If $\mu(\Omega)$ is finite, for every $\varepsilon > 0$, there exists a subdivision S of Ω such that $\|f_k - T(f_k, S)\|_\infty \leqslant \varepsilon$ for all $0 \leqslant k \leqslant n$.*

PROOF. Let $f \in L^\infty(\Omega)$. For $\varepsilon > 0$, we denote by r the entire part of $2\|f\|_\infty/\varepsilon$. We have $f = \Re(f) + i\Im(f)$, and for every $(k,l) \in \{-r-1,\dots,r\}^2$, we set

$$E_{kl} = \left\{x \in \Omega \;/\; \frac{k\varepsilon}{2} \leqslant \Re(f)(x) < \frac{(k+1)\varepsilon}{2} \text{ and } \frac{l\varepsilon}{2} \leqslant \Im(f)(x) < \frac{(l+1)\varepsilon}{2}\right\}.$$

Let Δ be the subset of indexes such that $\mu(E_{kl}) > 0$. We choose $(k_0, l_0) \in \Delta$ and we add to $E_{k_0 l_0}$ the elements of the negligible set $\Omega - \cup_{(k,l)\in\Delta} E_{kl}$. The resulting family $S = (E_{kl})_{(k,l)\in\Delta}$ is a subdivision of Ω and the function $g = \displaystyle\sum_{(k,l)\in\Delta}\left(\frac{k\varepsilon}{2} + i\frac{l\varepsilon}{2}\right)\chi_{E_{kl}}$ satisfies to $\|f-g\|_\infty \leqslant \varepsilon$. Now, for a finite family $(f_k)_{0\leqslant k\leqslant n}$, we can build such subdivisions $(S_k)_{0\leqslant k\leqslant n}$. The subdivision S obtained by taking the intersection of all elements of those subdivisions answer to the question. If $\mu(\Omega) < +\infty$, for the previous subdivision S, $T(f,S)$ is defined for every $f \in L^\infty(\Omega)$ and clearly verifies $\|f_k - T(f_k,S)\|_\infty \leqslant \varepsilon$. □

Definition 4.1. Let Γ be a subset of $L^\infty(\Omega)$. We say that Γ is *equi-∞-integrable* if for every $\varepsilon > 0$ there exists a subdivision S of Ω such that for all $F \in \Gamma$ we can find a simple function $F_{s\varepsilon}$ adapted to S and such that $\|F - F_{s\varepsilon}\|_\infty \leqslant \varepsilon$.

Theorem 4.2. *When $\mu(\Omega)$ is finite, Γ is equi-∞-integrable if and only if for every $\varepsilon > 0$ there exists a subdivision S of Ω such that for all $F \in \Gamma$, $\|F - T(F,S)\|_\infty \leqslant \varepsilon$.*

PROOF. We suppose $\mu(\Omega) < +\infty$. If Γ is equi-∞-integrable, let $S = (E_i)_{0\leqslant i\leqslant n}$ be a subdivision of Ω such that for every $F \in \Gamma$ there exists a simple function $F_{s\varepsilon} = \displaystyle\sum_{0\leqslant i\leqslant n} \alpha_i(F)\chi_{E_i}$ satisfying $\|F - F_{s\varepsilon}\|_\infty \leqslant \varepsilon/2$. For $i \in \{0,\dots,n\}$ and for almost all $x \in E_i$, we have $|F(x) - \alpha_i(F)| \leqslant \frac{\varepsilon}{2}$. Thus $|T(F,S)(x) - \alpha_i(F)| \leqslant \frac{\varepsilon}{2}$ and $\|F - T(F,S)\|_\infty \leqslant \varepsilon$. The converse implication is straightforward. □

Theorem 4.3. *Let Γ be a subset of $L^\infty(\Omega)$. The following assertions are equivalent:*

(i) Γ *is relatively compact ;*

(ii) Γ *is bounded and equi-∞-integrable.*

PROOF. $(i) \Rightarrow (ii)$:

We have to show that Γ is uniformly equi-∞-integrable. For $\varepsilon > 0$, there exist $G_1,\dots,G_n$ in $L^p(\Omega)$ such that $\Gamma \subset \displaystyle\bigcup_{k=0}^{n} B\Big(G_k, \frac{\varepsilon}{2}\Big)$. From the Theorem

4.1, there exists a subdivision S of Ω and simple functions $H_0, \dots, H_n$ adapted to S such that

$$\forall k \in \{0, \dots, n\}, \quad \|G_k - H_k\|_\infty \leqslant \frac{\varepsilon}{2}.$$

For $F \in \Gamma$, there exists $k \in \{0, \dots, n\}$ such that $\|F - G_k\|_\infty \leqslant \dfrac{\varepsilon}{2}$ and we find

$$\|F - H_k\|_\infty \leqslant \|F - G_k\|_p + \|G_k - H_k\|_p \leqslant \varepsilon.$$

(ii) $\Rightarrow$ *(i):*

Let Γ be an equi-∞-integrable bounded part of $L^\infty(\Omega)$ and define $M = \operatorname{Sup}_{F\in\Gamma} \|F\|_\infty$. Let (F_n) be a sequence of Γ. For every $q \in \mathbb{N}$, there exists a subdivision S^q such that, for every $n \in \mathbb{N}$, $\|F_n - F_n^q\|_\infty \leqslant 2^{-q}$. Now, just like for lemma 3.1, for every bounded sequence in $L^\infty(\Omega)$ of simple functions adapted to a fixed subdivision of Ω, we can extract a subsequence converging in $L^\infty(\Omega)$. The end of the proof is similar to the one of Theorem 3.1. □

To conclude this section, let us recall the usual characterization of the compact parts of $L^p(\Omega)$ ([1] p. 31 or [2] p. 72). Let Ω be an open subset of $\mathbb{R}^N$ and $1 \leqslant p < +\infty$. For every $f \in L^p(\Omega)$, we define an extension $\tilde{f}$ of f

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N - \Omega. \end{cases}$$

Theorem 4.4. *(Fréchet-Kolmogorov Theorem) A bounded part Γ of $L^p(\Omega)$ is relatively compact if and only if we can find, for every $\varepsilon > 0$, a real $\delta > 0$ and a compact part ω of Ω such that $\forall f \in \Gamma$,*

$$\forall h \in \mathbb{R}^N \text{ with } |h| < \delta, \quad \int_\omega \left|\tilde{f}(u+h) - \tilde{f}(u)\right|^p du \leqslant \varepsilon^p,$$

and

$$\int_{\Omega-\omega} |f(u)|^p \, du \leqslant \varepsilon^p.$$

Remark 4.1. *This theorem provides a direct characterization of bounded equi-p-integrable parts of $L^p(\Omega)$, when Ω is an open subset of $\mathbb{R}^N$.*

Remark 4.2. *The Fréchet-Kolmogorov theorem uses the additive structure of $\mathbb{R}^N$ which is not required in our approach.*

5 The Embedding $W^{1p}(\Omega) \to L^p(\Omega)$ is Compact for every Convex Bounded Subset of $\mathbb{R}^N$

Let Ω be an open subset of $\mathbb{R}^N$ and μ the Lebesgue measure on $\mathbb{R}^N$. For every $p \in [1, +\infty]$, let $W^{1p}(\Omega)$ be the usual Sobolev Spaces normed by

$$\forall f \in W^{1p}(\Omega), \quad \|f\|_{W^{1p}} = \|f\|_p + \||\nabla f|\|_p$$

with

$$\nabla f = (\partial_1 f, \ldots, \partial_N f) \quad \text{and} \quad |\nabla f| = \left(\sum_{i=1}^{N} |\partial_i f|^2 \right)^{1/2}.$$

We recall a well-known density theorem (Cf. [1] or [7]).

