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THE CENTERED COVERING MEASURES OF A CLASS OF SELF-SIMILAR SETS ON THE PLANE

Abstract

Let $S \subset R^2$ be the attractor of the iterated function system $\{f_1, f_2, f_3, f_4\}$, where $f_i(x) = \lambda_i x + b_i$, $i = 1, 2, 3, 4$, $x = (x_1, x_2) \in R^2$, $0 < \lambda_i \leq \frac{1}{2+\sqrt{2}}$,$b_1 = (0, 0)$, $b_2 = (1 - \lambda_2, 0)$, $b_3 = (1 - \lambda_3, 1 - \lambda_3)$, and $b_4 = (0, 1 - \lambda_4)$. This paper determines the exact centered covering measure of S under some conditions relating to the contraction parameters.

1 Introduction.

Computing and estimating the dimensions and measures of the fractal sets are one of the important problems in fractal geometry. Generally speaking, computing the Hausdorff and packing dimensions, especially the Hausdorff and packing measures, are very difficult. For a self-similar set satisfying the open set condition, we know that its Hausdorff dimension and packing dimension equal its self-similar dimension (see [6]), but there are almost no important results about the Hausdorff measure and packing measure except for sets like the Cantor set. The centered covering measure was introduced by Raymond and Tricot in [9]. The dimension defined by centered covering measure is equal to the Hausdorff dimension for any set and was used to give estimates for the

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multifractal spectrum of a measure. They have been used for other purposes as well and have recently become an object of study themselves (see [10]).

However, because of the difficulty in the definition, there are few results about the explicit computation of the above measures for fractal sets. Ayer and Strichartz in [1] discussed the computation of Hausdorff measure for line Cantor sets, and Dejun Feng in [3] gave a formula for packing measure for linear Cantor sets. In [13], we obtained the exact centered covering measure for the symmetry Cantor set. All of the above results are obtained for the self-similar sets on the line. This paper considers the exact centered covering measure for a class of self-similar sets on the plane.

Let $\delta > 0$, $s \geq 0$. Recall that a centered ball $\delta-$ cover of a given set $E \subset R^n$ is a $\delta-$ cover of E by the balls with center in E. Define

$$C_0^s(E) = \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty}(2r_i)^s : \{B(x_i, r_i)\}_{i>0} \text{ is a centered ball } \delta\text{-cover of } E\}$$

We call $C_0^s(E)$ the centered covering pre-measure of E. It is not a measure as $C_0^s(E)$ is not monotonic. However, it gives rise to a measure and is called the centered covering measure as follows:

$C^s(E) = sup\{C_0^s(F) : F \subset E\}$.

The following properties of the above measure can be found in [5],[9],[10], and [11]. Note that here $H^s(E)$ denotes the $s-$ dimensional Hausdorff measure of E.

Lemma 1.1. *Let $s \geq 0$, E be a subset of R^n, then*

(i) $2^{-s}C^s(E) \leq H^s(E) \leq C^s(E)$.

(ii) $\forall \lambda > 0$, $H^s(\lambda E) = \lambda^s H^s(E)$, $C^s(\lambda E) = \lambda^s C^s(E)$, *where* $\lambda E = \{\lambda x : x \in E\}$.

(iii) $H^s(E)$, $C^s(E)$ *are metric outer measures and Borel regular.*

It is easy to see that for any sets $E \subset R^n$, there is a number $dim_C(E)$, called the centered covering dimension of E such that

$dim_C(E) = inf\{s : C^s(E) = 0\} = sup\{s : C^s(E) = \infty\}$.

From Lemma 1.1, we have $dim_H(E) = dim_C(E)$ for any set E, where $dim_H(E)$ denotes the Hausdorff dimension of E.

Let μ be a finite measure defined on the Borel sets $E \subset R^n$, for $x \in R^n$, and the centered upper spherical density of x with respect to μ is defined by

$$\overline{D}^s(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{(2r)^s}. \tag{1.1}$$

The relation between the upper spherical density and the centered covering measure is given in the following Lemma.

Lemma 1.2. *[5][9] Let $E \subset R^n$, $C^s(E)$, $\overline{D}^s(\mu, x)$ defined as above. If we replace μ in (1.1) by $C^s(.)$ and $C^s(E) < \infty$, then $\overline{D}^s(C^s, x) = 1$ for C^s-almost all $x \in E$.*

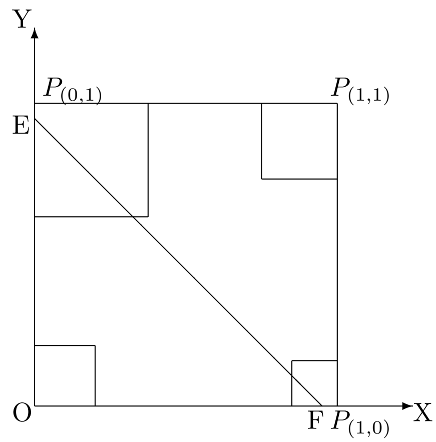

Figure 1

Let S_0 be the unit square on the plane. We establish an orthogonal coordinate system as follows. Take the origin to be a vertex of S_0 (see Figure 1), and $\{f_1, f_2, f_3, f_4\}$ be the iterated function system iterating on S_0, where $f_i(x) = \lambda_i x + b_i$, $i = 1, 2, 3, 4$, $x = (x_1, x_2) \in R^2$, $0 < \lambda_i \leq \frac{1}{2+\sqrt{2}}$, $b_1 = (0, 0)$, $b_2 = (1 - \lambda_2, 0)$, $b_3 = (1 - \lambda_3, 1 - \lambda_3)$, and $b_4 = (0, 1 - \lambda_4)$. Then there exists an unique non-empty compact set S that satisfies [6]

$$S = \bigcup_{i=1}^{4} f_i(S).$$

If the set S is a self-similar set satisfying the strong separation condition (that is, $f_i(S) \cap f_j(S) = \emptyset$ for $i \neq j$), then $dim_H(S) = dim_C(S) = s$, where s is the unique positive solution of the equation

$$\sum_{i=1}^{4} \lambda_i^s = 1. \tag{1.2}$$

Moreover, $0 < H^s(S) \leq C^s(S) < \infty$. That is, the above measures are positive and finite [6].

