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A THEOREM OF NAKANISHI FOR THE GENERAL DENJOY INTEGRAL

Abstract

In this paper, we give an example to show that a theorem of Nakanishi for the Henstock integral does not hold for the general Denjoy integral.

1 Introduction and Preliminaries

Shizu Nakanishi proved the following theorem [3].

Theorem 1.1. *Let f be a Henstock integrable function on an interval E of the real line. Then for any monotone null sequence $\{\varepsilon_k\}$, there exists a sequence $\{X_k\}$ of closed sets in E such that:*

1). $X_k \nearrow E$,

2). f_{X_k} is Lebesgue integrable on E for each k,

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3). for any k, if $\{I_i\}_{i=1}^{p}$ is a finite sequence of nonoverlapping intervals in E with at least one of the vertices of each I_i belonging to X_k, then we have

$$\sum_{i=1}^{p}\left|(L)\int_{I_i} f_{X_k} - (H)\int_{I_i} f\right| < \varepsilon_k,$$

where $X_k \nearrow E$ means that $X_k \subset X_{k+1}$ for any k and $\cup_{k=1}^{\infty} X_k = E$, and $f_{X_k}(x) = f(x)$ when $x \in X_k$ and 0 otherwise.

It is well-known that the Henstock integral is equivalent to the Denjoy integral in the restricted sense, and not to the Denjoy integral in the wide sense (general Denjoy integral). So a question arises naturally: Can Theorem 1.1 apply to the general Denjoy integral? The answer is negative. In this short paper, we give an example to illustrate this. We note that a modified version of Theorem 1.1 for the general Denjoy integral is given in Corollary 1 of [2].

2 Point Sets on the Real Line

Let $[0, 1]$ be the unit interval on the real line and X be the generalized Cantor set with $|X| = \frac{7}{8}$, [1, p.41], with the complementary open intervals given by $I_{i,j}^{\circ}$, $i = 1, 2, \ldots, j = 1, \ldots, 2^{i-1}$, in which $|I_{i,j}| = \dfrac{1}{2^{2i+2}}$. Suppose Y is another closed set with $|Y| \geq \frac{7}{8}$. Then it is obvious that $|X \cap Y| \geq \frac{3}{4}$. Moreover, we have the following lemmas.

Lemma 2.1. *There exists a point $x_0 \in X$ and an $r_0 > 0$ such that for any interval $I \subset B(x_0, r_0)$ with $x_0 \in I$, we have*

$$|X \cap I \cap Y| \geq \frac{3}{4}|X \cap I|, \tag{2.1}$$

where $B(x_0, r_0) = \{x : |x - x_0| < r_o\}$.

Proof. If not, then for any $x \in X$, there exists asequence of intervals $\{I_n(x)\}$ which satisfies $\cap_{n=1}^{\infty} I_n(x) = \{x\}$ and

$$|X \cap I_n(x) \cap Y| < \frac{3}{4}|X \cap I_n(x)|. \tag{2.2}$$

Then from the Vitali's Covering Theorem, [8, p.109], we know that for any $\varepsilon > 0$, there exists a finite sequence of disjoint intervals $\{I_{n_k}(x_k) : k = 1, 2, \ldots, s\}$ in $\{I_n(x) : n = 1, 2, \ldots$ and $x \in X\}$, such that $|X - (\cup_{k=1}^{s} I_{n_k}(x_k)) \cap X| < \varepsilon$. Thus

$$|(X - \cup_{k=1}^{s} I_{n_k}(x_k)) \cap X \cap Y| < \varepsilon. \tag{2.3}$$

On the other hand, using (2.2) we know that

$$\begin{aligned}|X \cap (\cup_{k=1}^{s} I_{n_k}(x_k)) \cap Y| =& \sum_{k=1}^{s} |X \cap I_{n_k}(x_k) \cap Y| \\ <& \frac{3}{4}\sum_{k=1}^{s} |X \cap I_{n_k}(x_k)| < \frac{3}{4}|X|.\end{aligned}$$

Combining the above relation with (2.3), we have $|X \cap Y| < \frac{21}{32} + \varepsilon$. Since ε can be arbitrarily small, then $|X \cap Y| < \frac{3}{4}$. This is a contradiction to the fact that $|X \cap Y| \geq \frac{3}{4}$. The proof is complete. □

Lemma 2.2. *Let x_0 and r_0 be a point and the corresponding positive number in Lemma 2.1. We denote by $I_{i,j(i)}$ the interval which is closest to x_0 among $I_{i,j}$, $j = 1, \dots, 2^{i-1}$, and by x_i , y_i and z_i the center, the left and the right endpoints of $I_{i,j(i)}$, for any $i = 1, 2, \dots$. Then there exists a positive integer i_0 and a sequence of points $\{u_i : i > i_0\}$ in $X \cap Y$, such that $\{< u_i, x_i >\}$ is a sequence of closed intervals, no two of which have common points, where $< u_i, x_i >$ denotes $[u_i,x_i]$ when $u_i \leq x_i$ and $[x_i, u_i]$ when $x_i < u_i$.*

