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LIMIT SUMMABILITY OF REAL FUNCTIONS

Abstract

Let f be a real (or complex) function with domain D_f containing the positive integers. We introduce the functional sequence $\{f_{\sigma_n}(x)\}$ as follows:$f_{\sigma_n}(x) = xf(n) + \sum_{k=1}^{n}(f(k) - f(x+k))$ and say that the function f *limit summable* at the point x_0 if the sequence $\{f_{\sigma_n}(x_0)\}$ is convergent, $(f_{\sigma_n}(x_0) \to f_\sigma(x_0))$ as $n \to \infty$, and we call the function $f_\sigma(x)$ as the limit summand function (of f). In this article, we first give a necessary condition for the limit summability of functions and present some elementary properties. Then we prove some tests about limit summability of functions and consider the relation between $f(x)$ and $f_\sigma(x)$. One of the main theorems in this paper gives a uniqueness conditions for a function to be a limit summand function. Finally, as a consequence of this theorem we deduce a generalization of a result due to Bohr-Mollerup [1].

1 Preliminaries

Definitions and theorems in this article are for complex functions, except when the real case is explicitly mentioned. In general we assume $f : D_f \to \mathbb{C}$, where $D_f \subseteq \mathbb{C}$. In the real case we take the function $f : D_f \to \mathbb{R}$, where $D_f \subseteq \mathbb{R}$. For a function with domain D_f,we put

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\};$$

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so $x \in \Sigma_f$ if and only if $\{x+1, x+2, \cdots\} \subseteq D_f$. If $\mathbb{N}^* \subseteq D_f$ (or equivalently $0 \in \Sigma_f$) for any positive integer n and $x \in \Sigma_f$, set

$$R_n(f,x) = R_n(x) = f(n) - f(x+n),$$
$$f_{\sigma_n}(x) = xf(n) + \sum_{k=1}^{n} R_k(x).$$

When $x \in D_f$, we may use the notation $\sigma_n(f(x))$ instead of $f_{\sigma_n}(x)$.

Definition 1.1. The function f is called limit summable at $x_0 \in \Sigma_f$ if the functional sequence $\{f_{\sigma_n}(x)\}$ is convergent at $x = x_0$. The function f is called limit summable on the set $S \subseteq \Sigma_f$ if it is limit summable at all the points of S.

Convention: For brevity we use the term *summable* for limit summable, and restrict ourselves to the assumption $\mathbb{N}^* \subseteq D_f$.

Now, put $D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is summable at } x\}$. The function f_σ is the same limit function f_{σ_n} with domain D_{f_σ}. We represent also, the limit function $R_n(f,x)$ as $R(f,x)$ or $R(x)$. Clearly $f_\sigma(0) = 0$, $0 \in D_{f_\sigma}$. If $0 \in D_f$, then $-1 \in D_{f_\sigma}$, and we have $f_\sigma(-1) = -f(0)$. Regarding the relations

$$f_{\sigma_n}(1) = f(1) + R_n(1),$$
$$f_{\sigma_n}(x) - f_{\sigma_{n-1}}(x) = R_n(x) - xR_{n-1}(1),$$

we get $1 \in D_{f_\sigma}$ if and only if the sequence $\{R_n(1)\}$ is convergent, and if $R(1) = 0$, then $f_\sigma(1) = f(1)$ (e.g. if $\{f(n)\}$ is convergent, then $R(1) = 0$ and so $f_\sigma(1) = f(1)$). Also it is inferred that a necessary condition for the summability of f at x is $\lim_{n\to\infty}(R_n(x) - xR_{n-1}(1)) = 0$. Therefore if $1 \in D_{f_\sigma}$, then the functional sequence $\{R_n(x)\}$ is convergent on D_{f_σ} and $R(x) = R(1)x$ (for all $x \in D_{f_\sigma}$). Now it is not difficult to show that

$$D_f \cap \Sigma_f = \Sigma_f + 1 = \{x+1 | x \in \Sigma_f\}.$$

An interesting fact is the similarities between the properties of D_{f_σ} and those of Σ_f. The next theorem shows a corresponding relation for D_{f_σ}.

