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# THE MEDIAN OF A CONTINUOUS FUNCTION

**Abstract**

Let $\Omega$ be a domain in $\mathbb{R}^n$ with finite Lebesgue measure and $f \in C(\Omega) \cap L^1(\Omega)$ a real-valued function on $\Omega$. It is shown that there exists a unique number $M \in \mathbb{R}$ at which the function $I(y) = \int_\Omega |f(x) - y| \, d\lambda^n(x)$ is minimized, where $\lambda^n$ is the Lebesgue measure on $\mathbb{R}^n$. We can define this number as *the* median of $f$ over $\Omega$ with respect to $\lambda^n$.

## 1 Introduction.

Given a random variable $X$ and a probability measure $P$ on a sample space $\Omega$, one can define the mean of $X$ as $E[X] = \int_\Omega x \, dP(x)$ and *a* median of $X$ as a real number $M$ such that $P(X \le M) \ge \frac{1}{2}$ and $P(X \ge M) \ge \frac{1}{2}$. For any real-valued measurable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, where $\Omega$ is a measurable set with finite Lebesgue measure, one can also define its mean as $\bar{f} = \int_\Omega f(x) \frac{d\lambda^n(x)}{\lambda^n(\Omega)}$, where $\lambda^n$ is the Lebesgue measure on $\mathbb{R}^n$. We could define *a* median of $f$ as a real number $M$ such that $\lambda^n\left(f^{-1}(-\infty, M] \cap \Omega\right) \ge \frac{\lambda^n(\Omega)}{2}$ and $\lambda^n\left(f^{-1}[M, \infty) \cap \Omega\right) \ge \frac{\lambda^n(\Omega)}{2}$. Notice that a median so defined may not be unique. For example, consider $f : [-2, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & -2 \le x \le -1 \\ 1 & -1 < x < 1 \\ 0 & 1 \le x \le 2 \end{cases} .$$

We see that

$$g(m) = \lambda\left(f^{-1}(-\infty, m] \cap [-2, 2]\right) = \begin{cases} 0 & m < 0 \\ 2 & 0 \le m < 1 \\ 4 & m \ge 1 \end{cases} .$$

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Consequently, every $m \in (0,1)$ is a median, as whenever $m \in (0,1)$, we have

$$\lambda\left(f^{-1}(-\infty, m] \cap [-2,2]\right) = \lambda\left(f^{-1}[m,\infty) \cap [-2,2]\right) = \frac{\lambda([-2,2])}{2} = 2.$$

Let $\operatorname{med}_{\Omega}(f)$ denote the set of all medians of $f$ over $\Omega$. If $f \in L^1(\Omega)$, it is known that $m \in \operatorname{med}_{\Omega}(f)$ if and only if

$$\int_{\Omega} |f(x) - m| \ d\lambda^n(x) = \min_{y \in \mathbb{R}} \int_{\Omega} |f(x) - y| \ d\lambda^n(x)$$

(cf. [2], [3]). If $\Omega$ is an open, connected subset of $\mathbb{R}^n$ with finite Lebesgue measure and $f$ is continuous on $\Omega$, we will show that there exists a unique number $M$ such that $I(M) < I(y)$ for all $y \neq M$.

## 2 Well-Posedness.

Henceforth, we will assume that $\Omega$ is an open, connected subset of $\mathbb{R}^n$ with finite Lebesgue measure. Suppose $f \in C(\Omega) \cap L^1(\Omega)$. We seek to minimize $I(y) = \int_{\Omega} |f(x) - y| \ d\lambda^n(x)$ over $\mathbb{R}$.

One can easily see that $I(y)$ is continuous in $y$.

**Theorem 2.1.** *If $f \in L^1(\Omega)$, then $I(y)$ as defined above is a continuous function in $y$.*

Proof. Let $y_i \to y$. We may assume without loss of generality that $|y_i| \leq |y| + 1$. Since we know that there exists a sufficiently large number $N$ such that for all $i > N$, $|y_i| \leq |y| + 1$ after discarding finitely many terms (if necessary), we will have the desired property. Next observe that

$$|f(x) - y_i| \leq |f(x)| + |y| + 1. \tag{1}$$

Furthermore, by virtue of the fact that $f \in L^1(\Omega)$ and that $\lambda^n(\Omega) < \infty$, we deduce that

$$\int_{\Omega} [|f(x)| + |y| + 1] \, d\lambda^n(x) = C < \infty, \tag{2}$$

where $C$ is a fixed constant. Moreover, for (almost) every $x \in \Omega$, $|f(x) - y_i| \to |f(x) - y|$ as $i \to \infty$. It follows from the Dominated Convergence Theorem (DCT) that

