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# ARE THE KANTOROVITCH POLYNOMIALS AREA DIMINISHING?

**Abstract**

The Bernstein-Bezier polynomials are known to possess total variation and length diminishing properties in one variable. We investigate the two dimensional generalizations to the square and the triangle. Simple counterexamples show that they do not diminish surface area. We consider Kantorovitch polynomials which seem to be a better choice to be area diminishing. A counterexample is given for the square. We then define the Kantorovitch polynomials on the triangle and give an area estimate for them.

## 1 Bernstein Type Polynomials in One Variable.

The one variable Bernstein polynomials $(B_nf)(x)$ of a function defined on $[0,1]$ are given by

$$(B_nf)(x) \equiv \sum_{r=0}^{n} f\left(\frac{r}{n}\right)p_{n,r}(x) \tag{1}$$

where

$$p_{n,r}(x) \equiv \binom{n}{r} x^r(1-x)^{n-r}.$$

For continuous $f$, the $B_nf$ converge uniformly to $f$ providing the standard proof of the Weierstrass theorem.We are interested, however, in the smoothing aspects of these polynomials.We will need three properties of the $B_nf$:

$$\sum_{i=0}^{n} p_{n,r}(x) = (x + (1-x))^n = 1 \tag{2}$$

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$$\int_0^1 p_{n.r}(x)\,dx = \frac{1}{n+1},\ \forall\, n, r \tag{3}$$

$$(B_n' f)(x) = \sum_{r=0}^{n-1}\Big(f\Big(\frac{r+1}{n}\Big) - f\Big(\frac{r}{n}\Big)\Big)p_{n-1,r}(x) \tag{4}$$

Using (3) and (4), we get the variation diminishing property:

$$VB_nf = \int_0^1 |(B_n' f)(x)|\ dx = \int_0^1 \Big| n\sum_{r=0}^{n-1}\Big(f\Big(\frac{r+1}{n}\Big) - f\Big(\frac{r}{n}\Big)\Big)p_{n-1,r}(x)\Big|\, dx \le$$

$$n\sum_{r=0}^{n-1}\Big|f\Big(\frac{r+1}{n}\Big) - f\Big(\frac{r}{n}\Big)\Big|\int_0^1 p_{n-1,r}(x)\,dx = \sum_{r=0}^{n-1}\Big|\, f\Big(\frac{r+1}{n}\Big) - f\Big(\frac{r}{n}\Big)\Big| \le Vf.$$

Schoenberg [9] showed that $B_nf$ have the following beautiful property: Let $Zf$ be the number of zeros of $f$, possibly infinite, on $[0, 1]$. Then

$$Z((B_nf)(x) - (ax + b)) \le Z(f(x) - (ax + b))$$

for all lines $y = ax + b$; i.e., $B_nf$ crosses every line no more often than $f$ does. The proof is based on Descartes' rule of signs. This property allows one to prove that the arc length $LB_nF \le LF$ for parametric space curves $F \equiv (f_1, f_2, f_3)$ where $B_nF \equiv (B_nf_1, B_nf_2, B_nf_3)$. The proof uses classical integral geometry, e.g., the plane case is an application of Crofton's formula for the length of a curve C,

$$2L(C) = \int_0^\pi \int_0^\infty N_C(\theta, \rho)\,d\rho\,d\theta,$$

where $N_C$ counts the number of times a line $l(\theta, \rho)$ crosses C. Similar techniques show that $KB_nF \le KF$, where $K$ indicates total integral curvature [7].

For functions in $L_1$, Kantorovitch introduced polynomials

$$(K_nf)(x) \equiv \sum_{r=0}^{n} M_{n,r}p_{n,r}(x) \tag{5}$$

where

$$M_{n,r} \equiv (n + 1)\int_{\frac{r}{n+1}}^{\frac{r+1}{n+1}} f(u)\,du.$$

The Kantorovitch polynomials converge in $L_1$ for $f \in L_1$ and converge uniformly for continuous functions [6]. They also possess the length and variation diminishing properties. We now consider the two dimensional generalizations of both $B_nf$ and $K_nf$.

