INROADS

Real Analysis Exchange
Vol. 33(1), 2007/2008, pp. 243–248

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TUBE-MEASURABILITY

Abstract

We investigate measurable sets of an outer measure defined using "tubes," and prove that the only tube-measurable sets are the tube-null sets and their complements.

In this note, we investigate measurable sets of an outer measure defined as follows.

Definition. In $\mathbb{R}^n$, let T_i denote an infinite tube of cross-sectional radius $r_i > 0$; that is, the closed r_i-neighbourhood of some straight line. For a set $E \subseteq \mathbb{R}^n$, we define its *tube-measure* by

$$\mu(E) := \inf\Big\{\sum_i \gamma_{n-1} r_i^{n-1} : \bigcup_i T_i \supseteq E\Big\},$$

where γ_{n-1} is the volume of the unit ball of $\mathbb{R}^{n-1}$. Call E *tube-null* if $\mu(E) = 0$.

A closely-related outer measure has been introduced by Carbery, Soria, and Vargas in connection with Fourier localisation. They showed that every tube-null set is a "set of divergence" for the localisation problem (see [2]). They observed that the tube-measure is very badly behaved in that Borel sets need not be measurable. Our main result is the following.

Theorem 1. *The only tube-measurable sets are the tube-null sets and their complements.*

Key Words: Measure
Mathematical Reviews subject classification: Primary 28A12
Received by the editors October 4, 2006
Communicated by: Udayan B. Darji

*Professor Wisewell passed away in July, 2007.

†Supported by a Royal Society Dorothy Hodgkin Fellowship

For the proof, we will need some estimates for the tube-measure of sets. Exact values are not known for even simple sets such as balls except in the case where $n = 2$ which corresponds to the famous plank problem [1].

Lemma 2. *For every set $E \subseteq \mathbb{R}^n$, we have the upper bound*

$$\mu(E) \leq \min |\operatorname{proj}(E)|,$$

where $\operatorname{proj}(E)$ *denotes a projection of E onto an $(n-1)$-dimensional subspace, and $|\cdot|$ denotes the Lebesgue outer measure in $\mathbb{R}^{n-1}$. For bounded sets E, we also have the lower bound*

$$\mu(E) \geq \frac{|E|}{\operatorname{diam}(E)},$$

where this time $|\cdot|$ denotes Lebesgue outer measure in $\mathbb{R}^n$.

The plank problem tells us that for convex $E \subseteq \mathbb{R}^2$, we actually have $\mu(E) = \min |\operatorname{proj}(E)|$.[1] However, the statement is not true for all convex bodies, with the tetrahedron in $\mathbb{R}^3$ providing a counterexample [1].

Proof of Lemma 2. The upper bound is obvious by covering E with parallel tubes. For the lower bound, note that for any tube T of cross-sectional radius r, we have $|E \cap T| \leq \operatorname{diam}(E)\gamma_{n-1}r^{n-1}$. So if $\bigcup_{i=0}^{\infty} T_i \supseteq E$, we have

$$|E| = \left|E \cap \bigcup_{i=0}^{\infty} T_i\right| \leq \sum_{i=0}^{\infty} |E \cap T_i| \leq \operatorname{diam}(E) \sum_{i=0}^{\infty} \gamma_{n-1} r_i^{n-1},$$

from which the inequality follows by taking the infimum. □

Next we show that the upper bound is the exact value in the case of sets that are Cartesian products with $\mathbb{R}$.

Lemma 3. *If $A \subseteq \mathbb{R}^{n-1}$, then $\mu(A \times \mathbb{R}) = |A|$.*

Proof. The upper bound is immediate from the previous lemma. For the lower bound, observe that for all $R > 0$, we have

$$\mu(A \times \mathbb{R}) \geq \mu(A \times [-R, R]) \geq \frac{2R|A|}{2R + \operatorname{diam}(A)}$$

by the previous lemma, which goes to $|A|$ as $R \to \infty$. □

[1] David Fremlin proved this statement for balls, and simplified our proof in his unpublished paper "Tube outer measure." Although his proof is not published, it is available from his website [3].

In particular, the μ-measure of a single tube is exactly its cross-sectional area, as expected. We are now ready to prove the theorem.