Theorem 5.1. *(Meyers-Serrin Theorem). For every open subset Ω of $\mathbb{R}^N$ and every $1 \leqslant p < +\infty$, $C^\infty(\Omega) \cap W^{1p}(\Omega)$ is a dense subset of $W^{1p}(\Omega)$.*

Lemma 5.1. *(Poincaré-Wirtinger Theorem). Let E be a bounded convex part of $\mathbb{R}^N$ and $1 \leqslant p < +\infty$. Then, there exists $\lambda_N \in \mathbb{R}_+^*$, such that for all $f \in W^{1p}(E)$,*

$$\int_{v\in E} \left| f(v) - \frac{1}{\mu(E)} \int_{u\in E} f(u)\, du \right|^p dv \leqslant \lambda_N \mathrm{diam}(E)^p \int_{u\in E} |\nabla f(u)|^p\, du$$

with $\lambda_1 = 2\ln(2)$ and $\lambda_N = \dfrac{2^N - 2}{N-1}$ for $N \geqslant 2$.

PROOF. Using Meyers-Serrin's theorem, we have only to prove the result for $f \in C^\infty(\Omega) \cap W^{1p}(\Omega)$. Let $D = \mu(E)^{p-1} \operatorname{diam}(E)^p$

$$\int_{v\in E} \left| \int_{u\in E} (f(v) - f(u))\, du \right|^p dv \leqslant \mu(E)^{p-1} \int_{v\in E} \int_{u\in E} |f(u) - f(v)|^p\, du\, dv$$

$$\leqslant \mu(E)^{p-1} \int_{v\in E} \int_{u\in E} \int_{t\in[0,1]} |\nabla f(u + t(v-u))|^p |v-u|^p\, dt\, du\, dv$$

$$\leqslant D \int_{v\in E} \int_{u\in E} \int_{t\in[1/2,1]} (|\nabla f(u + t(v-u))|^p + |\nabla f(v + t(u-v))|^p)\, dt\, du\, dv$$

$$\leqslant 2D \int_{u\in E} \int_{t\in[\frac{1}{2},1]} t^{-N} \int_{h\in u+t(-u+E)} |\nabla f(h)|^p\, dt dh\, du$$

$$\leqslant 2D \int_{u\in E} \int_{t\in[\frac{1}{2},1]} t^{-N} \int_{h\in E} |\nabla f(h)|^p\, dt dh\, du$$

$$\leqslant \lambda_N D \int_{h\in E} |\nabla f(h)|^p dh. \qquad \square$$

Theorem 5.2. *(Rellich-Kondrachov Theorem) Let Ω be a bounded convex open subset of $\mathbb{R}^N$. For every $p \in [1, +\infty]$, the canonical embedding of $W^{1p}(\Omega)$ into $L^p(\Omega)$ is compact.*

Proof. • First Case: $1 \leqslant p < +\infty$.

We will show that the unit ball $B_{W^{1p}}(0,1)$ of $W^{1p}(\Omega)$ is a relatively compact subset of $L^p(\Omega)$. This is a bounded subset and we have to prove that $B_{W^{1p}}(0,1)$ is an equi-p-Integrable subset of $L^p(\Omega)$.

Let $S = (E_i)_{1 \leqslant i \leqslant q}$ a subdivision of Ω composed of convex parts (the intersection of Ω with a regular lattice, for instance). We deduce from lemma 5.1

$$\begin{aligned}
\int_\Omega |f - T(f,S)|^p &= \sum_{i=1}^n \int_{E_i} \left| f(v) - \frac{1}{\mu(E_i)} \int_{E_i} f(u)\, du \right|^p dv \\
&\leqslant \sum_{i=1}^n \frac{1}{\mu(E_i)^p} \int_{E_i} \left[\int_{E_i} |f(v) - f(u)|\, du \right]^p dv \\
&\leqslant \lambda_N \sum_{i=1}^n \operatorname{diam}(E_i)^p \int_{u \in E_i} |\nabla f(u)|^p\, du \\
&\leqslant \lambda_N \tau(S)^p \sum_{i=1}^n \int_{u \in E_i} |\nabla f(u)|^p\, du \\
&\leqslant \lambda_N \tau(S)^p \int_{u \in \Omega} |\nabla f(u)|^p\, du \\
&\leqslant \lambda_N \tau(S)^p.
\end{aligned}$$

At last, for every $\eta > 0$, there exists such a subdivision S of Ω satisfying $\tau(S) < \eta$. We apply the previous inequality to conclude with theorem 3.2.

• Second Case: $p = +\infty$.

Let $f \in W^{1\infty}(\Omega)$. For every $p \in [1, +\infty[$, $f \in W^{1p}(\Omega)$ and

$$\lim_{p \to +\infty} \|f\|_p = \|f\|_\infty \quad \text{and} \quad \lim_{p \to +\infty} \|f\|_{W^{1p}} = \|f\|_{W^{1\infty}}.$$

From the first case, we know that for every subdivision S of Ω composed of convex parts, we have

$$\|f - T(f,S)\|_p \leqslant \lambda_N^{1/p} \tau(S) \|f\|_{W^{1p}}$$

and we deduce

$$\forall f \in W^{1\infty}(\Omega), \quad \|f - T(f,S)\|_\infty \leqslant \tau(S) \|f\|_{W^{1\infty}}.$$

The unit ball $B_{W^{1\infty}}(0,1)$ of $W^{1\infty}(\Omega)$ is a bounded and equi-∞-Integrable by theorem 4.2: it is a compact subset of $L^\infty(\Omega)$. □

Extension of theorem 5.2. The aim of this section and those following is to show how to deal with equi-Integrability and theorems 3.1, and 4.2. The Rellich-Kondrachov theorem presented above can be extended classically in two directions: compact embeddings between $W^{mp}(\Omega)$ spaces and compact embedding $W^{1p}(\Omega) \to L^q(\Omega)$ for $q \in [1, p^*[$, where p^* is the Sobolev conjugate exponent of p. The first extension can be done, as is usual, by the iteration of $W^{1p}(\Omega) \to L^q(\Omega)$ compact embeddings. The second one is more difficult.

Usually, we must prove the Sobolev-Gagliardo-Nirenberg inequality with $\Omega = \mathbb{R}^N$, and this result is extended to every extension domain. To conclude, we can prove that convex bounded open subsets of $\mathbb{R}^N$ are extension domains, but this "external" proof is not very satisfying in our "internal" approach. In fact, we shall prove a Sobolev-Gagliardo-Nirenberg inequality for convex bounded open subsets Ω of $\mathbb{R}^N$. First, we give two results allowing one to extend compact embedding results.

6 Compact Embedding Theorems for Other Domains

From a "puzzle" point of view, the following theorem will be useful later.

Theorem 6.1. *Let Ω be an open subset of $\mathbb{R}^N$ such that there exists a family $(\Omega_i)_{0 \leqslant i \leqslant m}$ of subsets of Ω satisfying $\Omega = \bigcup_{i=0}^{m} \Omega_i$ and, for all $0 \leqslant i \leqslant m$, $\mu(\partial\Omega_i) = 0$.*

We assume that the embeddings $W^{1p}(\mathring{\Omega}_i) \to L^q(\mathring{\Omega}_i)$ are compact for some $(p,q) \in [1,+\infty]^2$. Then the embedding $W^{1p}(\Omega) \to L^q(\Omega)$ is compact.

PROOF. Let (f_n) be a sequence of $B_{W^{1p}(\Omega)}(0,1)$. From the hypothesis, we can extract a sequence (g_n) such that, for every $0 \leqslant i \leqslant m$, the restriction of g_n to $\mathring{\Omega}_i$ converges to a limit G_i in $L^q(\mathring{\Omega}_i)$. If $\mathring{\Omega}_i \cap \mathring{\Omega}_j \neq \emptyset$, the restriction of G_i and G_j define the same class of functions.

Thus, we can define a unique class G over $\bigcup_{i=0}^{m} \mathring{\Omega}_i$, which can be extended to Ω because $\mu\left(\Omega - \bigcup_{i=1}^{m} \mathring{\Omega}_i\right) \leqslant \sum_{i=0}^{m} \mu(\partial\Omega_i) = 0$. It is easy to verify the convergence of (g_n) toward G in $L^q(\Omega)$. □

We shall also need a result concerning a change of variable on Sobolev functions.