Denote by μ the unique probability measure satisfying the self-similar relation

$$\mu = \sum_{i=1}^{4} \lambda_i^s \mu \circ f_i^{-1}. \tag{1.3}$$

By the scaling property of $C^s(S)$, $\mu = c_0 C^s|_S$, where c_0 is a constant. Then the self-similar set S is the support of μ.

We first give some notation. Denote by $P_{(x,y)}$ the point with coordinate (x,y). In Figure 1, EF is a straight line parallel to the diagonal $P_{(0,1)}P_{(1,0)}$. Set $t = dist(P_{(0,0)}, EF)$, $\Delta(P_{(0,0)}, t) = S_0 \cap (\Delta P_{(0,0)}EF)$ when $0 < t \leq \frac{\sqrt{2}}{2}$ and $\Delta(P_{(0,0)}, t) = S_0 - (S_0 \cap (\Delta P_{(1,1)}EF))$ when $\frac{\sqrt{2}}{2} \leq t < \sqrt{2}$, where $\Delta P_{(x,y)}EF$ denotes the triangle with the vertexes $P_{(x,y)}$, E, and F. Let μ be determined by (1.3), take $d^1_{min} = \inf\limits_{0<t\leq\sqrt{2}} \frac{\mu(\Delta(P_{(0,0)},t))}{t^s}$. We also define $d^2_{min} = \inf\limits_{0<t\leq\sqrt{2}} \frac{\mu(\Delta(P_{(1,0)},t))}{t^s}$, $d^3_{min} = \inf\limits_{0<t\leq\sqrt{2}} \frac{\mu(\Delta(P_{(1,1)},t))}{t^s}$ and $d^4_{min} = \inf\limits_{0<t\leq\sqrt{2}} \frac{\mu(\Delta(P_{(0,1)},t))}{t^s}$ in the same way.

Lemma 1.3. *Let $0 < \lambda_k \leq \frac{1}{2+\sqrt{2}}$, $k = 1,2,3,4$, s be determined as in (1.2), and $s < 1$. Then*

$$d^k_{min} = min\left\{\frac{\sqrt{2^s}\lambda_k^s}{(min\{1-\lambda_i, 1-\lambda_j\})^s}, \frac{\lambda_k^s + \lambda_i^s + \lambda_j^s}{(\sqrt{2}(1-\lambda_p))^s}\right\},$$

where $(k,i,j,p) \in \{(1,2,4,3),(2,1,3,4),(3,2,4,1),(4,1,3,2)\}$.

PROOF. Without loss of the generality, we only prove the case when $(k,i,j,p) = (1,2,4,3)$. By self-similarity, we only need to consider the interval $(0, \sqrt{2}]$ with $\sqrt{2}\lambda_1 < t \leq \sqrt{2}$. We shall prove that the infimum in the definition of d^1_{min} could not be attained when EF intersects $f_j(S_0)$ in the interior of $f_j(S_0)$ (j=2,3,4). As $0 < \lambda_k \leq \frac{1}{2+\sqrt{2}}$, the straight line EF doesn't intersect with three of $f_1(S_0)$, $f_2(S_0)$, $f_3(S_0)$, and $f_4(S_0)$.

If there exists $t_0 \in (min\{\frac{1-\lambda_2}{\sqrt{2}}, \frac{1-\lambda_4}{\sqrt{2}}\}, max\{\frac{1+\lambda_2}{\sqrt{2}}, \frac{1+\lambda_4}{\sqrt{2}}\})$ such that

$$\frac{\mu(\Delta(P_{(0,0)}, t_0))}{t_0^s} = \frac{\sqrt{2^s}\lambda_1^s}{(min\{1-\lambda_2, 1-\lambda_4\})^s},$$

EF must intersect at least one of $f_2(S_0)$, $f_4(S_0)$ in their interiors when $t = t_0$. Define $t_2 = dist(P_{(1-\lambda_2,0)}, EF)$ and $t_4 = dist(P_{(0,1-\lambda_4)}, EF)$. Note that if EF doesn't intersect $f_k(S_0)$ in its interior, we take $t_k = 0$, where $k = 2$ when $\lambda_2 \leq \lambda_4$ and $k = 4$ when $\lambda_2 \geq \lambda_4$. Then

$$\begin{aligned}
\frac{\mu(\Delta(P_{(0,0)},t_0))}{t_0^s} &\geq \frac{\lambda_1^s + max\{\mu(\Delta(P_{(1-\lambda_2,0)},t_2)),\mu(\Delta(P_{(0,1-\lambda_4)},t_4))\}}{(min\{\frac{1-\lambda_2}{\sqrt{2}},\frac{1-\lambda_4}{\sqrt{2}}\}+max\{t_2,t_4\})^s} \\
&> \frac{\lambda_1^s + max\{\mu(\Delta(P_{(1-\lambda_2,0)},t_2)),\mu(\Delta(P_{(0,1-\lambda_4)},t_4))\}}{(min\{\frac{1-\lambda_2}{\sqrt{2}},\frac{1-\lambda_4}{\sqrt{2}}\})^s+(max\{t_2,t_4\})^s} \\
&\geq min\{\frac{\lambda_1^s}{(min\{\frac{1-\lambda_2}{\sqrt{2}},\frac{1-\lambda_4}{\sqrt{2}}\})^s},\frac{max\{\mu(\Delta(P_{(1-\lambda_2,0)},t_2)),\mu(\Delta(P_{(0,1-\lambda_4)},t_4))\}}{(max\{t_2,t_4\})^s}\} \\
&= min\{\frac{\sqrt{2}^s\lambda_1^s}{(min\{1-\lambda_2,1-\lambda_4\})^s},\frac{max\{\mu(\Delta(P_{(1-\lambda_2,0)},t_2)),\mu(\Delta(P_{(0,1-\lambda_4)},t_4))\}}{(max\{t_2,t_4\})^s}\},
\end{aligned}$$

which contradicts the definition of d_{min}^1.