$$\begin{aligned}
\lim_{y_i \to y} I(y_i) &= \lim_{y_i \to y} \int_{\Omega} |f(x) - y_i| \ d\lambda^n(x) \\
&= \int_{\Omega} \lim_{y_i \to y} |f(x) - y_i| \ d\lambda^n(x) \ \text{(DCT)} \\
&= \int_{\Omega} |f(x) - y| \ d\lambda^n(x) = I(y),
\end{aligned}$$

yielding the continuity for $I(y)$. □

To minimize $I(y)$, we can restrict $y$ to a compact subset in $\mathbb{R}$.

**Lemma 2.2.** *If $f \in L^1(\Omega)$, then there exists a positive number $B > 0$ such that* $\min_{y\in\mathbb{R}} I(y) = \min_{y\in[-B,B]} I(y)$.

PROOF. It suffices to show that there exist positive numbers $B_1$ and $B_2$ such that for all $y > B_1$, $I(y) > I(B_1)$, and for all $y < -B_2$, $I(y) > I(-B_2)$, because then we can choose $B$ to be $\max\{B_1, B_2\}$. The argument for the existence of $B_1$ is identical to the one for that of $B_2$. Since $f \in L^1(\Omega)$, there exists $B_1$ sufficiently large and positive so that

$$\lambda^n \{x \in \Omega : f < B_1\} \geq \frac{3}{4}\lambda^n(\Omega). \tag{3}$$

We claim that for all $y > B_1$, $I(y) > I(B_1)$. Toward this end, we consider the following two expressions:

$$I(y) = \int_\Omega |f - y| \; d\lambda^n(x) = \tag{4}$$
$$= \int_{\{f<B_1\}\cap\Omega} (y - f) \; d\lambda^n(x) + \int_{\{B_1<f<y\}\cap\Omega} (y - f) \; d\lambda^n(x) + \int_{\{f>y\}\cap\Omega} (f - y) \; d\lambda^n(x).$$

$$I(B_1) = \int_\Omega |f - B_1| \; d\lambda^n(x) = \tag{5}$$
$$= \int_{\{f<B_1\}\cap\Omega} (B_1 - f) \; d\lambda^n(x) + \int_{\{B_1<f<y\}\cap\Omega} (f - B_1) \; d\lambda^n(x) + \int_{\{f>y\}\cap\Omega} (f - B_1) \; d\lambda^n(x).$$

Subtracting (5) from (4), we find that

$$I(y) - I(B_1) = \tag{6}$$
$$= \int_{\{f<B_1\}\cap\Omega} (y - B_1) \; d\lambda^n(x) + \int_{\{B_1<f<y\}\cap\Omega} (y + B_1 - 2f) \; d\lambda^n(x) + \int_{\{f>y\}\cap\Omega} (B_1 - y) \; d\lambda^n(x).$$

By (3) and the supposition that $y > B_1$, we have

$$\lambda^n(\{f > y\} \cap \Omega) \leq \frac{1}{4}\lambda^n(\Omega). \tag{7}$$

$$\lambda^n(\{B_1 < f < y\} \cap \Omega) \leq \frac{1}{4}\lambda^n(\Omega). \tag{8}$$

Applying inequalities (3), (7), and (8) to (6), one obtains

$$\begin{aligned}
&I(y) - I(B_1) \\
&\geq \frac{1}{2}\lambda^n(\Omega)(y - B_1) + \int_{\{B_1 < f < y\}\cap\Omega} (y + B_1 - 2f)\ d\lambda^n(x) \\
&\geq \frac{1}{2}\lambda^n(\Omega)(y - B_1) + \frac{1}{4}\lambda^n(\Omega)(y + B_1 - 2y) \\
&= \frac{1}{4}\lambda^n(\Omega)(y - B_1) > 0,
\end{aligned}$$

by the assumption that $y > B_1$. This proves the claim. To find $B_2$, we can retrace our steps. In this case, we will consider choosing $B_2$ sufficiently large and positive so that

$$\lambda^n\{x \in \Omega : f > -B_2\} \geq \frac{3}{4}\lambda^n(\Omega).$$

Going through the same argument, one arrives at the conclusion we stated at the outset. □

Therefore, to minimize $I(y)$ over $\mathbb{R}$ is the same as to minimize $I(y)$ over a compact subset $[-B, B]$. Since $I(y)$ is continuous by Theorem 2.1, the Extreme Value Theorem implies that there exists $m \in [-B, B]$ such that $I(m) \leq I(y)$ for all $y \in \mathbb{R}$. Thus, we have shown the following.