Proof. Let i_0 be the integer such that $I_{i,j(i)} \subset B(x_0, r_0)$ for any $i > i_0$ and $i_0 < i_1 < i_2 < \dots$ be all the integers such that $I_{i_k,j(i_k)} \subset [x_0, x_0 + r_0]$. Note that the generalized Cantor set X is symmetric. So that x_0 lies inside the left-hand half of an interval with center at $x_{i_{k+1}}$ and right-hand endpoint y_{i_k}. Thus, $|[x_0, x_{i_{k+1}}]| \leq |[x_{i_{k+1}}, y_{i_k}]|$ and $|[x_0, y_{i_{k+1}}]| \leq |[z_{i_{k+1}}, y_{i_k}]|$. Again from the symmetry of X we know that

$$|[x_0, y_{i_{k+1}}] \cap X| \leq |[z_{i_{k+1}}, y_{i_k}] \cap X|.$$

Then from (2.1) we have

$$\begin{aligned}|X \cap [x_0, y_{i_k}] \cap Y| \geq& \frac{3}{4}|X \cap [x_0, y_{i_k}]| \\ \geq& \frac{3}{2}|X \cap [x_0, y_{i_{k+1}}]| \\ >& \frac{3}{2}|X \cap [x_0, y_{i_{k+1}}] \cap Y|.\end{aligned}$$

From the above relation we know that $X \cap [z_{i_{k+1}}, y_{i_k}] \cap Y \neq \emptyset$. Choose a point u_{i_k} from $X \cap [z_{i_{k+1}}, y_{i_k}] \cap Y$. It is obvious that $[u_{i_k}, x_{i_k}] \cap [x_0, x_{i_{k+1}}] = \emptyset$. Do the same on the other side of x_0. Then we obtain the required sequence of points $\{u_i\}$. The proof is complete. □

3 A Counter Example

Let $I_{i,j}$ be any interval mentioned above. Suppose $I_{i,j} = [a,b]$. Then we can define easily a differentiable function $\Psi_{i,j}$ on $I_{i,j}$ such that $\Psi_{i,j}(\frac{a+b}{2}) = \frac{1}{i}$ and $\Psi_{i,j}(a) = \Psi_{i,j}(b) = 0$. Let $\psi_{i,j}(x) = \Psi'_{i,j}(x)$ for any $x \in I_{i,j}$. Then we obtain a D-integrable function $\psi_{i,j}$ on $I_{i,j}$. Let ϕ be a function defined on $[0,1]$ such that $\phi(x) = \psi_{i,j}(x)$ when $x \in I_{i,j}$ and 0 otherwise. Then from the theorem [4, p.257] we can verify that ϕ is D-integrable on $[0,1]$. Now we shall prove that the above D-integrable function f does not satisfy the conditions in Theorem 1.1. If f satisfies all the conditions in Theorem 1.1, then for given positive number 1, there exists a closed set $Y_0 \subset [0,1]\times[0,1]$ with its measure being bigger than $\frac{7}{8}$, such that f_{Y_0} is Lebesgue integrable on $[0,1]\times[0,1]$ and for any finite sequence of nonoverlapping intervals $\{I_k : k = 1,\ldots,p\}$ in $[0,1]$ with that at least one of vertices of each I_k belong to Y_0, we have $\sum_{k=1}^{p}\left|(L)\int_{I_k} f_{Y_0} - (D)\int_{I_k} f\right| < 1$. Thus $\sum_{k=1}^{p}\left|(D)\int_{I_k} f\right| < M$, where $|f_{Y_0}|$ denotes the absolute function of f_{Y_0} and $M = 1 + (L)\int_{[0,1]} |f_{Y_0}|$. From Lemma 2.2 we know that there exist a positive k_0 and a sequence of point-intervals $\{(x_k, < u_k, x_k >) : k = k_0, \ldots, \infty\}$ with $x_k \in X \cap Y$ and no two of $\{< u_k, x_k >\}$ have common points. It is obvious that $(D)\int_{<u_k,x_k>} \phi = \frac{1}{k}$ for any k. Let $I_k =< u_k, x_k >$. Then for the above given $M > 0$, there exists $p > 0$ such that $\sum_{k=1}^{p} |(D)\int_{I_k} \phi| > M$. This is a contradiction.

We note that an easy application of Theorem 1.1 shows that the function given above is not Henstock integrable on [0,1]. Thus, it is also an example of general Denjoy integrable but not Henstock integrable functions.

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