If there exists $t_0 \in (\sqrt{2}(1-\lambda_3),\sqrt{2})$ such that

$$\frac{\mu(\Delta(P_{(0,0)},t_0))}{t_0^s} = \frac{\lambda_1^s+\lambda_2^s+\lambda_4^s}{(\sqrt{2}(1-\lambda_3))^s},$$

EF must intersect $f_3(S_0)$ in its interior when $t=t_0$. Define

$$t_3 = dist(P_{(1-\lambda_3,1-\lambda_3)},EF).$$

Then

$$\begin{aligned}
\frac{\mu(\Delta(P_{(0,0)},t_0))}{t_0^s} &= \frac{\lambda_1^s+\lambda_2^s+\lambda_4^s+\mu(\Delta(P_{(1-\lambda_3,1-\lambda_3)},t_3))}{(\sqrt{2}(1-\lambda_3)+t_3)^s} \\
&> \frac{\lambda_1^s+\lambda_2^s+\lambda_4^s+\mu(\Delta(P_{(1-\lambda_3,1-\lambda_3)},t_3))}{(\sqrt{2}(1-\lambda_3))^s+t_3^s} \\
&\geq min\{\frac{\lambda_1^s+\lambda_2^s+\lambda_4^s}{\sqrt{2}^s(1-\lambda_3)^s},\frac{\mu(\Delta(P_{(1-\lambda_3,1-\lambda_3)},t_3))}{t_3^s}\},
\end{aligned}$$

which contradicts the definition of d_{min}^1. This completes the proof.

□

We now give the main result of this paper.

Main Theorem. *Let $0<\lambda_t<\frac{1}{2+\sqrt{2}}$, $t=1,2,3,4$, $0<s<1$. If the parameters λ_1, λ_2, λ_3, λ_4 satisfy the following conditions*

(i) $2\lambda_p \leq (\frac{(1-\lambda_k)d_{min}^p}{\sqrt{2}^s})^{\frac{1}{1-s}}$, *where* $k\neq p$, $k,p\in\{1,2,3,4\}$,

(ii) $\frac{\lambda_k^s+\lambda_i^s+\lambda_j^s}{(1-\lambda_k)^s} \leq \frac{1}{\sqrt{2}^s}$, *where* $k\neq i$, $k\neq j$, $i\neq j$, $k,i,j\in\{1,2,3,4\}$,

then

$$C^s(S) = D_{max}^{-1},$$

where

$$D_{max} = \max_{1\le t\le 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_t))^s}\}.$$

The proof of the main theorem will be given in next Section, we first give an example that the exact centered covering measure can be determined by the main theorem.

Example. Let $\lambda_1 = \lambda_3 = \frac{1}{400}$ and $\lambda_2 = \lambda_4 = \frac{1}{20}$ for the convenience of computation. Then $s = log_{20}(\sqrt{3}+1)$, $\frac{1}{\sqrt{2^s}} \approx 0.8902$ and the following can be easily computed.

$$d_{min}^1 = d_{min}^3 = (\frac{\sqrt{2}}{20\times 19})^s, \text{ and } d_{min}^2 = d_{min}^4 = (\frac{20\sqrt{2}}{399})^s.$$

$$2\lambda_1 = 2\lambda_3 = \frac{1}{200} = 0.0050, \text{ and } \ 2\lambda_2 = 2\lambda_4 = \frac{1}{10} = 0.1000.$$

$$(\frac{(1-\lambda_3)d_{min}^1}{\sqrt{2^s}})^{\frac{1}{1-s}} = (\frac{(1-\lambda_1)d_{min}^3}{\sqrt{2^s}})^{\frac{1}{1-s}} \approx 0.0496,$$

$$(\frac{(1-\lambda_4)d_{min}^2}{\sqrt{2^s}})^{\frac{1}{1-s}} = (\frac{(1-\lambda_2)d_{min}^4}{\sqrt{2^s}})^{\frac{1}{1-s}} \approx 0.2043.$$

$$(\frac{(1-\lambda_i)d_{min}^j}{\sqrt{2^s}})^{\frac{1}{1-s}} = (\frac{(1-\lambda_u)d_{min}^v}{\sqrt{2^s}})^{\frac{1}{1-s}} \approx 0.0461 \text{ for } i,u\in\{2,4\} \text{ and } j,v\in\{1,3\}.$$

$$(\frac{(1-\lambda_i)d_{min}^j}{\sqrt{2^s}})^{\frac{1}{1-s}} = (\frac{(1-\lambda_u)d_{min}^v}{\sqrt{2^s}})^{\frac{1}{1-s}} \approx 0.2198 \text{ for } i,u\in\{1,3\} \text{ and } j,v\in\{2,4\}.$$

$$\frac{\lambda_1^s+\lambda_2^s+\lambda_4^s}{(1-\lambda_1)^s} = \frac{\lambda_3^s+\lambda_2^s+\lambda_4^s}{(1-\lambda_3)^s} \approx 0.8668,$$

$$\frac{\lambda_2^s+\lambda_1^s+\lambda_3^s}{(1-\lambda_2)^s} = \frac{\lambda_4^s+\lambda_1^s+\lambda_3^s}{(1-\lambda_4)^s} \approx 0.6450.$$

$$\frac{\lambda_1^s+\lambda_2^s+\lambda_3^s}{(1-\lambda_1)^s} = \frac{\lambda_3^s+\lambda_1^s+\lambda_2^s}{(1-\lambda_3)^s} = \frac{\lambda_1^s+\lambda_3^s+\lambda_4^s}{(1-\lambda_1)^s} = \frac{\lambda_3^s+\lambda_1^s+\lambda_4^s}{(1-\lambda_3)^s} \approx 0.6345.$$

$$\frac{\lambda_2^s+\lambda_3^s+\lambda_4^s}{(1-\lambda_2)^s} = \frac{\lambda_4^s+\lambda_2^s+\lambda_3^s}{(1-\lambda_4)^s} = \frac{\lambda_2^s+\lambda_1^s+\lambda_4^s}{(1-\lambda_2)^s} = \frac{\lambda_4^s+\lambda_1^s+\lambda_2^s}{(1-\lambda_4)^s} \approx 0.8811.$$

The above equalities indicate that all of the conditions of the main theorem are satisfied so that

$$D_{max} = max\{\frac{1}{(2\sqrt{2}(1-\lambda_1))^s}, \frac{1}{(2\sqrt{2}(1-\lambda_2))^s}\} = (\frac{10}{19\sqrt{2}})^s,$$

and thus

$$C^s(S) = (\frac{19\sqrt{2}}{10})^s.$$

2 Proof of the Main Theorem.

This section gives the proof of the main theorem. The desired statement follows from the following Lemmas (see [1], Lemma 2.2).