## 2 Bernstein Type Polynomials on the Unit Square.

We define the two parameter generalizations on the unit square $Q \equiv [0,1] \times [0,1]$:

$$(B_{n,m}f)(x,y) \equiv \sum_{r=0}^{n} \sum_{s=0}^{m} f\Big(\frac{r}{n}, \frac{s}{m}\Big) p_{n,r}(x) p_{m,s}(y) \tag{6}$$

and

$$(K_{n,m}f)(x,y) \equiv \sum_{r=0}^{n} \sum_{s=0}^{m} M_{n,r,m,s} p_{n,r}(x) p_{m,s}(y) \tag{7}$$

where

$$M_{n,r,m.s} \equiv (n+1)(m+1) \int_{\frac{r}{n+1}}^{\frac{r+1}{n+1}} \int_{\frac{s}{m+1}}^{\frac{s+1}{m+1}} f(u,v)\, dv\, du.$$

Again, $B_{n,m}f$ and $K_{n,m}f$ converge uniformly for continuous $f$, and $K_{n,m}f$ converges to $f$ in $L_1$. See [6]. Note that in both cases the two variable operator is the successive iteration of the one variable operators. We consider the surface area properties of these operators, recalling the difficulty caused by the fact that in the limit, the sum of the elementary areas of inscribed triangles may exceed the Lebesgue area of the approximated surface as in the classical Schwarz-Peano example [1]. To see that $B_{n,m}f$ are not area diminishing, define $f(x,y) = x^t y^t$ on $Q$ and note that $f$ is zero at three of the corners and one at $(1,1)$ so that $(B_{1,1}f)(x,y) = xy$ for all $t > 0$. The surface area $AB_{1,1}f \approx 1.28$ and even for $t = 2$, $Af \approx 1.19$. In fact, $Af \to 1$ as $t \to \infty$. This happens because $B_{n,m}f$ depends only on the discrete values $f(\frac{r}{n}, \frac{s}{m})$. A key distinction is that $K_{n,m}f$ depends on the integral means of $f$ and consequently involves the isoperimetric relationship between the area of $f$ and the volume bounded by $f$. For example, for $f$ as above and $t = 2$,

$$(K_{1,1}f)(x,y) = \frac{1}{144}(36xy + 6x + 6y + 1)$$

which gives $AK_{1,1}f \approx 1.03 < Af$. A second indicator that the Kantorovitch polynomials may be area diminishing is that they are derived from integral means which are area diminishing [10]. In fact, in the one variable case, we have

$$K_n f = B'_{n+1}\Big(\int_0^x f\Big)$$

so

$$K_{n,m}f = \frac{\partial^2 B_{n+1\,,\,m+1}}{\partial x \partial y}\Big(\int_0^x \int_0^y f\Big).$$

The Tonelli variations of $f$ are given by

$$V_x f \equiv \int_0^1 Vf(\cdot, y)\, dy,$$

where $V_x f(\cdot, y)$ is the linear variation of $f$ on $[0, 1]$ for a fixed $y$. Similarly, $V_y f$. This furnishes a third suggestive difference between the Bernstein polynomials and the Kantorovitch polynomials: Kantorovitch polynomials are Tonelli variation diminishing [8] (See also [4]). However, even for $f(x, y) = x^2 y^2$, $V_x B_{1,1} f = V_y B_{1,1} f = \frac{1}{2}$ with $V_x f = V_y f = \frac{1}{3}$. In Section 3, we construct a counterexample showing that $K_{n,m}$ is not area diminishing without some restriction on $n$ and $m$.