**Theorem 2.3.** *If $f \in C(\Omega) \cap L^1(\Omega)$, then there exists $m \in \mathbb{R}$ such that $I(m) \leq I(y)$ for all $y \in \mathbb{R}$.*

We may further restrict $y$ to a subset $[a, b] \subset [-B, B]$ such that $f : \Omega \to [a, b]$ is surjective, except possibly at $a$ and $b$.

Next, we show that there exists a unique $m \in \mathbb{R}$ that minimizes $I(y)$.

**Theorem 2.4.** *If $f \in C(\Omega) \cap L^1(\Omega)$, then there exists a unique $m \in \mathbb{R}$ such that $I(m) < I(y)$ for all $y \neq m \in \mathbb{R}$.*

Proof. Suppose there exist two absolute minima denoted by $m_1$ and $m_2$ with $m_1 < m_2$ and $m_1, m_2 \in [a, b]$ defined as above. Thus, $I(m_1) = I(m_2)$. Now, we have

$$
\begin{aligned}
& I\left(\frac{m_1+m_2}{2}\right)-\frac{I\left(m_1\right)+I\left(m_2\right)}{2}= \\
& =\int_{\Omega}\left|\frac{f-m_1}{2}+\frac{f-m_2}{2}\right| \, d\lambda^n(x) \\
& \quad-\frac{1}{2}\left[\int_{\Omega}\left|f-m_1\right| d\lambda^n(x)+\int_{\Omega}\left|f-m_2\right| d\lambda^n(x)\right]= \\
& =\int_{\left\{f \leq m_1\right\} \cap \Omega}\left(\frac{m_1-f}{2}+\frac{m_2-f}{2}\right) \, d\lambda^n(x) \\
& \quad+\int_{\left\{f \geq m_2\right\} \cap \Omega}\left(\frac{f-m_1}{2}+\frac{f-m_2}{2}\right) \, d\lambda^n(x) \\
& \quad\quad+\int_{\left\{m_1<f<m_2\right\} \cap \Omega}\left|\frac{f-m_1}{2}+\frac{f-m_2}{2}\right| \, d\lambda^n(x) \\
& \quad\quad\quad-\frac{1}{2}\left[\int_{\left\{f \leq m_1\right\} \cap \Omega}\left(m_1-f\right) \, d\lambda^n(x)+\int_{\left\{f>m_1\right\} \cap \Omega}\left(f-m_1\right) \, d\lambda^n(x)\right] \\
& \quad\quad\quad\quad-\frac{1}{2}\left[\int_{\left\{f \leq m_2\right\} \cap \Omega}\left(m_2-f\right) \, d\lambda^n(x)+\int_{\left\{f>m_2\right\} \cap \Omega}\left(f-m_2\right) \, d\lambda^n(x)\right]= \\
& =\int_{\left\{m_1<f<m_2\right\} \cap \Omega}\left|\frac{f-m_1}{2}+\frac{f-m_2}{2}\right| \, d\lambda^n(x) \\
& \quad-\int_{\left\{m_1<f<m_2\right\} \cap \Omega} \frac{f-m_1}{2} \, d\lambda^n(x)-\int_{\left\{m_1<f<m_2\right\} \cap \Omega} \frac{m_2-f}{2} \, d\lambda^n(x)<0,
\end{aligned}
$$

where the last inequality follows from the continuity of $f$, the triangle equality (if we call $\frac{f-m_1}{2}=A$ and $\frac{f-m_2}{2}=B$, then $|A+B| \leq |A|+|B|$. However, equality holds if and only if $A$ and $B$ have the same sign; as $A>0$ and $B<0$ in our situation, we arrive at the strict inequality), and the fact that $\lambda^n\left(\{m_1<f<m_2\} \cap \Omega\right)>0$.

This contradicts the fact that $m_1$ and $m_2$ are absolute minima. In summary, we conclude that there exists a unique $m$ such that $I(m)<I(y)$ for all $y \in \mathbb{R} \setminus \{m\}$. □

The above approach gives an elementary proof of the existence of a unique minimizer to the function $I(y)=\int_{\Omega}|f-y| \, d\lambda^n(x)$ for any continuous, absolutely integrable function on an open, bounded, connected subset $\Omega$ of $\mathbb{R}^n$.

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