Lemma 2.1. *[1] Suppose* $0 < \alpha < 1$*,* $p \le p_0$*,* $a \ge a_0$ *and* $y \ge \lambda_0 x^\alpha$*. Then* $0 < x \le (\frac{a_0\lambda_0}{p_0})^{\frac{1}{1-\alpha}}$ *implies* $\frac{p-y}{(a-x)^\alpha} < \frac{p}{a^\alpha}$*.*

For any $P_{(x,y)} \in f_i(S_0)$, $i = 1,2,3,4$, and let $\overline{P}_{jq}^{(x,y)}$ be the farthest vertex of $f_j(S_0)$ from $P_{(x,y)}$, where $j \ne i$, $j \in \{1,2,3,4\}$, and the lower label jq in $\overline{P}_{jq}^{(x,y)}$ denotes $\{\overline{P}_{jq}^{(x,y)}\} = \lim_{n\to\infty} f_j \circ f_q^n(S_0)$. For example, in Figure 1, if $P_{(x,y)} = P_{(0,\lambda_1)} \in f_1(S_0)$, then $\overline{P}_{22}^{(0,\lambda_1)} = P_{(1,0)}$, $\overline{P}_{33}^{(0,\lambda_1)} = P_{(1,1)}$, $\overline{P}_{43}^{(0,\lambda_1)} = P_{(\lambda_4,1)}$, etc. With above notation, we have

Lemma 2.2. *Under the conditions of the main theorem, for any* $P_{(x,y)} \in f_k(S)$ *and any real* r *with* $\sqrt{2}\lambda_k \le r < \sqrt{2}$*,* $k = 1,2,3,4$*, we have*

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \le \frac{1}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{pq}^{(x,y)}))^s},$$

where $(k,p) \in \{(1,3),(2,4),(3,1),(4,2)\}$*. Moreover,*

$$\sup_{\sqrt{2}\lambda_k \le r < \sqrt{2}} \frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} = \frac{1}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{pq}^{(x,y)}))^s}.$$

PROOF. Without loss of the generality, we only prove the case when $(k,p) = (1,3)$. For any $P_{(x,y)} \in f_1(S_0)$, $\overline{P}_{jq}^{(x,y)}$ must be a vertex of $f_j(S_0)$, $j = 2,3,4$ (see Figure 2). If the circular arc of $B(P_{(x,y)}, r)$ intersects $f_j(S_0)$ in its interior, let

$$t_{jq} = sup\{t \in (0, \sqrt{2}\lambda_j] : \Delta(\overline{P}_{jq}^{(x,y)}, t) \cap B(P_{(x,y)}, r) = \emptyset\}, \quad q \in \{1,2,3,4\}.$$

Then $\Delta(\overline{P}_{jq}^{(x,y)}, t_{jq})$ intersects the circular arc of $B(P_{(x,y)}, r)$ at an unique point which is denoted by Q_j, and $Q_j \in f_j(S_0)$. It is easy to see that

$$r = dist(P_{(x,y)}, Q_j) \ge dist(P_{(x,y)}, \overline{P}_{jq}^{(x,y)}) - dist(\overline{P}_{jq}^{(x,y)}, Q_j).$$

Note that the angle between the sides of square and its diagonal is $\frac{\pi}{4}$. Thus, the above inequality implies

$$r = dist(P_{(x,y)}, Q_j) \geq dist(P_{(x,y)}, \overline{P}_{jq}^{(x,y)}) - \sqrt{2}t_{jq}. \tag{2.1}$$

We first note that $f_1(S_0) \subset B(P_{(x,y)}, r)$, and we may suppose that

$$dist(P_{(x,y)}, \overline{P}_{4q}^{(x,y)}) \leq dist(P_{(x,y)}, \overline{P}_{2q}^{(x,y)}) \leq dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)})$$

without loss of the generality. We continue our proof in the following four cases:

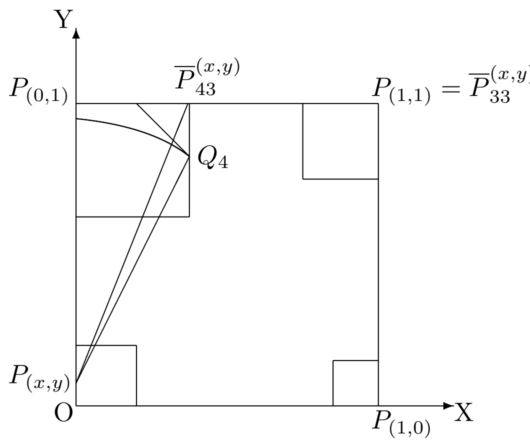

Figure 2

Case 1. When $\sqrt{2}\lambda_1 \leq r < dist(P_{(x,y)}, \overline{P}_{4q}^{(x,y)})$, then either $B(P_{(x,y)}, r) \cap f_{j_0}(S_0) = \emptyset (j_0 = 2, 3, 4)$, or the circular arc of $B(P_{(x,y)}, r)$ intersects only one of the squares $f_2(S_0)$, $f_3(S_0)$, $f_4(S_0)$, or the circular arc of $B(P_{(x,y)}, r)$ intersects two of the squares $f_2(S_0)$, $f_3(S_0)$, $f_4(S_0)$, or the circular arc of $B(P_{(x,y)}, r)$ intersects all of the squares $f_2(S_0)$, $f_3(S_0)$, $f_4(S_0)$.