A one-to-one mapping $T : \Omega \to \Omega'$ is bi-Lipschitzian if T and T^{-1} are Lipschitzian maps. As an application of Rademacher's theorem, we find in [7] p. 52 the following result.

Lemma 6.1. *Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a bi-Lipschitzian mapping. If $f \in W^{1p}(\Omega)$, $p \in [1, +\infty]$, then $g = f \circ T \in W^{1p}(T^{-1}(\Omega))$, and $\nabla f(T(x)).dT_x = \nabla g(x)$ for a.e. $x \in \Omega$, where dT_x is the differential of T at point x.*

We deduce easily the following theorem.

Theorem 6.2. *Let $(p, q) \in [1, +\infty]^2$, $T : \mathbb{R}^N \to \mathbb{R}^N$ be a bi-Lipschitzian mapping, Ω_1 an open subset of $\mathbb{R}^N$ and $\Omega_2 = T(\Omega_1)$. If the canonical embedding $W^{1p}(\Omega_1) \to L^q(\Omega_1)$ is compact, then the embedding $W^{1p}(\Omega_2) \to L^q(\Omega_2)$ is compact.*

PROOF. The applications $\Phi_q : L^q(\Omega_1) \to L^q(\Omega_2)$ and $\Psi_p : W^{1p}(\Omega_2) \to W^{1p}(\Omega_1)$ defined by $\Phi_p(f) = f \circ T^{-1}$ and $\Psi_p(g) = g \circ T$ are linear and continuous. The result follows by chain rule since $W^{1p}(\Omega_1) \to L^q(\Omega_1)$ is compact. □

7 A Sobolev-Gagliardo-Nirenberg Inequality

Let us recall a well-known result (See [2] for instance).

Lemma 7.1. *Let $N \geqslant 2$ and $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For $x \in \mathbb{R}^N$ we set*

$$\tilde{x}_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-1}.$$

Then, the function $f(x) = f_1(\tilde{x}_1) \ldots f(\tilde{x}_N)$ is in $L^1(\mathbb{R}^N)$ and

$$\|f\|_{L^1(\mathbb{R}^N)} \leqslant \prod_{k=1}^{N} \|f_k\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Lemma 7.2. *Let Ω be a convex bounded open subset of $\mathbb{R}^N$. For every $f \in W^{1,1}(\Omega)$ we have*

$$\|f - T(f, \Omega)\|_{L^{\frac{N}{N-1}}(\Omega)} \leqslant N \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} \int_\Omega |\nabla f|$$

where $T(f, \Omega)$ is the average of f over Ω.

PROOF. The proof is a simple adaptation of the corresponding estimation in the usual Sobolev-Gagliardo-Nirenberg inequality.

We have to prove the result for $f \in C^\infty(\Omega)$. We first remark that we can find a box $E = I_1 \times \ldots \times I_N$, $\Omega \subset E$, where every I_k are non empty open intervals of $\mathbb{R}$ with $l(I_1) = \operatorname{diam}(\Omega)$ and $l(I_k) \leqslant \operatorname{diam}(\Omega)$ for $2 \leqslant k \leqslant N$ (we denote by $l(I)$ the length of I). We extend f *and* ∇f by null functions over $E - \Omega$ and we set $\tilde{I}_k = I_1 \times \ldots I_{k-1} \times I_{k+1} \times \ldots \times I_N$.

Thanks to the convexity, for $(u, v) \in \Omega^2$ we have

$$\begin{aligned} &|f(v_1, \ldots, v_N) - f(u_1, \ldots, u_N)| \\ &\leqslant \sum_{k=1}^{N} \int_{I_k} |\nabla f(u_1, \ldots, u_{k-1}, t_k, v_{k+1}, \ldots, v_N)| \, dt_k \end{aligned}$$

and we deduce for $v \in E$,

$$\begin{aligned} &|f(v) - T(f, \Omega)| \\ &\leqslant \frac{1}{\mu(\Omega)} \sum_{k=1}^{N} \prod_{i=k}^{N} l(I_i) \int_{I_1 \times \ldots \times I_k} |\nabla f(t_1, \ldots, t_k, v_{k+1}, \ldots, v_N)| \, dt_1 \ldots dt_k \\ &= f_1(\tilde{v}_1). \end{aligned}$$

By a permutation of indexes, we also have $|f(v) - T(f, \Omega)| \leqslant f_k(\tilde{v}_k)$ for every $1 \leqslant k \leqslant N$.

A simple computation gives

$$\|f_k\|_{L^1(\tilde{I}_k)} \leqslant N \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} \int_\Omega |\nabla f|.$$

Now, since $|f(v) - T(f, \Omega)|^N \leqslant \prod_{k=1}^{N} f_k(\tilde{v}_k)$, we deduce from Lemma 7.1 the inequality

$$\|f - T(f, \Omega)\|_{L^{\frac{N}{N-1}}(\Omega)} \leqslant \prod_{k=1}^{N} \|f_k\|_{L^1(\tilde{I}_k)}^{1/N} \leqslant N \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} \int_\Omega |\nabla f|.$$

□

We recall that the Sobolev conjugate of $p \in [1, N[$ is defined by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$.

Theorem 7.1. *Let $N \geqslant 2$ and Ω be a convex bounded open subset of $\mathbb{R}^N$. For every $p \in [1, N[$ we have $W^{1p}(\Omega) \subset L^{p^*}(\Omega)$ with continuous embedding and $\forall f \in W^{1p}(\Omega)$,*

$$\|f - T(f,\Omega)\|_{p^*} \leqslant \left((N-1)p^* \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} + \lambda_N^{1/p} \frac{\operatorname{diam}(\Omega)}{\mu(\Omega)^{1/N}} \right) \| |\nabla f| \|_p.$$

Proof. It is enough to prove this inequality for $f \in C^1(\Omega)$ that satisfy $T(f, \Omega) = 0$. For $t > 1$,

$$\begin{aligned}
\|f\|^t_{\frac{tN}{N-1}} &= \left\| f|f|^{t-1} \right\|_{\frac{N}{N-1}} \\
&\leqslant \left\| f|f|^{t-1} - T(f|f|^{t-1}, \Omega) \right\|_{\frac{N}{N-1}} + \left\| T(f|f|^{t-1}, \Omega) \right\|_{\frac{N}{N-1}} \\
&\leqslant tN \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} \left\| |f|^{t-1} |\nabla f| \right\|_1 + \frac{1}{\mu(\Omega)^{1/N}} \|f\|^t_t.
\end{aligned}$$

We have
$$\left\| |f|^{t-1} |\nabla f| \right\|_1 \leqslant \|f\|^{t-1}_{p'(t-1)} \| |\nabla f| \|_p$$
and, thanks to the Poincaré-Wirtinger inequality,

$$\|f\|^t_t \leqslant \|f\|_p \|f\|^{t-1}_{p'(t-1)} \leqslant \lambda_N^{1/p} \operatorname{diam}(\Omega) \|f\|^{t-1}_{p'(t-1)} \| |\nabla f| \|_p.$$

Choosing t such that $\dfrac{tN}{N-1} = p'(t-1)$ we have $p^* = \frac{tN}{N-1}$ and previous inequalities give

$$\|f\|_{p^*} \leqslant \left((N-1)p^* \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} + \lambda_N^{1/p} \frac{\operatorname{diam}(\Omega)}{\mu(\Omega)^{1/N}} \right) \| |\nabla f| \|_p.$$

Now, for every $f \in C^1(\Omega)$, we have

$$\begin{aligned}
\|f\|_{p^*} &\leqslant \left((N-1)p^* \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} + \lambda_N^{1/p} \frac{\operatorname{diam}(\Omega)}{\mu(\Omega)^{1/N}} \right) \| |\nabla f| \|_p + \|T(f,\Omega)\|_{p^*} \\
&\leqslant \left((N-1)p^* \frac{\operatorname{diam}(\Omega)^N}{\mu(\Omega)} + \lambda_N^{1/p} \frac{\operatorname{diam}(\Omega)}{\mu(\Omega)^{1/N}} \right) \| |\nabla f| \|_p + \mu(\Omega)^{1/p^* - 1/p} \|f\|_p
\end{aligned}$$

which proves the continuity of the embedding $W^{1p}(\Omega) \to L^{p^*}(\Omega)$. □

Now, we can extend theorem 5.2.