## 3 A Counterexample for $K_{n,m}$.

We construct a function $g(x, y)$ such that for some large $m$, $AK_{1,m} g > Ag$. First, we will approximate $g$ by the linear Kantorovitch polynomial in $x$, $(K_1^x g)(x, y)$ and show that its area exceeds that of $g$. The construction of $g$ is based on the apparently unrelated observation that, although the arc length of a parametric Bernstein curve is less than or equal to that of the approximated curve, this is not true if one only approximates one component. For example, let $d_1(t) = d_2(t) = t^2$ on $[0, 1]$ so that $(B_1 d)(t) = (B_2 d)(t) = t$. Thus, the arc lengths are $L(B_1 d_1, B_1 d_2) = L(d_1, d_2) = \sqrt{2}$ since the paths are linear and monotone from $(0, 0)$ to $(1, 1)$, but

$$L(B_1 d_1, d_2) = L(t, t^2) = \int_0^1 \sqrt{1 + 4t^2}\, dt \approx 1.479 > \sqrt{2}.$$

Now we modify this example to construct a function $f(t)$ such that

$$L((K_1 f)(t), (B_2 f)(t)) > L(f(t), f(t)).$$

Define $f(t)$ to be 0 on $[0, .5]$, $10000(-200t^3 + 303t^2 - 153t + 25.75)$ on $[.5, .51]$ and 1 on $[.51, 1]$. The cubic is chosen to make $f(.5) = f'(.5) = 0, f(.51) = 1$, $f'(.51) = 0$ so that $f$ is $C^1$ and monotone nondecreasing, and that $f''$ exists and is continuous except at .5 and .51. We have

$$(K_1 f)(t) = 2\Big(\int_0^{.5} f(u) du\Big)(1 - t) + 2\Big(\int_{.5}^1 f(u)\, du\Big)t \approx .990t,$$

and $(B_2 f)(t) = 0(1 - t)^2 + 2(0)t(1 - t) + 1t^2 = t^2$. Thus, $L(f(t), f(t)) = \sqrt{2}$, but

$$L((K_1 f)(t), (B_2 f)(t)) \approx \int_0^1 \sqrt{(.990)^2 + 4t^2}\, dt \approx 1.472 > \sqrt{2} = L(f(t), f(t)).$$

Note that since $(K_n f')(t) = (B'_{n+1} f)(t)$, we can replace $(B'_2 f)(t)$ by $(K_1 f')(t)$. Thus, the length inequality may be written

$$\int_0^1 \sqrt{((K'_1 f)(t))^2 + ((K_1 f')(t))^2}\, dt > \int_0^1 \sqrt{(f'(t))^2 + (f'(t))^2}\, dt.$$

We may perturb the inequality slightly so that it becomes

$$\int_0^1 \sqrt{\epsilon^2 + ((K'_1 f)(t))^2 + ((K_1 f')(t))^2}\, dt > \int_0^1 \sqrt{\epsilon^2 + (f'(t))^2 + (f'(t))^2}\, dt.$$

In particular, $\epsilon = .01$ will work. Using the linearity of the Kantorovitch operator and its derivative, this may be rewritten as

$$\int_0^1 \sqrt{1 + (100(K'_1 f)(t))^2 + (100(K_1 f')(t))^2}\, dt > \int_0^1 \sqrt{1 + (100 f'(t))^2 + (100 f'(t))^2}\, dt.$$

Now, we define $g(x, y) = 100(f(x) + f'(x)y)$ on $[0, 1] \times [0, \delta]$ with $\delta$ to be determined. Thus,

$$\frac{\partial g}{\partial x} = 100[f'(x) + f''(x)y] \text{ and } \frac{\partial g}{\partial y} = 100 f'(x).$$

Also

$$(K_1^x g)(x, y) = 100[(K_1 f)(x) + (K_1 f')(x)y]$$

so that

$$\frac{\partial K_1^x g}{\partial x} = 100[(K_1 f')(x) + (K'_1 f)(x)y] \text{ and } \frac{\partial K_1^x g}{\partial y} = 100(K_1 f')(x).$$

With this notation, the last inequality becomes

$$\int_0^1 \sqrt{1 + \Big(\frac{\partial K_1^x g}{\partial x}\Big)^2 + \Big(\frac{\partial K_1^x g}{\partial y}\Big)^2}\, dx > \int_0^1 \sqrt{1 + \Big(\frac{\partial g}{\partial x}\Big)^2 + \Big(\frac{\partial g}{\partial y}\Big)^2}\, dx$$

evaluated at $y = 0$. This inequality remains true on some interval $0 \le y \le \delta$ because the integrals are continuous in $y$. For convenience, choose an integer