If $B(P_{(x,y)}, r) \cap f_{j_0}(S_0) = \emptyset (j_0 = 2, 3, 4)$, it is obvious that

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \frac{\lambda_1^s}{(2\sqrt{2}\lambda_1)^s} = \frac{1}{(2\sqrt{2})^s}. \tag{2.2}$$

If the circular arc of $B(P_{(x,y)}, r)$ intersects only one of the squares $f_2(S_0)$, $f_3(S_0)$, $f_4(S_0)$. With the above notation, $r = dist(P_{(x,y)}, Q_{j_0})$, $j_0 = 2, 3, 4$, and $\mu(B(P_{(x,y)}, r)) \leq \lambda_1^s + \lambda_{j_0}^s - \mu(\Delta(\overline{P}_{j_0 q}^{(x,y)}, t_{j_0 q}))$. Combining with (2.1), we obtain

$$\frac{\mu(B(P_{(x,y)},r))}{(2r)^s} \leq \frac{\lambda_1^s+\lambda_{j_0}^s-\mu(\Delta(\overline{P}_{j_0q}^{(x,y)},t_{j_0q}))}{(2\cdot(dist(P_{(x,y)},\overline{P}_{j_0q}^{(x,y)})-\sqrt{2}t_{j_0q}))^s}$$
$$= \frac{\lambda_1^s+\lambda_{j_0}^s-\mu(\Delta(\overline{P}_{j_0q}^{(x,y)},t_{j_0q}))}{(2\cdot dist(P_{(x,y)},\overline{P}_{j_0q}^{(x,y)})-2\sqrt{2}t_{j_0q})^s}.$$

We take in Lemma 2.1, $\alpha = s$, $p_0 = p = \lambda_1^s + \lambda_{j_0}^s$, $a = 2\cdot dist(P_{(x,y)},\overline{P}_{j_0q}^{(x,y)})$, $a_0 = 2(1-\lambda_1)$ when $j_0 = 2,4$ and $a_0 = 2\sqrt{2}(1-\lambda_1)$ when $j_0 = 3$, $y = \mu(\Delta(\overline{P}_{j_0q}^{(x,y)},t_{j_0q}))$, $x = 2\sqrt{2}t_{j_0q}$. By Lemma 1.3, we have

$$\frac{y}{x^\alpha} = \frac{\mu(\Delta(\overline{P}_{j_0q}^{(x,y)},t_{j_0q}))}{(2\sqrt{2}t_{j_0q})^s} \geq \frac{1}{2^s\sqrt{2^s}}d_{min}^q.$$

Take λ_0 the right hand side of the above inequality, and condition (i) of the main theorem ensures $2\sqrt{2}t_{j_0q} \leq (\frac{a_0\lambda_0}{p_0})^{\frac{1}{1-s}}$. By Lemma 2.1, we have

$$\frac{\mu(B(P_{(x,y)},r))}{(2r)^s} \leq \frac{\lambda_1^s+\lambda_{j_0}^s}{(2\cdot dist(P_{(x,y)},\overline{P}_{j_0q}^{(x,y)}))^s}. \tag{2.3}$$

If the circular arc of $B(P_{(x,y)},r)$ intersects two of the squares $f_2(S_0)$, $f_3(S_0)$, $f_4(S_0)$. Let

$dist(P_{(x,y)},\overline{P}_{m_0q}^{(x,y)}) = max\{dist(P_{(x,y)},\overline{P}_{i_0q}^{(x,y)}), dist(P_{(x,y)},\overline{P}_{j_0q}^{(x,y)})\}$, where $(i_0,j_0) \in \{(2,3),(2,4),(3,4)\}$. With the above notation, we have

$$\frac{\mu(B(P_{(x,y)},r))}{(2r)^s} \leq \frac{\lambda_1^s+\lambda_{i_0}^s+\lambda_{j_0}^s-\mu(\Delta(\overline{P}_{i_0q}^{(x,y)},t_{i_0q}))-\mu(\Delta(\overline{P}_{j_0q}^{(x,y)},t_{j_0q}))}{(2\cdot(dist(P_{(x,y)},\overline{P}_{m_0q}^{(x,y)})-\sqrt{2}t_{m_0q}))^s}$$
$$\leq \frac{\lambda_1^s+\lambda_{i_0}^s+\lambda_{j_0}^s-\mu(\Delta(\overline{P}_{m_0q}^{(x,y)},t_{m_0q}))}{(2\cdot dist(P_{(x,y)},\overline{P}_{m_0q}^{(x,y)})-2\sqrt{2}t_{m_0q})^s}$$

We take in Lemma 2.1, $\alpha = s$, $p_0 = p = \lambda_1^s + \lambda_{i_0}^s + \lambda_{j_0}^s$, $a = 2\cdot dist(P_{(x,y)},\overline{P}_{m_0q}^{(x,y)})$, $a_0 = 2(1-\lambda_1)$ when $m_0 = 2,4$, and $a_0 = 2\sqrt{2}(1-\lambda_1)$ when $m_0 = 3$, $y = \mu(\Delta(\overline{P}_{m_0q}^{(x,y)},t_{m_0q}))$, $x = 2\sqrt{2}t_{m_0q}$. By Lemma 1.3, we have

$$\frac{y}{x^\alpha} = \frac{\mu(\Delta(\overline{P}_{m_0q}^{(x,y)},t_{m_0q}))}{(2\sqrt{2}t_{m_0q})^s} \geq \frac{1}{2^s\sqrt{2^s}}d_{min}^q.$$

Take λ_0 on the right hand side of the above inequality, and condition (i) of the main theorem ensures $2\sqrt{2}t_{m_0 q} \le (\frac{a_0\lambda_0}{p_0})^{\frac{1}{1-s}}$. By Lemma 2.1, we have

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \le \frac{\lambda_1^s + \lambda_{i_0}^s + \lambda_{j_0}^s}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{m_0 q}^{(x,y)}))^s}. \tag{2.4}$$

If the circular arc of $B(P_{(x,y)}, r)$ intersects all of the squares $f_2(S_0)$, $f_3(S_0)$, and $f_4(S_0)$. Note that

$$dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) \ge dist(P_{(x,y)}, \overline{P}_{2q}^{(x,y)})$$

and

$$dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) \ge dist(P_{(x,y)}, \overline{P}_{4q}^{(x,y)}),$$

and then

$$\begin{aligned} \frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} &\le \frac{1 - \mu(\Delta(\overline{P}_{2q}^{(x,y)}, t_{2q})) - \mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q})) - \mu(\Delta(\overline{P}_{4q}^{(x,y)}, t_{4q}))}{(2 \cdot (dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) - \sqrt{2}t_{3q}))^s} \\ &\le \frac{1 - \mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q}))}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) - 2\sqrt{2}t_{3q})^s}. \end{aligned}$$

We take Lemma 2.1 $\alpha = s$, $p_0 = p = 1$, $a = 2 \cdot dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)})$, $a_0 = 2\sqrt{2}(1 - \lambda_1)$, $y = \mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q}))$, $x = 2\sqrt{2}t_{3q}$.