Theorem 1.2. *If $R_n(1,f)$ is convergent, then $D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1$.*

Proof. Take an x in $D_f \cap D_{f_\sigma}$. Then $x \in \Sigma_f + 1$ and so, both x and $x-1$ belong to Σ_f and we have

$$f_{\sigma_n}(x-1) = f_{\sigma_n}(x) - f(x) - R_n(x).$$

From $x \in D_{f_\sigma}$ and $R_n(1) \to R(1)$ we conclude that $R(x) = R(1)x$; so $f_{\sigma_n}(x-1)$ is convergent; that is, $x-1 \in D_{f_\sigma}$ and so $x \in D_{f_\sigma} + 1$.

Now if $x \in D_{f_\sigma} + 1$, then $x \in D_f \cap \Sigma_f$, and $R_n(x)$ is convergent, because $R_n(x) = R_n(1) + R_{n+1}(x-1)$, and $x - 1 \in D_{f_\sigma}$. Hence $f_{\sigma_n}(x)$ is convergent and $f_\sigma(x) = f(x) + f_\sigma(x-1) + R(1)x$; so that $x \in D_{f_\sigma} \cap D_f$. □

Remark. The converse of the above theorem is clearly true.

Corollary 1.3. *If $R(1) = 0$, then*

(a) $f_\sigma(x) = f(x) + f_\sigma(x-1)$, for all $x \in D_{f_\sigma} + 1$.

(b) f is summable on $\mathbb{N}$ and on $\Sigma_f \cap \mathbb{Z}^-$, and we have

$$f_\sigma(m) = \begin{cases} \sum_{j=1}^{m} f(j) & \text{if } m \in \mathbb{N}^* \\ -\sum_{j=0}^{-m-1} f(-j) & \text{if } m \in \mathbb{Z}^- \cap \Sigma_f. \end{cases}$$

2 Limit Summable Functions

Lemma 2.1. *The followings are equivalent:*

(a) $D_f \subseteq D_{f_\sigma},\ R(1) = 0$.

(b) $D_{f_\sigma} = \Sigma_f,\ D_f \subseteq D_f - 1,\ R(1) = 0$.

(c) $f_\sigma(x) = f(x) + f_\sigma(x-1)$, for all $x \in D_f$.

PROOF. $(a) \Longrightarrow (b)$: Since $D_f \subseteq D_{f_\sigma}$, we have $D_f \subseteq D_{f_\sigma} \subseteq \Sigma_f \subseteq D_f - 1$. Hence $D_f \subseteq D_f - 1$, and consequently $\Sigma_f = D_f - 1$. Now, by Theorem 1.2. we get $\Sigma_f = (D_f \cap D_{f_\sigma}) - 1 = D_{f_\sigma}$.

(b)$\Longrightarrow$ (c): This clearly follows from Corollary 1.3.

(c)$\Longrightarrow$ (a): From the assumption we conclude that $D_f \subseteq D_{f_\sigma}$. Now putting $x = 1$ we get $f_\sigma(1) = f(1) + f_\sigma(0) = f(1)$, and this yields $R(1) = 0$. □

Definition 2.2. The function f is called limit summable (or more briefly summable) if it is summable on its domain and $R(1) = 0$. In this case the function f_σ is referred to as the limit summand function of f (or the summand function of f).

Because a summable function f satisfies condition (a) of Lemma 2.1, one has $D_f = D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1$, i.e. $D_{f_\sigma} = D_f - 1$.