Theorem 7.2. *Let Ω be a bounded convex open subset of $\mathbb{R}^N$, $N \geqslant 2$. For every $p \in [1, +\infty]$, the canonical embedding of $W^{1p}(\Omega)$ into $L^q(\Omega)$ is compact for $1 \leqslant q < p^*$.*

PROOF. These embeddings are continuous. Without loss of generality, we can assume that $0 \in \Omega$. There exists $h \in C^0(S_{N-1}, \mathbb{R}_+^*)$ such that $\Omega = \{th(y)y,\ y \in S_{N-1},\ t \in [0, 1[\}$. The application h is Lipschitzian and the application defined by $\Phi_h(0) = 0$ and $\Phi_h(x) = h(\frac{x}{|x|})x$ for $x \in \mathbb{R}^N - \{0\}$ is bi-Lipschitzian (a proof is given in appendix B). Obviously, Φ_h sends the unit open ball B_N onto Ω.

The hypercube $C =]-1, 1[^N$ also satisfies this condition for an application Φ_{h_0}. Then $\Phi = \Phi_h \circ \Phi_{h_0}^{-1}$ is a bi-Lipschitzian application sending C onto Ω. Considering theorem 6.2., we have merely to prove the compacity of the embeddings $W^{1p}(C) \to L^q(C)$ for $1 \leqslant q < p^*$.

For every $n \in \mathbb{N}^*$, we consider a subdivision $S_n = (C_{in})_{i \in \Delta_n}$ of C by half-Open hypercubes with sides of size $1/n$. From theorem 7.1, there exists a constant $\alpha = \alpha(N, p)$ such that for every C_{in},

$$\forall f \in W^{1p}(C), \quad \int_{C_{in}} \left| f(u) - \frac{1}{\mu(C_{in})} \int_{C_{in}} f \right|^{p^*} \leqslant \alpha \left(\int_{C_{in}} |\nabla f|^p \right)^{p^*/p}.$$

We deduce, for all $f \in W^{1p}(C)$,

$$\begin{aligned} \int_C |f(u) - T(f, S_n)(u)|^{p^*} &\leqslant \alpha \sum_{i \in \Delta_n} \left(\int_{C_{in}} |\nabla f|^p \right)^{p^*/p} \\ &\leqslant \alpha \left(\int_C |\nabla f|^p \right)^{p^*/p} \end{aligned}$$

since $p \leqslant p^*$ and $\|f - T(f, S_n)\|_{p^*} \leqslant \alpha^{1/p^*} \||\nabla f|\|_p$.

The end of the proof is classical: for every $1 \leqslant q < p^*$, we can write

$$\frac{1}{q} = \frac{\eta}{1} + \frac{1-\eta}{p^*} \quad \text{with } 0 < \eta \leqslant 1.$$

Using Hölder's interpolation inequality, we find

$$\begin{aligned}\|f - T(f,S_n)\|_q &\leqslant \|f - T(f,S_n)\|_1^{\eta}\|f - T(f,S_n)\|_{p^*}^{1-\eta}\\ &\leqslant \alpha^{(1-\eta)/p^*}\||\nabla f|\|_p^{1-\eta}\|f - T(f,S_n)\|_1^{\eta}\\ &\leqslant \alpha^{(1-\eta)/p^*}\mu(C)^{\eta(p-1)/p}\||\nabla f|\|_p^{1-\eta}\|f - T(f,S_n)\|_p^{\eta}\\ &\leqslant \tau(S_n)^{\eta}\alpha^{(1-\eta)/p^*}\mu(C)^{\eta(p-1)/p}\lambda_N^{\eta/p}\||\nabla f|\|_p.\end{aligned}$$

thanks to the final estimation in the proof of theorem 5.2. Now, the conclusion follows from theorem 3.2. □

Remark 7.1. *In the special case $N = 2$, we need not to use Lipschitzian and bi-Lipschjitzian mappings. Indeed, the reader will easily see that for every $\varepsilon > 0$, there exists a subdivision S_ε of Ω such that* $\operatorname{diam}(E)^2 \leqslant 2\mu(E)$ *for every part of this subdivision, and $\tau(S_\varepsilon) \leqslant \varepsilon$. The end of the proof is straightforward.*

8 Extension domain and the Rellich-Kondrachov theorem

Let us recall that a bounded open subset $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain if each point on $\partial\Omega$ has a neighborhood U_x such that $\partial\Omega \cap U_x$ is the graph of a Lipschitz function. For a general definition of Lipschitz domains, see [1] or [3].

It is well known that embeddings $W^{1p}(\Omega) \to L^p(\Omega)$ are compact for Lipschitz domains because they are extension domains [3]. In this way, P. W. Jones characterized all finitely connected extension domains in the plane (for the Sobolev embedding) and proved that it is exactly the (ε, δ)-Domains [5]. Using this characterization, we will show that a very simple subset Ω of $\mathbb{R}^2$ which is not an extension domain can satisfy the conclusion of the Rellich-Kondrachov theorem.

An open subset Ω of $\mathbb{R}^N$ is an extension domain if for every $(k,p) \in \mathbb{N} \times [1,+\infty]$ there exists a bounded linear operator $\Lambda_{kp} : W^{kp}(\Omega) \to W^{kp}(\mathbb{R}^N)$ such that $\Lambda_{kp}(f)_{|\Omega} = f$ for all $f \in W^{kp}(\Omega)$.

An open subset Ω of $\mathbb{R}^N$ is an (ε,δ)-Domain if, $\forall (x,y) \in \Omega^2$ such that $|x-y| < \delta$, there exists a rectifiable arc $\gamma \subset \Omega$ joining x to y and satisfying

$$l(\gamma) \leqslant \frac{1}{\varepsilon}|x-y| \quad \text{and} \quad d(z,\Omega^c) \geqslant \frac{\varepsilon|x-z||y-z|}{|x-y|}, \quad \forall z \in \gamma,$$

where $l(\gamma)$ is the length of γ.

Jones proved the following theorem: Let $\Omega \subset \mathbb{R}^2$ be an open finitely connected set. Then Ω is an extension domain if and only if it is an (ε, δ)-Domain for some values of $\varepsilon, \delta > 0$.

Let us consider the open set of the plane $\Omega = \{(x, y),\ x \in]-1, 1[,\ 0 < y < 1 + \sqrt{|x|}\}$. The set Ω can be split in two convex subsets by the line $x = 0$. We can apply theorems 5.2 and 6.1 to deduce that the canonical embeddings $W^{1p}(\Omega) \to L^p(\Omega)$ are compact for $p \in [1, +\infty]$. Nevertheless, Ω is not an extension domain.

Indeed, for $n \in \mathbb{N}^*$, $n \geqslant 2$, we consider $X_n = \left(-\frac{1}{n}, 1 + \sqrt{\frac{1}{2n}}\right)$ and $Y_n = \left(\frac{1}{n}, 1 + \sqrt{\frac{1}{2n}}\right)$. We easily verify that every path $\gamma \subset \Omega$ joining X_n and Y_n is such that $l(\gamma) \geqslant 2\left(\frac{1}{n^2} + \frac{1}{2n}\right)^{\frac{1}{2}} \sim \sqrt{\frac{2}{n}}$.

But we have $|X_n - Y_n| = \frac{2}{n}$ and the (ε, δ) condition cannot be verified for any ε and δ in $\mathbb{R}_+^*$.