By Lemma 1.3, we have $\frac{y}{x^\alpha} = \frac{\mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q}))}{(2\sqrt{2}t_{3q})^s} \ge \frac{1}{2^s\sqrt{2^s}} d_{min}^q$.
Take λ_0 the right hand side of the above inequality, and condition (i) of the main theorem ensures that $2\sqrt{2}t_{3q} \le (\frac{a_0\lambda_0}{p_0})^{\frac{1}{1-s}}$. By Lemma 2.1, we have

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \le \frac{1}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}))^s}. \tag{2.5}$$

Case 2. When $dist(P_{(x,y)}, \overline{P}_{4q}^{(x,y)}) \le r < dist(P_{(x,y)}, \overline{P}_{2q}^{(x,y)})$, then

$$B(P_{(x,y)}, r) \supset f_4(S_0).$$

In this case, either $B(P_{(x,y)}, r) \cap f_2(S_0) = \emptyset$ and $B(P_{(x,y)}, r) \cap f_3(S_0) = \emptyset$, or the circular arc of $B(P_{(x,y)}, r)$ intersects one of the $f_2(S_0)$ and $f_3(S_0)$, or the circular arc of $B(P_{(x,y)}, r)$ intersects all of the $f_2(S_0)$ and $f_3(S_0)$.

If $B(P_{(x,y)}, r) \cap f_2(S_0) = \emptyset$ and $B(P_{(x,y)}, r) \cap f_3(S_0) = \emptyset$, then

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \frac{\lambda_1^s + \lambda_4^s}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{4q}^{(x,y)}))^s}. \tag{2.6}$$

If the circular arc of $B(P_{(x,y)}, r)$ intersects one of $f_2(S_0)$ or $f_3(S_0)$, then

$$\begin{aligned}\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} &\leq \frac{\lambda_1^s + \lambda_4^s + \lambda_{j_0}^s - \mu(\Delta(\overline{P}_{j_0 q}^{(x,y)}, t_{j_0 q}))}{(2 \cdot (dist(P_{(x,y)}, \overline{P}_{j_0 q}^{(x,y)}) - \sqrt{2} t_{j_0 q}))^s} \\ &= \frac{\lambda_1^s + \lambda_4^s + \lambda_{j_0}^s - \mu(\Delta(\overline{P}_{j_0 q}^{(x,y)}, t_{j_0 q}))}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{j_0 q}^{(x,y)}) - 2\sqrt{2} t_{j_0 q})^s},\end{aligned}$$

where $j_0 \in \{2, 3\}$. We take in Lemma 2.1, $\alpha = s$, $p_0 = p = \lambda_1^s + \lambda_4^s + \lambda_{j_0}^s$, $a = 2 \cdot dist(P_{(x,y)}, \overline{P}_{j_0 q}^{(x,y)})$, $a_0 = 2(1 - \lambda_1)$ when $j_0 = 2$, and $a_0 = 2\sqrt{2}(1 - \lambda_1)$ when $j_0 = 3$, $y = \mu(\Delta(\overline{P}_{j_0 q}^{(x,y)}, t_{j_0 q}))$, $x = 2\sqrt{2} t_{j_0 q}$. By Lemma 1.3, we have

$$\frac{y}{x^\alpha} = \frac{\mu(\Delta(\overline{P}_{j_0 q}^{(x,y)}, t_{j_0 q}))}{(2\sqrt{2} t_{j_0 q})^s} \geq \frac{1}{2^s \sqrt{2^s}} d_{min}^q.$$

Take λ_0 on the right hand side of the above inequality, and condition (i) of the main theorem ensures $2\sqrt{2} t_{j_0 q} \leq (\frac{a_0 \lambda_0}{p_0})^{\frac{1}{1-s}}$. By Lemma 2.1, we have

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \frac{\lambda_1^s + \lambda_4^s + \lambda_{j_0}^s}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{j_0 q}^{(x,y)}))^s}, j_0 \in \{2, 3\}. \tag{2.7}$$

If the circular arc of $B(P_{(x,y)}, r)$ intersects all of the $f_2(S_0)$ and $f_3(S_0)$, it is obvious that $dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) \geq dist(P_{(x,y)}, \overline{P}_{2q}^{(x,y)})$. Then

$$\begin{aligned}\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} &\leq \frac{1 - \mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q}))}{(2 \cdot (dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) - \sqrt{2} t_{3q}))^s} \\ &= \frac{1 - \mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q}))}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) - 2\sqrt{2} t_{3q})^s}.\end{aligned}$$

As in Lemma 2.1, take $\alpha = s$, $p_0 = p = 1$, $a = 2 \cdot dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)})$, $a_0 = 2\sqrt{2}(1 - \lambda_1)$, $y = \mu(\Delta(\overline{P}_{3q}^{(x,y)}, t_{3q}))$, $x = 2\sqrt{2} t_{3q}$. By Lemma 1.3, we have

$$\frac{y}{x^\alpha}=\frac{\mu(\Delta(\overline{P}_{3q}^{(x,y)},t_{3q}))}{(2\sqrt{2}t_{3q})^s}\geq\frac{1}{2^s\sqrt{2^s}}d_{min}^q.$$

Taking λ_0 on the right hand side of the above inequality, the condition (i) of the main theorem ensures $2\sqrt{2}t_{3q}\leq(\frac{a_0\lambda_0}{p_0})^{\frac{1}{1-s}}$. By Lemma 2.1, we have

$$\frac{\mu(B(P_{(x,y)},r))}{(2r)^s}\leq\frac{1}{(2\cdot dist(P_{(x,y)},\overline{P}_{3q}^{(x,y)}))^s}. \tag{2.8}$$