Example 2.3. *If $|a| < 1$, then the function a^x is summable and we have*

$$\sigma(a^x) = \frac{a}{a-1}(a^x - 1).$$

Example 2.4. *The function $f(x) = 1/x$ is not summable. But from the fact that it is summable on $D = D_f \setminus \mathbb{Z}^-$, the restricted function $g = f|_D$ is summable and we have*

$$g_\sigma(x) = \sum_{n=1}^{\infty} \frac{x}{nx + n^2}.$$

The domain of g_σ is the set $\mathbb{C} \setminus \mathbb{Z}^-$ (if x is complex) or the set $\mathbb{R} \setminus \mathbb{Z}^-$ (if x is real).

Example 2.5. *The real function $\ln x$ (with domain $\mathbb{R}_+^*$) is summable and $\ln_\sigma(x) = \ln \Gamma(x+1)$.*

Lemma 2.6. *If the functions f and g are summable, then $\alpha f + \beta g$ is and we have $(\alpha f + \beta g)_\sigma = \alpha f_\sigma + \beta g_\sigma$.*

Proof. For any x belonging to $\Sigma_f \cap \Sigma_g = \Sigma_{\alpha f + \beta g}$ we have

$$(\alpha f + \beta g)_{\sigma_n}(x) = \alpha f_{\sigma_n}(x) + \beta g_{\sigma_n}(x),$$

and

$$R_n(\alpha f + \beta g, x) = \alpha R_n(f, x) + \beta R_n(g, x).$$

Now, since f and g are summable, by the above relations we conclude that $R(\alpha f + \beta g, 1) = 0$ and

$$D_{\alpha f + \beta g} = D_f \cap D_g \subseteq D_{f_\sigma} \cap D_{g_\sigma} \subseteq D_{(\alpha f + \beta g)_\sigma}. \qquad \square$$

Corollary 2.7. *Let $f = u + iv$ and $D_u = D_v$. The complex function f is summable if and only if the functions $u = Re(f)$ and $v = Im(f)$ are summable, and $f_\sigma = u_\sigma + iv_\sigma$.*

Example 2.8. *If $0 < a < 1$, then the real function $f(x) = ca^x + \log_b x$ is summable and we have*

$$f_\sigma(x) = \frac{ca}{a-1}(a^x - 1) + \log_b \Gamma(x+1),$$

$(D_{f_\sigma} = (-1, +\infty))$.

Very often it is sufficient to consider the summability of a real function on an interval of length 1. We prove this fact through a theorem preceded by the following definition.

Definition 2.9. The real function f is given. The set Σ_f is called concentrable if $\Sigma_f \setminus D_f$ is bounded above. In this case we set

$$\sigma_f = \sup(\Sigma_f \setminus D_f) \ \text{ if } \Sigma_f \setminus D_f \neq \emptyset,$$

and if $\Sigma_f \setminus D_f = \emptyset$, then we set $\sigma_f = 0$. The set $\Sigma_f \cap [\sigma_f, \sigma_f + 1)$ is called the center of Σ_f.

Usually, for the so called important functions, Σ_f is concentrable. For instance, in case the domain of f is one of the sets $(M, +\infty)$ or $[M, +\infty)$, or is a subgroup of $\mathbb{R}$ with identity, then Σ_f is concentrable. However the following represents a non-concentrable Σ_f.

Example 2.10. *Let E be a subset of $\mathbb{R}$ that is unbounded above, contains 0 and such that the subtraction of any two distinct elements of E is not an integer. Put $D = E + \mathbb{N}$, and take the function f such that $D_f = D$. So $\Sigma_f = E \cup D$ and $\Sigma_f \setminus D_f = E$, whence Σ_f is non concentrable.*

Theorem 2.11. *Let f be a real function for which $R_n(1)$ is convergent and Σ_f is concentrable. Then f is summable on Σ_f if and only if it is summable on the center of Σ_f.*

PROOF. Suppose that f is summable on the center of Σ_f and take a $x \in \Sigma_f$. Consider the following cases.