9 Compact embedding theorem for non Lipschitz domains

In this section, we will extend the Lipschitz boundary condition to provide more general domains satisfying the conclusion of the Rellich-Kondrachov theorem. The reader will easily verify that those domains are not (ε, δ)-Domains in general. In fact, if a boundary of an (ε, δ)-Domain must be rather smooth, we will show that the boundary of domains satisfying the conclusion of the Rellich-Kondrachov theorem can be wilder.

For every $p \in [1, +\infty]$, we denote by p' the conjugate exponent of p. For $r > 0$, we set $Q_r =]-r, r[^{N-1}$ and $\overline{Q_r} = [-r, r]^{N-1}$.

For every $h \in C^0(\overline{Q_r}, \mathbb{R}_+^*)$, we set

$$E(h) = \{(y_1, \ldots, y_N),\ (y_1, \ldots, y_{N-1}) \in Q_r,\ \text{and } 0 < y_N < h(y_1, \ldots, y_{N-1})\}$$

and

$$F(h) = \{(y, h(y)),\ y \in Q_r\}.$$

Definition 9.1. Let $q \in [1, \infty]$, Ω be an open subset of $\mathbb{R}^N$ and $x \in \partial\Omega$. We say that Ω has a q-Rectifiable boundary at point x if there exists $r > 0$, $h \in C^0(\overline{Q_r}, \mathbb{R}_+^*) \cap W^{1q}(Q_r)$, an affine rotation L and an open neighborhood U of x such that $L(F(h)) \subset U$, $\Omega \cap U = L(E(h))$ and

(a) For $q = +\infty$: There exists $\alpha > 0$ such that for all $1 \leqslant k \leqslant N-1$, and almost all $(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_{N-1}) \in]-r, r[^{N-2}$, the application $z \to h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})$ is in $W^{1\infty}(]-r, r[)$ with

$$\text{ess. sup}_{\mathbb{R}} \{|\nabla h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})|,\ z \in]-r, r[\} \leqslant \alpha.$$

(b) For $1 < q < +\infty$: There exists $\alpha > 0$ such that for all $1 \leqslant k \leqslant N-1$, and almost all $(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_{N-1}) \in]-r, r[^{N-2}$, the application $z \to h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})$ is in $W^{1q}(]-r, r[)$ with

$$\left[\int_{-r}^{r} |\partial_k h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})|^q dz\right]^{1/q} \leqslant \alpha$$

(c) For $q = 1$: $\forall \alpha > 0$, $\exists \eta > 0$ such that for every $1 \leqslant k \leqslant N-1$, and for almost all $(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_{N-1}) \in]-r, r[^{N-2}$, the application $z \to h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})$ is in $W^{1q}(]-r, r[)$ with

$$\int_{a}^{b} |\partial_k h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})| dz \leqslant \alpha$$

as soon as $-r < a \leqslant b < r$ with $b - a \leqslant \eta$.

Remark 9.1. *If $N = 2$, the conditions become $h \in C^0([-r, r], \mathbb{R}_+^*) \cap W^{1q}(]-r, r[)$.*

Remark 9.2. *In the general case, the hypothesis on functions h impose a condition on the length of paths drawn on the surface $F(h)$ staying on the parallels to the coordinate axes.*

Lemma 9.1. *Let $p \in [1, +\infty]$, $r > 0$ and $h \in W^{1p'}(Q_r)$ satisfying the hypothesis of definition 9.1 for $q = p'$. Then the canonical embedding $W^{1p}(E(h)) \to L^p(E(h))$ is compact.*

Proof. Without loss of generality, we can assume $r = 1$.

• First Case: $1 < p < +\infty$.

For every $0 < t < 1$ and $x \in \overline{Q}$, we set $\gamma(t, x) = (x, th(x))$. We denote $m = \min_{x \in \overline{Q}} h(x) > 0$ and $M = \max_{x \in \overline{Q}} h(x) > 0$.

From the hypothesis, there exists $\alpha > 0$ such that for all $0 \leqslant k \leqslant N-1$, almost every $y \in]-1, 1[^{N-2}$

$$\left[\int_{-1}^{1} \left(1 + |\partial_k h(y_1, \dots, y_{k-1}, z, y_{k+1}, \dots, y_{N-1})|^2\right)^{p'/2} dz\right]^{1/p'} \leqslant \alpha.$$

For every $n \in \mathbb{N}^*$, we denote $(C_\lambda^n)_{\lambda\in\Lambda_n}$ a subdivision of Q by half-Open hypercubes of size $1/n$. In the following, for the sake of simplicity, we denote $(C_i)_{0\leqslant i\leqslant r}$ for the partition $(C_\lambda^n)_{\lambda\in\Lambda_n}$.

We consider a sequence $0 = t_0 < t_1 < \ldots < t_q = 1$, and for $0 \leqslant i \leqslant r$ and $1 \leqslant j < q$, we set $\Omega_{ij} = \{\gamma(t,x),\ x \in C_i,\ t_j \leqslant t < t_{j+1}\}$ and $\Omega_{i0} = \{\gamma(t,x),\ x \in C_i,\ 0 < t < t_1\}$. Thus, $S_{n,t_0,\ldots,t_q} = (\Omega_{ij})_{\substack{0\leq i\leq r\\ 0\leq j<q}}$ is a subdivision of $\Omega = E(h)$.

Let us fix $0 \leqslant i \leqslant r$ and $0 \leqslant j < q$. For $g \in L^1(\Omega_{ij}) \cap C^1(\Omega_{ij})$, we have

$$\int_{\Omega_{ij}} g(u)\,du = \int_{x\in C_i}\int_{t_j}^{t_{j+1}} g(\gamma(t,x))h(x)\,dtdx.$$

For $f \in W^{1p}(\Omega) \cap C^1(\Omega)$, $0 \leqslant i \leqslant r$ and $0 \leqslant j < q$, we set

$$\Delta^{i,j} = \int_{\Omega_{ij}} \left| f(v) - \frac{1}{\mu(\Omega_{ij})}\int_{\Omega_{ij}} f(u)\,du \right|^p dv$$

$$\leqslant \frac{1}{\mu(\Omega_{ij})}\int_{t_j}^{t_{j+1}}\!\!\int_{C_i}\int_{t_j}^{t_{j+1}}\!\!\int_{C_i} |f(\gamma(t,x)) - f(\gamma(t',x'))|^p h(x)h(x')dx'\,dt'dx\,dt.$$

Now, we must join the points $\gamma(t,x)$ and $\gamma(t',x')$ with a path staying in Ω_{ij}.

We have

$$|f(\gamma(t,x)) - f(\gamma(t',x'))|$$
$$\leqslant |f(\gamma(t,x)) - f(\gamma(t',x))| + |f(\gamma(t',x)) - f(\gamma(t',x'))|$$

and

$$|f(\gamma(t,x)) - f(\gamma(t',x'))|^p$$
$$\leqslant 2^{p-1}(|f(\gamma(t,x)) - f(\gamma(t',x))|^p + |f(\gamma(t',x)) - f(\gamma(t',x'))|^p).$$

On one hand

$$|f(\gamma(t,x)) - f(\gamma(t',x))|^p \leqslant \left|\int_{a=t}^{t'} |\nabla f(\gamma(a,x))|h(x)da\right|^p$$
$$\leqslant (t_{j+1}-t_j)^{p-1}M^p\int_{a=t_j}^{t_{j+1}} |\nabla f(\gamma(a,x))|^p da.$$

Then,

$$\Delta_1^{i,j} = \frac{1}{\mu(\Omega_{ij})}\int_{t_j}^{t_{j+1}}\!\!\int_{C_i}\int_{t_j}^{t_{j+1}}\!\!\int_{C_i} |f(\gamma(t,x)) - f(\gamma(t',x))|^p h(x)h(x')dx'\,dt'dx\,dt$$

$$\leqslant (t_{j+1}-t_j)^p M^p \int_{x\in C_i}\int_{a=t_j}^{t_{j+1}} |\nabla f(\gamma(a,x)|^p h(x)dadx$$
$$\leqslant (t_{j+1}-t_j)^p M^p \int_{\Omega_{ij}} |\nabla f|^p$$