Case 3. When $dist(P_{(x,y)},\overline{P}_{2q}^{(x,y)})\leq r<dist(P_{(x,y)},\overline{P}_{3q}^{(x,y)})$, then

$$B(P_{(x,y)},r)\supset f_4(S_0)\text{ and }B(P_{(x,y)},r)\supset f_2(S_0).$$

In this case, either $B(P_{(x,y)},r)\cap f_3(S_0)=\emptyset$, or the circular arc of $B(P_{(x,y)},r)$ intersects $f_3(S_0)$. If $B(P_{(x,y)},r)\cap f_3(S_0)=\emptyset$, then

$$\frac{\mu(B(P_{(x,y)},r))}{(2r)^s}\leq\frac{\lambda_1^s+\lambda_2^s+\lambda_4^s}{(2\cdot dist(P_{(x,y)},\overline{P}_{2q}^{(x,y)}))^s}. \tag{2.9}$$

If the circular arc of $B(P_{(x,y)},r)$ intersects $f_3(S_0)$, then

$$\begin{aligned}\frac{\mu(B(P_{(x,y)},r))}{(2r)^s}&\leq\frac{1-\mu(\Delta(\overline{P}_{3q}^{(x,y)},t_{3q}))}{(2\cdot(dist(P_{(x,y)},\overline{P}_{3q}^{(x,y)})-\sqrt{2}t_{3q}))^s}\\&=\frac{1-\mu(\Delta(\overline{P}_{3q}^{(x,y)},t_{3q}))}{(2\cdot dist(P_{(x,y)},\overline{P}_{3q}^{(x,y)})-2\sqrt{2}t_{3q})^s}.\end{aligned}$$

The discussion is now the same as above, take in Lemma 2.1 $\alpha=s$, $p_0=p=1$, $a=2\cdot dist(P_{(x,y)},\overline{P}_{3q}^{(x,y)})$, $a_0=2\sqrt{2}(1-\lambda_1)$, $y=\mu(\Delta(\overline{P}_{3q}^{(x,y)},t_{3q}))$, $x=2\sqrt{2}t_{3q}$. By Lemma 1.3, we have

$$\frac{y}{x^\alpha}=\frac{\mu(\Delta(\overline{P}_{3q}^{(x,y)},t_{3q}))}{(2\sqrt{2}t_{3q})^s}\geq\frac{1}{2^s\sqrt{2^s}}d_{min}^q.$$

Taking λ_0 on the right hand side of the above inequality, and condition (i) of the main theorem ensures $2\sqrt{2}t_{3q}\leq(\frac{a_0\lambda_0}{p_0})^{\frac{1}{1-s}}$. By Lemma 2.1, we have

$$\frac{\mu(B(P_{(x,y)},r))}{(2r)^s}\leq\frac{1}{(2\cdot dist(P_{(x,y)},\overline{P}_{3q}^{(x,y)}))^s}. \tag{2.10}$$

Case 4. When $dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}) \leq r < \sqrt{2}$, then $B(P_{(x,y)}, r) \supset f_k(S_0)$, $k = 1, 2, 3, 4$. It is obvious that

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \frac{1}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{3q}^{(x,y)}))^s}. \tag{2.11}$$

The above discussion indicated that the right hand sides of the above inequalities (2.2) – (2.11) are all no more than

$$\frac{1}{(2 \cdot dist(P_{(x,y)}, \overline{P}_{pq}^{(x,y)}))^s},$$

p=2,3,4. Note that $P_{(x,y)} \in f_1(S_0)$, and $\overline{P}_{pq}^{(x,y)}$ is the farthest vertex of the square $f_p(S_0)$ from $P_{(x,y)}$. Then $dist(P_{(x,y)}, \overline{P}_{pq}^{(x,y)}) \geq 1 - \lambda_1$ when $p \in \{2, 4\}$, and $dist(P_{(x,y)}, \overline{P}_{pq}^{(x,y)}) \geq \sqrt{2}(1-\lambda_1)$ when $p = 3$. Moreover, $dist(P_{(x,y)}, \overline{P}_{pq}^{(x,y)}) \leq \sqrt{2}$. Combining condition (ii) of the main theorem with the above inequalities, we complete the proof of Lemma 2.2. □

Corollary 2.3. *For any $P_{(x,y)} \in f_k(S)$, we have*

$$\sup_{\sqrt{2}\lambda_k \leq r < \sqrt{2}} \frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \frac{1}{(2\sqrt{2}(1 - \lambda_k))^s},$$

where $k = 1, 2, 3, 4$.

PROOF. Without loss of the generality, we only consider the case $k = 1$. Note that the point $P_{(\lambda_1,\lambda_1)}$ is the nearest point from $P_{(1,1)}$ in all of the points of $f_1(S_0)$. By Lemma 2.2, we have

$$\sup_{\sqrt{2}\lambda_1 \leq r < \sqrt{2}} \frac{\mu(B(P_{(\lambda_1,\lambda_1)}, r))}{(2r)^s} = \frac{1}{(2\sqrt{2}(1 - \lambda_1)^s}.$$

Then,

$$\sup_{\sqrt{2}\lambda_1 \leq r < \sqrt{2}} \frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \sup_{\sqrt{2}\lambda_1 \leq r < \sqrt{2}} \frac{\mu(B(P_{(\lambda_1,\lambda_1)}, r))}{(2r)^s} = \frac{1}{(2\sqrt{2}(1 - \lambda_1)^s}.$$

This completes the proof. □

Lemma 2.4. *Let $0 < \lambda_k \leq \frac{1}{2+\sqrt{2}}$, $k = 1, 2, 3, 4$, μ be the self-similar measure defined as in (1.3), and s be determined as in (1.2), then*

(i) For any point $P_{(x,y)} \in S$, $\overline{D}^s(\mu, P_{(x,y)}) \leq \max\limits_{1 \leq k \leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_k))^s}\}$.

(ii) $\overline{D}^s(\mu, P_{(x,y)}) \geq \max\limits_{1 \leq k \leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_k))^s}\}$ for $\mu-$ almost all $P_{(x,y)} \in S$.