Case (1) $x > \sigma_f$. There exists a non-negative integer m with $\sigma_f < x - m < \sigma_f + 1$; so we have $\{x, \cdots, x-m\} \subseteq \Sigma_f$, because if for a $t \in \Sigma_f$ the condition $t \notin \Sigma_f \setminus D_f$ holds, then $t \in \Sigma_f \cap D_f = \Sigma_f + 1$ and hence $t - 1 \in \Sigma_f$. Therefore $x - m \in (\sigma_f, \sigma_f + 1) \cap \Sigma_f \subseteq D_{f_\sigma}$, and this yields

$$f_{\sigma_n}(x) = f_{\sigma_n}(x-m) + \sum_{j=1}^{m} (f(x-m+j) + R_n(x-m+j)),$$

(note that $\sum_{j=1}^{0} a_j = 0$). Now, since $(x-m) \in D_{f_\sigma}$ and $R_n(1) \to R(1)$ as $n \to \infty$ and since $R_n(x-m+j) = R_n(j) + R_{n+j}(x-m)$, for $j = 1, \ldots, m$, we see that

$$R_n(x-m+j) \to R(1)(x-m+j),$$

as $n \to \infty$ for each $j = 1, \cdots, m$, and so $\{f_{\sigma_n}(x)\}_{n \geq 1}$ is convergent.

Case (2) $x \leq \sigma_f$. There exists an non-negative integer m with $\sigma_f \leq x+m < \sigma_f + 1$. Since $x \in \Sigma_f$, and since $m \geq 0$, we have $\{x, \cdots, x+m\} \subseteq \Sigma_f$, and therefore

$$f_{\sigma_n}(x) = f_{\sigma_n}(x+m) - \sum_{j=0}^{m-1} (f(x+m-j) + R_n(x+m-j)),$$

(note that $\sum_{j=0}^{-1} a_j = 0$). Now the verification of the convergence of $f_{\sigma_n}(x)$ is rendered as in case (1). □

Corollary 2.12. *Let f be a real function for which $R(1) = 0, D_f \subseteq D_f - 1$ and Σ_f is concentrable. If f is summable on the center of Σ_f, then f is summable.*

Corollary 2.13. *If* $\{1, x\} \subseteq D_{f_\sigma}$*, then* $x + \mathbb{N}^* \subseteq D_{f_\sigma}, (x + \mathbb{Z}^-) \cap \Sigma_f \subseteq D_{f_\sigma}$ *and for any integer* m

$$f_\sigma(x+m) = \begin{cases} \sum_{j=1}^{m} f(x+j) + f_\sigma(x) + R(1)mx + R(1)\dfrac{m^2+m}{2} \\ \textit{if } m \in \mathbb{N}^* \\ -\sum_{j=0}^{-m-1} f(x-j) + f_\sigma(x) + R(1)mx + R(1)\dfrac{m^2+m}{2} \\ \textit{if } m \in \mathbb{Z}^-. \end{cases}$$

3 A Test for Summability of Real Functions and Uniqueness Conditions for a Summand Function

Let E be a subset of $\mathbb{R}$ (not necessarily an interval) and suppose that a real function f is defined on E. The function f is called *convex on* E if for every three elements x_1, x_2, x_3 of E with $x_1 < x_2 < x_3$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

If the above inequalities are reversed, then f is called *concave*. Therefore a function f is concave if and only if the function $-f$ is convex. If f is convex on E, then it is so on each subset of E. For example if f' is increasing on (a, b), then f is convex on each subset of (a, b).

Theorem 3.1. *Let* f *be a real function for which* $R_n(f, 1)$ *is convergent. Suppose there exists a function* λ *such that*

$$(\star) \;\; \lambda(x) = f(x) + \lambda(x-1) \textit{ for all } x \in \Sigma_f + 1.$$

(a) If $R(1) \ge 0$ *and* λ *is convex on* $\Sigma_f + 1$ *from a number on, then* f *is summable on* Σ_f*.*
(b) If $R(1) \le 0$ *and* λ *is concave on* $\Sigma_f + 1$ *from a number on, then* f *is summable on* Σ_f*.*
In each of the above cases we have

$$f_\sigma(x) = \lambda(x) + R(1)\frac{x^2+x}{2} - \lambda(0) \textit{ for all } x \in \Sigma_f + 1.$$

Proof. (Notice that since $R_n(f, 1)$ is convergent, f is summable on the integer points of Σ_f.)