On the other hand, if we set $C_i = I_{i1}\times\ldots\times I_{iN-1}$, then for $1\leqslant k\leqslant N-1$,

$$\begin{aligned}&\big|f(\gamma(t',(x_1,\ldots,x_{k-1},x'_k,\ldots,x'_{N-1}))) - f(\gamma(t',(x_1,\ldots,x_k,x'_{k+1},\ldots,x'_{N-1})))\big|\\ &\qquad\leqslant\left[\int_{I_{ik}}\big|\nabla f(\gamma(t',(x_1,\ldots,x_{k-1},z,x'_{k+1},\ldots,x'_{N-1})))\big|^p dz\right]^{1/p}\\ &\qquad\quad\times\left[\int_{I_{ik}}\Big(1+\big|\partial_k h(x_1,\ldots,x_{k-1},z,x'_{k+1},\ldots,x'_{N-1})\big|^2\Big)^{p'/2}dz\right]^{1/p'}\\ &\qquad\leqslant\alpha\left[\int_{I_{ik}}\big|\nabla f(\gamma(t',(x_1,\ldots,x_{k-1},z,x'_{k+1},\ldots,x'_{N-1})))\big|^p dz\right]^{1/p}.\end{aligned}$$

We obtain

$$\begin{aligned}&|f(\gamma(t',x)) - f(\gamma(t',x'))|^p\\ &\leqslant\left[\sum_{k=1}^{N-1}\Big|f(\gamma(t',(x_1,\ldots,x_{k-1},x'_k,\ldots,x'_{N-1})))\right.\\ &\qquad\qquad\left. - f(\gamma(t',(x_1,\ldots,x_k,x'_{k+1},\ldots,x'_{N-1})))\Big|\right]^p\\ &\leqslant\alpha^p N^{p-1}\sum_{k=1}^{N-1}\int_{I_{ik}}\big|\nabla f(\gamma(t',(x_1,\ldots,x_{k-1},z,x'_{k+1},\ldots,x'_{N-1})))\big|^p dz\end{aligned}$$

and, if $\Delta f\gamma = f(\gamma(t',x)) - f(\gamma(t',x'))$, then

$$\begin{aligned}\Delta_2^{i,j} &= \frac{1}{\mu(\Omega_{ij})}\int_{t=t_j}^{t_{j+1}}\int_{x\in C_i}\int_{t'=t_j}^{t_{j+1}}\int_{x'\in C_i}\Big|\Delta f\gamma\Big|^p h(x)h(x')dx'\,dt'dx\,dt\\ &\leqslant\frac{\alpha^p M^2 N^{p-1}}{nm}\sum_{k=1}^{N-1}\int_{t'=t_j}^{t_{j+1}}\int_{C_i}\Big|\Theta(f,\gamma)\Big|^p dx_1\ldots dx_{k-1}dzdx'_{k+1}dx_{N-1}\,dt'\\ &\leqslant\frac{\alpha^p M^2 N^p}{nm^2}\int_{\Omega_{ij}}|\nabla f|^p.\end{aligned}$$

where $\Theta(f,\gamma) = \nabla f(\gamma(t',(x_1,\dots,x_{k-1},z,x'_{k+1},\dots,x'_{N-1})))$. Now, $\Delta^{i,j} \leqslant 2^{p-1}(\Delta_1^{i,j}+\Delta_2^{i,j})$, and

$$\left\|f - T(f, S_{n,t_0,\dots,t_q})\right\|_p^p = \sum_{\substack{0\le i\le r\\ 0\le j<q}} \Delta^{ij}$$
$$\leqslant 2^{p-1} \sum_{\substack{0\le i\le r\\ 0\le j<q}} \left[(t_{j+1}-t_j)^p M^p + \frac{\alpha^p M^2 N^{p+1}}{nm}\right] \int_{\Omega_{ij}} |\nabla f|^p.$$

We choose $(t_j)_{0\leqslant j\leqslant q}$ such that $\mathrm{Max}_{0\leqslant j<q}(t_{j+1}-t_j) \leqslant \dfrac{\varepsilon}{2M}$ and $n \geqslant 1$ such that $n \geqslant \dfrac{2^p\alpha^p M^2 N^p}{m^2\varepsilon^p}$. The subdivision $S_{n,t_0,\dots,t_q}$ satisfies

$$\|f - T(S,f)\|_p \leqslant \varepsilon\left(\int_\Omega |\nabla f(u)|^p\,du\right)^{1/p}$$

for every $f \in W^{1p}(\Omega)$ (with the usual density argument). Thus, the embedding $W^{1p}(\Omega) \to L^p(\Omega)$ is compact, and the proof is complete.

• SECOND CASE: $p = 1$.

The proof is very similar. Indeed,

$$\Delta_1^{i,j} \leqslant (t_{j+1}-t_j)M \int_{\Omega_{ij}} |\nabla f(u)|\,du, \quad \text{and}$$
$$\Delta_2^{i,j} \leqslant \frac{M^2 N(1+\|h\|_{W^{1\infty}})}{nm^2}\mu(\Omega_{ij}) \int_{\Omega_{ij}} |\nabla f|.$$

• THIRD CASE: $p = +\infty$.

Let $f \in W^{1\infty}(\Omega)\cap C^1(\Omega)$. For $(t,t') \in [t_j,t_{j+1}]^2$, $t \leqslant t'$ and $(x,x') \in C_i^2$, the estimations become

$$|f(\gamma(t,x) - f(\gamma(t',x))| \leqslant \int_{a=t}^{t'} |\nabla f(\gamma(a,x))| h(x) da \leqslant (t_{j+1}-t_j)M\|f\|_{W^{1\infty}}$$

and

$$|f(\gamma(t',x) - f\gamma(t',x'))|$$
$$\leqslant \|f\|_{W^{1\infty}} \sum_{k=1}^N \int_{I_{ik}} \left(1 + \left|\partial_k h(x_1,\dots,x_{k-1},z,x'_{k+1},\dots,x'_{N-1})\right|^2\right)^{1/2} dz.$$

Now, for $\varepsilon > 0$, we can choose $(t_j)_{0\leqslant j\leqslant q}$ satisfying $\mathrm{Max}_{0\leqslant j<q}(t_{j+1}-t_j) \leqslant \dfrac{\varepsilon}{2M}$ and $n \geqslant 1$ such that for every $1 \leqslant k \leqslant N-1$, and for almost all $(y_1,\ldots,y_{k-1},y_{k+1},\ldots,y_{N-1}) \in [-1,1]^{N-2}$

$$\int_a^b \left(1+\left|\partial_k h(x_1,\ldots,x_{k-1},z,x'_{k+1},\ldots,x'_{N-1})\right|^2\right)^{1/2} \leqslant \frac{\varepsilon}{2N}$$

as soon as $-1 < a \leqslant b < 1$ with $b-a \leqslant \frac{1}{n}$. And we have

$$f \in W^{1\infty}(\Omega)\cap C^1(\Omega), \quad \left\|f - T(f, S_{n,t_0,\ldots,t_q})\right\|_\infty \leqslant \varepsilon\|f\|_{W^{1\infty}}.$$

Using an usual application of regularization, for every $f \in W^{1\infty}(\Omega)$, there exists a sequence $(f_r)_{r\in\mathbb{N}}$ of $W^{1\infty}(\Omega)\cap C^1(\Omega)$ such that

$$\lim_{r\to+\infty}\|f_r - f\|_\infty = 0 \text{ and } \forall r\in\mathbb{N},\ \|f_r\|_{W^{1\infty}} \leqslant \|f\|_{W^{1\infty}}.$$

We obtain

$$f \in W^{1\infty}(\Omega), \quad \left\|f - T(f, S_{n,t_0,\ldots,t_q})\right\|_\infty \leqslant \varepsilon\|f\|_{W^{1\infty}}$$

and the result follows.