$i$ such that $\frac{1}{2i} < \delta$, and now define $g(x, y)$ on $[0, 1] \times [\frac{1}{2i}, \frac{1}{i}]$ by reflection; i.e., $g(x, \frac{1}{2i} + a) = g(x, \frac{1}{2i} - a)$. Now extend $g$ continuously to $Q$ by repetition on the remaining strips $[0, 1] \times [\frac{j}{i}, \frac{j+1}{i}]$ for $j = 2, \cdots, (i - 1)$. Note that $g$ is continuous everywhere and is $C^1$ except along the lines $x = .5, x = .51$, and $y = \frac{j}{2i}$, and that the inequality holds for all $y \neq \frac{j}{2i}$. If we integrate the last inequality with respect to $y$ between 0 and 1, it yields the surface area inequality $AK_1^x g > Ag$.

Now we approximate $K_1^x g$ by $K_m^y(K_1^x g) = K_{1,m} g$. Since for each $(x, y)$, $K_{1,m} g$ converges to the continuous function $K_1^x g$, the convergence is uniform on Q. Lebesgue area is lower semicontinuous so that $\liminf\limits_{m\to\infty} AK_{1,m} g \geq AK_1^x g$. Now let $\gamma < AK_1^x g - Ag$. Then there is an $M > 0$ such that for $m > M$,

$$AK_{1,m} g > \liminf AK_{1,m} g - \gamma \geq AK_1^x g - \gamma > Ag.$$

This counterexample leads however to the question of whether $K_{n,m}$ may be area diminishing if some restrictions are put on $n$ and $m$. The Peano-Schwarz example shows that if one restricts the ratio of the base $\frac{1}{n}$ and height $\frac{1}{m}$ to be bounded, then the elementary areas of the inscribed triangles approach the Lebesgue area. On the other hand, the Fefferman example on multiple Fourier series [3] shows that even if $n$ and $m$ satisfy $\frac{1}{2} < \frac{m}{n} < 2$, there is a function in $L_2$ whose Fourier series does not converge anywhere, but if $|m|$ and $|n| < W$ as $W \to \infty$, the Fourier series converges a.e. This suggests two possibilities to consider for Kantorovitch polynomials: $\alpha < \frac{m}{n} < \beta$ and $n = m$.

## 4 Bernstein Type Polynomials on the Triangle.

We now consider the generalizations of $B_n$ and $K_n$ to the triangle. The domain will be

$$T \equiv \{(x, y) | x \geq 0, y \geq 0, x + y \leq 1\}$$

with domain area $\frac{1}{2}$. The triangular Bernstein polynomials are defined as a one parameter sequence

$$(B_n^T f)(x, y) \equiv \sum_{i=0,j=0}^{i+j=n} f\Big(\frac{i}{n}, \frac{j}{n}\Big) p_{nij}(x, y) \tag{8}$$

where

$$p_{nij}(x, y) \equiv \binom{n}{i, j} x^i y^j (1 - x - y)^{n-i-j}$$

with trinomial weights

$$\binom{n}{i, j} \equiv \frac{n!}{i! j! (n - i - j)!} \ .$$

Again, we have uniform convergence for continuous functions [6]. Several important papers discuss the convexity of $B_n^T f$ using barycentric coordinates on a general triangle instead of on $T$, [2, 4].

As in the square case, it is easy to find an $f$ such that $AB_n^T f > Af$. Let $f(x,y) = (1-x-y)^2$. Then

$$Af = \int_0^1 \int_0^{1-x} \sqrt{1+8(1-x-y)^2}\, dy\, dx \approx .728.$$

But $(B_1^T f)(x) = 1-x-y$ so that $AB_1^T f = \frac{\sqrt{3}}{2} \approx .866$.