(a) Firstly, assume that $R(1) = 0$ and $\lambda(1) = f(1)$. There exists an M such that λ is convex on $\Sigma_f + 1 \cap (M, +\infty)$. Now for a fixed non-integer $x \in \Sigma_f$ and every natural number n with $n > \max\{[x], M\} + 1$, we have

$$\{n-1, n, n+x-[x], n+1\} \subseteq (\Sigma_f + 1) \cap (M, +\infty),$$

and so the convexity of λ gives

$$\lambda(n) - \lambda(n-1) \leq \frac{\lambda(n+x-[x]) - \lambda(n)}{x-[x]} \leq \lambda(n+1) - \lambda(n).$$

Condition $(\star)$ with $\lambda(1) = f(1)$ implies the equalities

$$\lambda(n) = \sum_{j=1}^{n} f(j), \text{ and } \lambda(x+n-[x]) = \lambda(x) + \sum_{j=1}^{n-[x]} f(x+j).$$

From the latter we deduce that, if $[x] \geq 0$,

$$0 \leq \lambda(x) - f_{\sigma_n}(x) + \sum_{j=1}^{[x]} R_n(x-[x]+j) \leq ([x]-x)R_n(1),$$

and if $[x] \leq -1$,

$$0 \leq \lambda(x) - f_{\sigma_n}(x) - \sum_{j=1}^{-[x]} R_n(x+j) \leq ([x]-x)R_n(1).$$

When $[x] \geq 0$ we write

$$f_{\sigma_{n-[x]}}(x) = f_{\sigma_n}(x) - xR_n(-[x]) - \sum_{j=0}^{[x]-1} R_{n-j}(x),$$

and if $[x] \leq -1$,

$$f_{\sigma_{n-[x]}}(x) = f_{\sigma_n}(x) - xR_n(-[x]) + \sum_{j=1}^{-[x]} R_{n+j}(x).$$

Combining these with previous inequalities, we have if $[x] \geq 0$,

$$xR_n(-[x]) + \sum_{j=0}^{[x]-1} R_{n-j}(j) \leq \lambda(x) - f_{\sigma_{n-[x]}}(x)$$
$$\leq ([x]-x)R_n(1) + xR_n(-[x]) + \sum_{j=0}^{[x]-1} R_{n-j}(j),$$

and if $[x] \le -1$

$$
\begin{aligned}
xR_n(-[x]) - \sum_{j=1}^{-[x]} R_n(j) &\le \lambda(x) - f_{\sigma_{n-[x]}}(x) \\
&\le ([x]-x)R_n(1) + xR_n(-[x]) - \sum_{j=1}^{-[x]} R_n(j),
\end{aligned}
$$

Letting $n \to \infty$ and using the fact that $R_n(1) \to 0$, one sees that the right and left hand sides of the above inequalities tend to 0, and consequently f is summable at x with $f_\sigma(x) = \lambda(x)$.

Now to prove (a) in general put

$$f^*(x) = f(x) + R(1)x \text{ and } \lambda^*(x) = \lambda(x) + R(1)\frac{x^2+x}{2} - \lambda(0).$$

The conditions on f and λ imply that

$$\lambda^*(x) = f^*(x) + \lambda^*(x-1) \text{ for all } x \in \Sigma_{f^*} + 1 = \Sigma_f + 1.$$

On the other hand, since $R(1) \ge 0$, λ^* is convex (from a number on) and $R(f^*, 1) = 0, \lambda^*(1) = f^*(1)$. Thus, by the previous part we conclude that f^* is summable at x and $f^*_\sigma(x) = \lambda^*(x)$. But from $f_{\sigma_n}(x) = f^*_{\sigma_n}(x)$ we derive the summability of f at x, and we have

$$f_\sigma(x) = f^*_\sigma(x) = \lambda(x) + R(1)\frac{x^2+x}{2} - \lambda(0) \text{ for all } x \in \Sigma_f.$$