□

Definition 9.2. We say that an open set Ω has a q-Rectifiable boundary if there exists a finite family $(x_k)_{0\leqslant k\leqslant m}$ of $\partial\Omega$ such that Ω is q-Rectifiable at every point x_k with $\partial\Omega = \bigcup_{k=0}^{m} \overline{L_{x_k}(F(h_{x_k}))}$.

Theorem 9.1. *Let $p \in [1,+\infty]$. If Ω is a bounded open subset of $\mathbb{R}^N$ with a q-Rectifiable boundary, then the embedding $W^{1p}(\Omega) \to L^p(\Omega)$ is compact.*

PROOF. Let Ω be a bounded open subset of $\mathbb{R}^N$ with a q-Rectifiable boundary and a family $(x_i)_{0\leqslant i\leqslant m}$ of points of $\partial\Omega$ such that $\partial\Omega = \bigcup_{k=0}^{m} \overline{L_{x_k}(F(h_{x_k}))}$.

The reader may wish to convince himself that $\Omega' = \Omega - \bigcup_{i=0}^{m} \overline{L_{x_i}(E(h_{x_i}))}$ is a an open polytope of $\mathbb{R}^N$ which can be split (up to a neglideable set) in a finite partition of convex polytopes $(\Omega_k)_{0\leqslant k\leqslant r}$. A proof of this result can be found in appendix B. Then, from theorems 5.2 and 6.1, the embedding $W^{1p}(\Omega') \to L^p(\Omega')$ is compact.

We have a partition of Ω in $m+2$ parts. From lemma 9.1 and theorem 6.2, the embeddings $W^{1p}(L_{x_i}(E(h_{x_i}))) \to L^p(L_{x_i}(E(h_{x_i})))$ are compact, and since $\mu(\partial E(h_{x_i})) = 0$ for $0 \leqslant i \leqslant m$, we conclude from theorem 6.1 that the embedding $W^{1p}(\Omega) \to L^p(\Omega)$ is compact. □

We can give a nice formulation of this result in the case $N = 2$.

Corollary 9.1. *Let $p \in [1, +\infty[$. If Ω is a bounded open subset of $\mathbb{R}^2$ for whom the boundary is locally a graph of continuous applications in $W^{1p'}$, then the embedding $W^{1p}(\Omega) \to L^p(\Omega)$ is compact.*

10 APPENDIX A Extension to Locally Compact Metric Spaces

In this appendix, we give without proof the straightforward extension of compactness theorem when Ω is a general metric locally compact set and $1 \leqslant p < +\infty$.

To extend the result of section 3, we must specify notions of subdivision and simple function.

- A restricted subdivision of Ω is a couple (Ω_0, S), where $\Omega_0 \in \mathfrak{M}$ is relatively compact, and S is a subdivision of Ω_0.
- A function $f : \Omega \to \mathbf{C}$ is a restricted simple function if there is a restricted subdivision (Ω_0, S) of Ω such that the restriction of f to Ω_0 is simple, and f is the null function over Ω_0^c. In this case, we will say that f and (Ω_0, S) are adapted to each other. We denote by $E(\Omega)$ the classes of restricted simple functions modulo the negligibility relation. We will say that $F \in E(\Omega)$ and a restricted subdivision (Ω_0, S) are adapted to each other if there is $f \in F$ adapted to (Ω_0, S).
- Let $f : \Omega \to X$ be a measurable function and (Ω_0, S) a restricted subdivision such that f is integrable on Ω_0. We denote by $T(f, \Omega_0, S)$ the restricted simple function null outside of Ω_0 and such that the restriction to Ω_0 is $T(f, S)$. For every $F \in L^p(\Omega)$, $f \in F$ and (Ω_0, S) a restricted subdivision of Ω, we still denote by $T(F, \Omega_0, S)$ the class of $T(f, S)$.

Remark 10.1. *If Ω is relatively compact, for every subdivision S of Ω, (Ω, S) is a restricted subdivision of Ω. Moreover, simple functions and restricted simple functions are the same.*

Theorem 10.1. *Let Γ be a subset of $L^p(\Omega)$. We say that Γ is equi-p-Integrable if one of those equivalent properties is satisfied.*

- *For every $\varepsilon > 0$ there exists a restricted subdivision (Ω_0, S) of Ω such that*
$$\forall F \in \Gamma, \quad \|F - T(F, \Omega_0, S)\|_p \leqslant \varepsilon.$$
- *For every $\varepsilon > 0$ there exists a restricted subdivision (Ω_0, S) of Ω such that for all $F \in \Gamma$ we can find $F_{s\varepsilon} \in E(\Omega)$ adapted to (Ω_0, S) and such that $\|F - F_{s\varepsilon}\|_p \leqslant \varepsilon$.*

We say that Γ is weakly uniformly equi-p-Integrable if one of those equivalent properties is satisfied.

- *For every $\varepsilon > 0$ there exists $\eta > 0$ and $\Omega_0 \in \mathfrak{M}$ such that for every restricted subdivision (Ω_0, S) of Ω satisfying $\tau(S) \leqslant \eta$ we have*
$$\forall F \in \Gamma, \quad \|F - T(F, \Omega_0, S)\|_p \leqslant \varepsilon.$$
- *For every $\varepsilon > 0$ there exists $\eta > 0$ and $\Omega_0 \in \mathfrak{M}$ such that for every restricted subdivision (Ω_0, S) of Ω satisfying $\tau(S) \leqslant \eta$, and for every $F \in \Gamma$, we can find $F_{s\varepsilon} \in E(\Omega)$ adapted to (Ω_0, S) such that $\|F - F_{s\varepsilon}\|_p \leqslant \varepsilon$.*

Remark 10.2. *The equivalence of the two definitions of equi-p-Integrability is obvious for a relatively compact subset Ω, but uniform equi-p-Integrability and weak uniform equi-p-Integrability are not exactly the same notion.*

Clearly, uniform equi-p-Integrability implies weak uniform equi-p-Integrability but the converse is false. For instance, if $f : [0,1] \to \mathbb{R}$ with $f(x) = 0$ for $0 \leqslant x \leqslant \frac{1}{2}$ and $f(x) = 1$ for $\frac{1}{2} < x \leqslant 1$, the set of functions $\Gamma = \{\lambda f, \ \lambda \in \mathbb{R}\} \subset L^p(\Omega)$ is weakly uniformly equi-p-Integrable (we set $\Omega_0 = [\frac{1}{2}, 1] \ldots$), but it is not uniformly equi-p-Integrable.

Nevertheless, as a consequence of theorem 10.3, those concepts are equivalent for a bounded subset of $\mathcal{L}^p(\Omega)$.

The proofs of the following results are straightforward adaptations of previous theorems and will be omitted.