PROOF. (i) For any point $P_{(x,y)} \in S$, there exists a sequence $i_1, i_2, \cdots, i_n, \cdots$ such that $\{P_{(x,y)}\} = \lim_{n\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(S_0)$, where $i_j \in \{1,2,3,4\}$, $j = 1, 2, \cdots, n$. If a real r satisfies $0 < r \leq \sqrt{2}$, there is an integer $n > 0$ such that $\sqrt{2}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{i_{n+1}} < r \leq \sqrt{2}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}$. It is easy to see that

$$(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n})^{-1}(B(P_{(x,y)}, r))$$
$$= B((f_{i_1 \circ f_{i_2} \circ \cdots \circ f_{i_n}})^{-1}(P_{(x,y)}), (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n})^{-1}r),$$

then $\sqrt{2}\lambda_{i_{n+1}} < (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n})^{-1}r \leq \sqrt{2}$. Note that for any point $P_{(x,y)} \in f_i(S)$ and any radius $0 < r \leq \sqrt{2}\lambda_i$, the condition $0 < \lambda_i \leq \frac{1}{2+\sqrt{2}}$ implies that $B(P_{(x,y)}, r) \cap f_j(S) = \emptyset$, $j \neq i$, $i, j \in \{1,2,3,4\}$. Combining this fact with the self-similarity of S,

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} = \frac{\mu(B((f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n})^{-1}(P_{(x,y)}), (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n})^{-1}r))}{(2(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n})^{-1}r)^s}.$$

By Corollary 2.3,

$$\frac{\mu(B(P_{(x,y)}, r))}{(2r)^s} \leq \max_{1\leq k\leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_k))^s}\}.$$

Then,

$$\overline{D}^s(\mu, P_{(x,y)}) \leq \max_{1\leq k\leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_k))^s}\}.$$

(ii) Take $\frac{1}{(2\sqrt{2}(1-\lambda_{k_0}))^s} = \max_{1\leq i\leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_i))^s}\}$. For any $p \geq 1$, $k \geq 1$, let $f_{l_0}^k = f_{l_0} \circ f_{l_0} \circ \cdots \circ f_{l_0}$, where $(k_0, l_0) \in \{(1,3), (2,4), (3,1), (4,2)\}$. We construct a subset $F_{p,k}$ of S_0 that satisfies $\mu(F_{p,k}) = 1$ as follows

$$F_{p,k} = \bigcup_{n=p}^{\infty} \bigcup_{(i_1 i_2 \cdots i_n)\in I_n} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} \circ f_{k_0} \circ f_{l_0}^k(S_0),$$

where $I_n = \{(i_1 i_2 \cdots i_n) : i_j \in \{1,2,3,4\}, j = 1, 2, \cdots, n\}$. According to [2], we have $\mu(\bigcap_{k\geq 1}(\bigcap_{p\geq 1} F_{p,k})) = 1$. By the above results, if $P_{(x,y)} \in S \cap (\bigcap_{k\geq 1}(\bigcap_{p\geq 1} F_{p,k}))$, then $P_{(x,y)} \in S \cap F_{p,k}$ for any $k \geq 1$ and $p \geq 1$. The definition of $F_{p,k}$ implies that there exists some $n > p$ (n relative to p) such that $dist(P_{(x,y)}, P_n^0) \leq \sqrt{2}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0}\lambda_{l_0}^k$, where $\{P_n^0\} = \lim_{k\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} \circ f_{k_0} \circ f_{l_0}^k(S_0)$. Take $r_p = \sqrt{2}(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0}) + \sqrt{2}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0}\lambda_{l_0}^k$, then

$$B(P_{(x,y)}, r_p) \supset B(P_n^0, \sqrt{2}(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0})).$$

Thus,

$$\mu(B(P_{(x,y)}, r_p)) \geq \mu(B(P_n^0, \sqrt{2}(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0}))),$$

and

$$\frac{\mu(B(P_{(x,y)}, r_p))}{(2r_p)^s} \geq \frac{\lambda_{i_1}^s\lambda_{i_2}^s\cdots\lambda_{i_n}^s}{2^s\sqrt{2^s}(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0} + \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{k_0}\lambda_{l_0}^k)^s} = \frac{1}{2^s\sqrt{2^s}(1-\lambda_{k_0}+\lambda_{k_0}\lambda_{l_0}^k)^s}.$$

Note that $r_p \to 0$ when $p \to \infty$, and $P_{(x,y)} \in S \cap (\bigcap\limits_{p\geq 1} F_{p,k})$ for any $k \geq 1$, which implies

$$\overline{D}^s(\mu, P_{(x,y)}) \geq \frac{1}{2^s\sqrt{2^s}(1-\lambda_{k_0}+\lambda_{k_0}\lambda_{l_0}^k)^s}.$$

Combining $P_{(x,y)} \in S \cap (\bigcap\limits_{k\geq 1}(\bigcap\limits_{p\geq 1} F_{p,k}))$ with the above inequality, let $k \to \infty$, then

$$\overline{D}^s(\mu, P_{(x,y)}) \geq \frac{1}{2^s\sqrt{2^s}(1-\lambda_{k_0})^s}.$$

As $\mu(\bigcap\limits_{k\geq 1}(\bigcap\limits_{p\geq 1} F_{p,k})) = 1$ and S is the support of the measure μ, then the above inequality is valid for $\mu-$ almost all $P_{(x,y)} \in S$. This completes the proof of Lemma 2.4. □

Proof of the Main theorem. By the definition of the self-similar measure μ, for any measurable subset $E \subset R^2$, $\mu(E) = \frac{C^s(E\cap S)}{C^s(S)}$. Combining with the definition of the centered upper spherical density and Lemma 1.2, we have $\overline{D}^s(\mu, P_{(x,y)}) = \frac{1}{C^s(S)}$ for $\mu-$ almost all $P_{(x,y)} \in S$. Lemma 2.4 showed that if $\overline{D}^s(\mu, P_{(x,y)}) = \max\limits_{1\leq i\leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_i))^s}\}$ for $\mu-$ almost all $P_{(x,y)} \in S$, then $C^s(S) = (\max\limits_{1\leq i\leq 4}\{\frac{1}{(2\sqrt{2}(1-\lambda_i))^s}\})^{-1}$. This complete the proof of the main theorem. □

Remark. The method developed in this paper can be generalized to other self-similar sets with a larger number of generators and dimensions no more than one, but the computations are more tedious.

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