PROOF OF THE MAIN THEOREM. Let $E \subseteq \mathbb{R}^n$ be μ-measurable and, for a contradiction, suppose that both $\mu(E) > 0$ and $\mu(\mathbb{R}^n \setminus E) > 0$. Choose a ball for which $\mu(E \cap \text{ball}) > 0$. Then for any $\varepsilon > 0$, we can find a family of tubes T_i covering $E \cap \text{ball}$ such that

$$(1-\varepsilon)\sum_i \mu(T_i) < \mu(E \cap \text{ball}) \leq \sum_i \mu(E \cap \text{ball} \cap T_i).$$

Therefore there exists a tube $T = T_i$ with

$$(1-\varepsilon)\mu(T) \leq \mu(E \cap \text{ball} \cap T) \leq \mu(E \cap T).$$

Subdivide this T into the union of countably many non-overlapping "square tubes" $T = \bigcup R_i$ (where each R_i is a shifted and rotated copy of $[-\delta_i, \delta_i]^{n-1} \times \mathbb{R}$). Without loss of generality, we can assume that $\delta_i \in \mathbb{Q}$ for all i. Then, using Lemma 3,

$$(1-\varepsilon)\mu(T) = (1-\varepsilon)\sum_i \mu(R_i), \tag{1}$$

hence

$$(1-\varepsilon)\sum_i \mu(R_i) \leq \mu(E \cap T) \leq \sum_i \mu(E \cap R_i).$$

Therefore we can choose a square tube $R = R_i$ with

$$(1-\varepsilon)\mu(R) \leq \mu(E \cap R). \tag{2}$$

We can similarly do this for $\mathbb{R}^n \setminus E$. The two square tubes we have found may be of different widths, but since both of the widths are rational, we can subdivide as before into the union of square tubes of some common, smaller width $\delta > 0$, and from each of the two collections, select a tube still satisfying (2).

So we now have some $\delta > 0$ (which depends on ε) and two copies of $[-\delta, \delta]^{n-1} \times \mathbb{R}$ that we denote by R_1 and R_2 such that

$$\begin{aligned}(1-\varepsilon)\mu(R_1) &\leq \mu(R_1 \cap E)\\ (1-\varepsilon)\mu(R_2) &\leq \mu(R_2 \setminus E).\end{aligned} \tag{3}$$

Choose disjoint δ-balls $B_1 \subseteq R_1$, $B_2 \subseteq R_2$. We now want to pass from balls to very eccentric sets because for such sets the upper and lower bounds of Lemma 2 are almost equal. So into each ball B_i ($i = 1, 2$), place a cuboid

C_i of diameter 2δ with $n-1$ of its edges all equal to some small $\eta > 0$ to be chosen later. By Pythagoras' Theorem, their measure is

$$|C_1| = |C_2| = \eta^{n-1}\sqrt{4\delta^2 - (n-1)\eta^2}.$$

Orient the two cuboids within the balls so that they both lie in the Cartesian product of $\mathbb{R}$ with a common cube of side η. Then

$$\begin{aligned}
\eta^{n-1} &\geq \mu(C_1 \cup C_2) && \text{by Lemma 2} \\
&= \mu\big((C_1 \cup C_2) \cap E\big) + \mu\big((C_1 \cup C_2) \setminus E\big) && \text{by measurability of } E \\
&\geq \mu(C_1 \cap E) + \mu(C_2 \setminus E) && \text{by monotonicity} \\
&= \mu(C_1) - \mu(C_1 \setminus E) + \mu(C_2) - \mu(C_2 \cap E) && \text{by measurability of } E.
\end{aligned}$$

Now by Lemma 2, we have $\mu(C_2) = \mu(C_1) \geq |C_1|/\operatorname{diam}(C_1) = |C_1|/2\delta$. Also, by measurability of E and (3),

$$\begin{aligned}
\mu(C_1 \setminus E) \leq \mu(R_1 \setminus E) \leq \varepsilon\mu(R_1) = \varepsilon(2\delta)^{n-1} \\
\mu(C_2 \cap E) \leq \mu(R_2 \cap E) \leq \varepsilon\mu(R_2) = \varepsilon(2\delta)^{n-1}.
\end{aligned}$$

Putting these together, we find

$$\eta^{n-1} \geq 2\Big(\frac{|C_1|}{2\delta} - \varepsilon(2\delta)^{n-1}\Big),$$

that is,

$$1 \geq 2\Big(\frac{|C_1|}{2\delta\eta^{n-1}} - \varepsilon(2\delta/\eta)^{n-1}\Big). \tag{4}$$

We must choose suitable values of the parameters so that this is a contradiction. Note that

$$|C_1|/2\delta\eta^{n-1} = \sqrt{1-(n-1)(\eta/2\delta)^2}.$$

So we choose our ε and δ as follows. First, let

$$p < \sqrt{\frac{3}{4(n-1)}}$$

so that

$$\sqrt{1-(n-1)p^2} > 1/2.$$

Choose ε so small that

$$\sqrt{1-(n-1)p^2} - \varepsilon/p^{n-1} > 1/2.$$

Then in the above construction, use this ε and let $\eta = 2\delta p$. Then (4) gives

$$1 \geq 2\big(\sqrt{1-(n-1)(\eta/2\delta)^2} - \varepsilon(2\delta/\eta)^{n-1}\big) > 1,$$

a contradiction. $\square$

References

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