Kantorovitch polynomials have not been defined for triangular domains. We note that $B_n^T f$ contains $\frac{(n+1)(n+2)}{2}$ terms, and a difficulty arises if one tries to partition $T$ into that many equiarea pieces. It is natural, however, to partition $T$ into $(n+1)^2$ congruent triangles using the lines $x = \frac{i}{n+1}, y = \frac{j}{n+1}, x+y = \frac{i+j}{n+1}$ for $i \geq 0, j \geq 0, 1 \leq i+j \leq n$. Of these, $\frac{(n+1)(n+2)}{2}$, which we will denote by $T_{i,j}$, are oriented in the same manner as T, and the remaining $\frac{n(n+1)}{2}$, which will not be used, are rotated by $\pi$. One may visualize a triangular checkerboard pattern. Note that each Bernstein node $(\frac{i}{n}, \frac{j}{n})$ lies in

$$T_{i,j} = \left\{(x,y) \middle| x \geq \frac{i}{n+1}, y \geq \frac{j}{n+1}, \text{ and } x+y \leq \frac{i+j+1}{n+1}\right\}.$$

The Kantorovitch polynomials will be defined as

$$(K_n^T f)(x,y) \equiv \sum_{i=0,j=0}^{i+j=n} M_{n,i,j}^T p_{nij}(x,y) \tag{9}$$

where

$$M_{n,i,j}^T \equiv 2(n+1)^2 \int\int_{T_{ij}} f(u,v)\, dv\, du.$$

Then $K_n^T f$ converge uniformly for continuous $f$ because $|f(\frac{i}{n}, \frac{j}{n}) - M_{n,i,j}^T| < \epsilon$ for large $n$. $K_n^T f$ will also converge in $L_1$, for $f \in L_1$, using an $\frac{\epsilon}{3}$ argument.

We compute the surface area of $(K_1^T f)(x,y)$ for $f(x,y) = (1-x-y)^2$.

$$(K_1^T f)(x,y) = \frac{1}{24}(11 - 10x - 10y),$$

so $AK_1^T f = \frac{\sqrt{194}}{24} \approx .580 \leq Af$ in contrast to $B_1^T f$. The Tonelli variations on the triangle are

$$V_x f \equiv \int_0^1 \int_0^{1-x} \left|\frac{\partial f}{\partial x}\right| dy\, dx,$$

and similarly $V_y f$. As in the square case $B_n^T$, do not diminish Tonelli variation. For $f(x,y) = (1-x-y)^2, V_x f = V_y f = \frac{1}{3}$ while $V_x B_1^T f = V_y B_1^T f = \frac{1}{2}$.

In contrast to the square case, Kantorovitch polynomials also may not diminish Tonelli variation. On $T$, let

$$f(x,y) = \begin{cases} (1-2y)x & \text{for } 0 \le y \le .5 \\ 0 & \text{for } .5 \le y \le 1 \end{cases}$$

which is continuous along the line $y = \frac{1}{2}$. Then $(K_1^T f)(x,y) = \frac{1}{8} + \frac{1}{3}x - \frac{1}{8}y$ and

$$V_x f = \int_0^{\frac{1}{2}} \int_0^{1-x} (1-2y)\, dy\, dx = \frac{1}{12}\,.$$

But

$$V_x K_1^T f = \int_0^1 \int_0^{1-x} \frac{1}{3}\, dy\, dx = \frac{1}{6} > V_x f.$$

It should be noted that

$$V_y K_1^T f = \frac{1}{16} < \frac{7}{24} = V_y f$$

and that

$$AK_1^T f \approx .53 < .67 \approx Af.$$

## 5 Linear Functions and an Estimate for $AK_n^T f$.

In both the square and triangular cases, the Bernstein polynomials of linear functions are identical to the function, so they have the same surface area. In both cases, the Kantorovitch polynomials diminish the surface area of linear functions. We consider the triangular case. In what follows, the following properties of $K_n^T f$ are useful :