Theorem 10.2. *Let $(f_k)_{0 \leqslant k \leqslant n}$ be a finite family of $\mathcal{L}^p(\Omega)$. For every $\varepsilon > 0$, there exits a relatively compact part Ω_0 in $\mathfrak{M}$ and $\eta > 0$ such that for every restricted subdivision (Ω_0, S) of Ω satisfying $\tau(S) < \eta$ we have*
$$\forall k \in \{0, \ldots, n\}, \quad \|f_k - T(f_k, \Omega_0, S)\|_p \leqslant \varepsilon.$$

Lemma 10.1. *Let (F_n) be a sequence of $E(\Omega)$ adapted to a same restricted subdivision of Ω. If $(\|F_n\|_p)$ is bounded, we can extract a subsequence $(F_{\varphi(n)})$ converging in $(E(\Omega), \| \ \|_p)$.*

Theorem 10.3. *Let Γ be a subset of $L^p(\Omega)$. The following assertions are equivalent:*

(i) Γ *is relatively compact ;*

(ii) Γ *is bounded and weakly uniformly equi-p-Integrable ;*

(iii) Γ *is bounded and equi-p-Integrable.*

11 APPENDIX B Topological Results

Theorem 11.1. *Let Ω be an open subset of $\mathbb{R}^N$ with a q-Rectifiable boundary, and let a finite family $(x_k)_{0\leqslant k\leqslant m}$ of $\partial\Omega$ be such that Ω is q-Rectifiable at every point x_k with $\partial\Omega = \cup_{k=0}^m \overline{L_{x_k}(F(h_{x_k}))}$. Then the open set $\Omega' = \Omega - \cup_{k=0}^m \overline{L_{x_k}(E(h_{x_k}))}$ is the union of a finite family of convex polytopes, with a negligible set lying on boundaries of those convex polytopes.*

PROOF. For $0 \leqslant i \leqslant m$ we set $\Omega_i = L_{x_i}(E(h_{x_i}))$ and denote by $(G_{ik})_{1\leqslant k\leqslant 2^N-1}$ the hyperplanes limiting the boundary of Ω_i. We set $G_{ik} = \{x \in \mathbb{R}^N,\ g_{ik}(x) = 0\}$, where g_{ik} are non null linear forms, and $G_{ik}^+ = \{x \in \mathbb{R}^N,\ g_{ik}(x) > 0\}$, $G_{ik}^- = \{x \in \mathbb{R}^N,\ g_{ik}(x) < 0\}$.

We first prove that $\partial\Omega' \subset \cup_{i,k} G_{ik}$.

Indeed, since ${\Omega'}^c = \Omega^c \cup (\cup_{0\leqslant i\leqslant m}\overline{\Omega_i})$ we deduce that for every $x \in \partial\Omega'$, there exists an index i such that $x \in \overline{\Omega_i} \cap \overline{\Omega'}$, and more precisely, $x \in \partial\Omega_i \cap \overline{\Omega'}$.

For every index i, we have $\partial\Omega_i \subset (\cup_k G_{ik}) \cup L_{x_i}(F(h_{x_i}))$. But $U_{x_i} \subset \overline{\Omega'}^c$ and $L_{x_i}(F(h_{x_i})) \subset U_{x_i}$. We deduce $x \in \cup_k G_{ik} \subset \cup_{j,k} G_{jk}$ and the result follows.

For $\varepsilon = (\varepsilon_{ik}) \in \{+,-\}^{(m+1)(2^N-1)}$, we set $V_\varepsilon = \cap_{i,k} G_{ik}^{\varepsilon_{ik}}$. We want to prove the following alternative: $V_\varepsilon \cap \Omega' = \emptyset$ or $V_\varepsilon \subset \Omega'$.

If there exist $x_0 \in V_\varepsilon \cap \Omega'$ and $x_1 \in V_\varepsilon \cap {\Omega'}^c$, we can find $x_2 \in [x_0, x_1] \cap \partial\Omega'$. Then, there exists an index (i,k) such that $x_2 \in G_{ik}$ which is a contradiction with $x_2 \in V_\varepsilon$.

Now, let Δ be the set of indexes ε such that $V_\varepsilon \subset \Omega'$. From the partition $\mathbb{R}^N = (\cup_\varepsilon V_\varepsilon) \cup (\cup_{i,k} G_{i,k})$, we deduce $\Omega' = (\cup_{\varepsilon\in\Delta} V_\varepsilon) \cup (\Omega' \cap (\cup_{i,k} G_{i,k}))$.

□

Lemma 11.1. *Let Ω be a bounded convex open subset of $\mathbb{R}^N$ with $0 \in \Omega$. The function $h : S_{N-1} \to \mathbb{R}_+^*$ defined by $h(u)u \in \partial\Omega$ is Lipschitzian.*

PROOF. Let $\alpha > 0$ be such that $B(0,\alpha) \subset \Omega$. If h is not a Lipschitzian application, we can find two sequences (u_n) and (v_n) of S_{N-1} with the same limit u, such that

$$\forall n \in \mathbb{N},\quad u_n \neq v_n,\quad u_n \neq -v_n \quad\text{and}\quad h(u_n) - h(v_n) \geqslant n|u_n - v_n|.$$

Let $w_n \in S_{N-1}$ be such that (v_n, w_n) is an orthonormal basis of $vect(u_n, w_n)$ satisfying $u_n = \cos(\theta_n)v_n + \sin(\theta_n)w_n$ with $\theta_n \in]0, \pi[$ and $\lim_{n\to+\infty} \theta_n = 0$. We have $\theta_n < \pi/2$ for $n \geqslant n_0$. We are going to show that $h(v_n)v_n$ falls into the triangle $(O, h(u_n)u_n, -\alpha w_n)$ for some n. This will be in contradiction with $h(v_n)u_n \in \partial\Omega$.

For $n \geqslant n_0$, $h(v_n)v_n = \dfrac{h(v_n)}{\cos(\theta_n)h(u_n)} h(u_n)u_n + \dfrac{\sin(\theta_n)h(v_n)}{\alpha\cos(\theta_n)}(-\alpha v_n)$.

We have

$$\frac{h(v_n)}{\cos(\theta_n)h(u_n)} > 0 \text{ and } \frac{\sin(\theta_n)h(v_n)}{\alpha\cos(\theta_n)} > 0;$$

we have to prove

$$\frac{h(v_n)}{\cos(\theta_n)h(u_n)} + \frac{\sin(\theta_n)h(v_n)}{\alpha\cos(\theta_n)} < 1 \text{ for large } n.$$

From the hypothesis,

$$\begin{aligned}\frac{h(v_n)}{\cos(\theta_n)h(u_n)} + \frac{\sin(\theta_n)h(v_n)}{\alpha\cos(\theta_n)} &\leqslant \frac{1}{\cos(\theta_n)}\left(1 - \frac{2n\sin(\theta_n/2)}{h(u_n)} + \frac{\sin(\theta_n)}{\alpha}h(v_n)\right) \\ &\leqslant 1 - \frac{n\theta_n}{h(u)} + o(n\theta_n)\end{aligned}$$

which concludes the proof. □

Theorem 11.2. *Let Ω be a bounded convex open subset of $\mathbb{R}^N$ with $0 \in \Omega$ and h as in lemma 11.1. The function $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ defined by $\Phi(0) = 0$ and, for $x \in \mathbb{R}^N - \{0\}$, $\Phi(x) = h(\frac{x}{|x|})x$ is bi-Lipschitzian.*

PROOF. M denotes a Lipschitz ratio for h. For x and x' in $\mathbb{R}^N - \{0\}$,

$$\begin{aligned}|\Phi(x) - \Phi(x')| &= \left|h(\frac{x}{|x|})x - h(\frac{x'}{|x'|})x'\right| \\ &\leqslant \|h\|_\infty|x - x'| + |x'|\left|h(\frac{x}{|x|}) - h(\frac{x'}{|x'|})\right| \\ &\leqslant \|h\|_\infty|x - x'| + M|x'|\left|\frac{x}{|x|} - \frac{x'}{|x'|}\right| \\ &\leqslant \|h\|_\infty|x - x'| + \frac{M}{|x|}|x|x'| - x'|x||\end{aligned}$$

$$\leqslant \|h\|_\infty |x - x'| + \frac{M}{|x|}(|x|||x'| - |x|| + |x||x - x'|)$$
$$\leqslant (\|h\|_\infty + 2M)|x - x'|$$

which is still valid when $x = 0$ or $x' = 0$.

Now, $\Phi^{-1}(x) = \left[h(\frac{x}{|x|})\right]^{-1} x$ for $x \in \mathbb{R} - \{0\}$ and the previous calculus still stands for Φ^{-1} because

$$\forall (u, v) \in S_{N-1}^2, \quad \left|\frac{1}{h(u)} - \frac{1}{h(v)}\right| \leqslant \frac{M}{m^2}|u - v|$$

where $m = \min\{h(u),\ u \in S_{N-1}\} > 0$. □

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