(b) If the two functions f and λ satisfy the said conditions, then the functions $-f$ and $-\lambda$ satisfy the conditions of (a), and so

$$
\begin{aligned}
-f_\sigma(x) =&(-f)_\sigma(x) = (-\lambda)(x) + R(-f,1)\frac{x^2+x}{2} - (-\lambda)(0) \\
=& -\lambda(x) - R(f,1)\frac{x^2+x}{2} + \lambda(0),
\end{aligned}
$$

which gives $f_\sigma(x) = \lambda(x) + R(1)\frac{x^2+x}{2} - \lambda(0)$, for all $x \in \Sigma_f$. □

Corollary 3.2. *Suppose f satisfies $R(f,1) = 0$ and $D_f \subseteq D_f - 1$. If there exists a function λ which is convex (concave) on D_f such that $\lambda(x) = f(x) + \lambda(x-1)$for all $x \in D_f$, then f is summable, and $f_\sigma(x) = \lambda^0(x)$ for every $x \in D_f - 1$, where $\lambda^0 = \lambda - \lambda(0)$.*

The above corollary contains a result which may be viewed as a generalization of the Bohr-Mollerup theorem about the Gamma function.

Corollary 3.3. (A GENERALIZATION OF THE BOHR-MOLLERUP THEOREM). *Let f be a positive function on $(M, +\infty)$ satisfying*

$$\lim_{n\to\infty} \frac{f(n)}{f(n+1)} = 1.$$

If there is a positive function ϕ defined on $(M-1, +\infty)$ with:

(a) $\phi(1) = f(1)$,

(b) $\phi(x) = f(x)\phi(x-1)$ *for all* $x \in (M, +\infty)$,

(c) $\ln \phi$ *is convex on* $(M, +\infty)$, *from a number on*

then the function $\ln f$ *is summable, and*

$$\phi(x) = e^{(\ln f)_\sigma(x)} \text{ for all } x \in (M-1, +\infty).$$

Corollary 3.4. *Let f be a summable and f_σ be convex (concave) on D_f. Then f_σ is the only function satisfying:*

(a) $f_\sigma(1) = f(1)$.

(b) f_σ *is convex (concave) on* D_f.

(c) $f_\sigma(x) = f(x) + f_\sigma(x-1)$ *for all* $x \in D_f$.

(This means that if another function λ satisfies the above condition, then $\lambda(x) = f_\sigma(x)$ for all $x \in D_{f_\sigma} = D_f - 1$).

Example 3.5. *If λ is a function concave on $\mathbb{R}^+$ satisfying*

$$\lambda(x) = a^x + \frac{1}{x} + \lambda(x-1) \text{ for all } x \in \mathbb{R}^+,$$

then there is a constant c such that one could write

$$\lambda(x) = \frac{a}{a-1}(a^x - 1) + \sum_{n=1}^{\infty} \frac{x}{nx + n^2} + c \text{ for all } x \in (-1, +\infty).$$

This follows easily from Example 2.3 and 2.4 along with Corollary 3.2 or 3.4.

Remark. One can easily deduce Theorem 3.1 of [2] from Corollary 3.3 (by taking $\phi(x) = f(x+1)$, $f(x) = g(x)$, $M = 0$).

References

[1] E. Artin, *The Gamma Function*, Holt Rhinehart & Wilson, New York, 1964; transl. by M. Butler from *Einführung un der Theorie der Gamma-fonktion*, Teubner, Leipzig, 1931.

[2] R. J. Webster, *Log-convex solutions to the functional equation* $f(x+1) = g(x)f(x)$*:* Γ*-type functions,* J. Math. Anal. Appl., **209** (1997), 605–623.