$$\sum_{i=0,j=0}^{i+j=n} p_{nij}(x,y) = [x+y+(1-x-y)]^n = 1 \tag{10}$$

$$\sum_{i=0,j=0}^{i+j=n} i p_{nij}(x,y) = nx \ \text{ and } \sum_{i=0,j=0}^{i+j=n} j p_{nij}(x,y) = ny \tag{11}$$

$$\int_0^1 \int_0^{1-x} p_{nij}(x,y)\, dy\, dx = \frac{1}{(n+1)(n+2)},\ \forall\, n,i,j \tag{12}$$

$$\frac{\partial K_n^T f}{\partial x} = \sum_{i=0,j=0}^{i+j=n-1} n(M_{n,i+1,j}^T - M_{n,i,j}^T) p_{n-1,ij}(x,y)$$
$$\frac{\partial K_n^T f}{\partial y} = \sum_{i=0,j=0}^{i+j=n-1} n(M_{n,i,j+1}^T - M_{n,i,j}^T) p_{n-1,ij}(x,y) \tag{13}$$

(The partials of $B_n^T f$ are the same as those of $K_n^T f$ except $f(\frac{i}{n}, \frac{j}{n})$ replaces $M_{n,i,j}^T$ .) The proofs are natural extensions of the one variable case found in [6]. Now suppose $f(x,y) = ax + by + c$, so that

$$Af = \int\int_T \sqrt{1 + a^2 + b^2}\, dy\, dx = \frac{1}{2}\sqrt{1 + a^2 + b^2}.$$

Then

$$(K_n^T f)(x,y) = \sum_{i=0,j=0}^{i+j=n} \left[\frac{1}{3(n+1)}[a(3i+1) + b(3j+1)] + c\right] p_{nij}(x,y).$$

Using (11) and (12), we obtain

$$(K_n^T f)(x,y) = \frac{n}{n+1}(ax + by) + c^*$$

so that

$$AK_n^T f = \frac{1}{2}\sqrt{1 + \left(\frac{na}{n+1}\right)^2 + \left(\frac{nb}{n+1}\right)^2} < Af.$$

We conclude with an estimate for $AK_n^T f$ which depends on the convexity of the function $s(\alpha, \beta) = \sqrt{1 + \alpha^2 + \beta^2}$. The integrand of $AK_n^T f$ is the square root of

$$1 + \left(\sum_{i=0,j=0}^{i+j=n-1} n(M_{n,i+1,j}^T - M_{n,i,j}^T) p_{n-1,ij}(x,y)\right)^2 + \left(\sum_{i=0,j=0}^{i+j=n-1} n(M_{n,i,j+1}^T - M_{n,i,j}^T) p_{n-1,ij}(x,y)\right)^2.$$

Using Jensen's inequality and (10), this is less than or equal to

$$\sum_{i=0,j=0}^{i+j=n-1} \sqrt{1 + \left(n(M_{n,i+1,j}^T - M_{n,i,j}^T)\right)^2 + \left(n(M_{n,i,j+1}^T - M_{n,i,j}^T)\right)^2}\ \ p_{n-1,ij}(x,y).$$

Integrating this expression over $T$ and using (12), we get $AK_n^T f$ less than or equal to

$$\frac{1}{n(n+1)}\sum_{i=0,j=0}^{i+j=n-1}\sqrt{1+\left(n(M^T_{n,i+1,j}-M^T_{n,i,j})\right)^2+\left(n(M^T_{n,i,j+1}-M^T_{n,i,j})\right)^2}.$$

This estimate is not possible in the square case since the weight functions $p_{n,r}(x)$ and $p_{m,s}(y)$ are independent, so Jensen's inequality does not apply. If $f$ is continuous, we may use the mean value theorem for integrals to replace the differences by point values, e.g., in $x$ we get $n[f(x_i+\frac{1}{n+1},y_j)-f(x_i,y_j)]$ where $(x_i,y_j)\in T_{ij}$. If $f$ is differentiable, then we may further replace the point differences by, for example, $\frac{n}{n+1}\frac{\partial f}{\partial x}(x_i^*,y_j)$ where $x_i^*\in(x_i,\, x_i+\frac{1}{n+1})$.

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