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Wojciech Wojdowski, Institute of Mathematics, Technical University of Łódż, 90 - 924 Łódż, Poland. email: wojwoj@gmail.com

ON A GENERALIZATION OF THE DENSITY TOPOLOGY ON THE REAL LINE

Abstract

Wilczynski's definition of Lebesgue density point given in [19] created a new tool for the study of the more subtle properties of the notion of density point and the density topology, their various modifications and most of all category analogues. In the paper we develop further properties of the $\mathcal{A}_d$-density topology on the real line, introduced in [22]. The topology is a generalization of the Lebesgue density topology and is based on the definition given by Wilczyński. We consider the properties of continuos functions with respect to the $\mathcal{A}_d$-density topology and prove that the topology is completely regular but not normal.

Let S be the σ-algebra of Lebesgue measurable subsets of the real line $\mathbb{R}$, and I the σ-ideal of null sets. We shall say that the sets A, $B \in S$ are equivalent ($A \sim B$), if and only if $\lambda(A \triangle B) = 0$, where λ stands for Lebesgue measure on the real line. Recall that a point $x \in \mathbb{R}$ is a density point of a set $A \in S$, if and only if

$$\lim_{h \to 0} \frac{\lambda\left(A \cap [x-h, x+h]\right)}{2h} = 1. \qquad (*)$$

The notion of density point has been studied and developed extensively since the notion of the density topology $\mathcal{T}$ was introduced by Haupt and Pauc in 1952 [9]. It is interesting that the related notion of approximate continuity, as defined by Denjoy in 1915 [5], had been known far earlier and utilized in the study of the theory of integration. The properties of the density topology were discovered gradually by Goffman and Waterman [8], Goffman, Neugebauer and Nishiura [7] and Tall [17]. The theory seemed to be mostly complete in late seventies. However, in 1981 W. Wilczyński in [19] reformulated the notion of density point. It was a turning point in the development of the theory of

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density topologies. Wilczyński no longer employed the notion of measure in his version of the definition, replacing it with the notion of a null set.

He observed that the condition (∗) in the definition is equivalent to each of the following:

$$\lim_{n\to\infty} \frac{\lambda\left(A \cap \left[x - \frac{1}{n}, x + \frac{1}{n}\right]\right)}{\frac{2}{n}} = 1$$

or

$$\lim_{n\to\infty} \lambda\left(n \cdot (A - x) \cap [-1, 1]\right) = 2$$

or

$$\left\{\chi_{(n\cdot(A-x))\cap[-1,1]}\right\}_{n\in\mathbb{N}} \text{ converges in measure to } \chi_{[-1,1]}.$$

With this last condition in hand and the Riesz theorem, he could give the definition of a density point of a set $A \in S$, in terms of convergence almost everywhere of characteristic functions of dilations of the set A.

A point $x \in \mathbb{R}$ is a density point of a set $A \in S$, if and only if every subsequence $\left\{\chi_{(n_m\cdot(A-x))\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of $\left\{\chi_{(n\cdot(A-x))\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ contains a subsequence $\left\{\chi_{\left(n_{m_p}\cdot(A-x)\right)\cap[-1,1]}\right\}_{p\in\mathbb{N}}$, which converges to $\chi_{[-1,1]}$ I-almost everywhere on $[-1, 1]$ (which means except on a set belonging to I).

The above definition, as proved in [14] (Corollary 1 p. 556), is equivalent to the following (for the detailed discussion see [20] p. 680–681):

A point $x \in \mathbb{R}$ is a density point of a set $A \in S$, if and only if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1, 1]$ to $\chi_{[-1,1]}$.

Wilczynski's definition created a new tool for the study of the subtler properties of the notion of density point and the density topology, their various modifications and most of all category analogues (see [13], [14], [4] and [16]).

Recently, the notions of simple density point and complete density point have been introduced with associated $\mathcal{T}_s$ and $\mathcal{T}_c$ density topologies, respectively, essentially different from density topology $\mathcal{T}$ (see [1] and [21]). Actually, we have the inclusions

$$\mathcal{T}_n \subsetneqq \mathcal{T}_c \subsetneqq \mathcal{T}_s \subsetneqq \mathcal{T},$$

where $\mathcal{T}_n$ is the natural topology on the real line.

Following this approach, in [22] we gave a new generalization of density point, leading to a new density topology $\mathcal{T}_{\mathcal{A}_d}$ that extends the sequence of inclusions to

$$\mathcal{T}_n \subsetneqq \mathcal{T}_c \subsetneqq \mathcal{T}_s \subsetneqq \mathcal{T} \subsetneqq \mathcal{T}_{\mathcal{A}_d}.$$

We consider the following families of sets:

a) $\mathcal{A}_{[-1,1]}$- the family of subsets of interval $[-1, 1]$ of Lebesgue measure two,

b) $\mathcal{A}_{seg} = \{E \in S : \lambda(E \cap J) = \lambda(J)$ for some segment J centered at $0\}$,
c) $\mathcal{A}_d$- the family of measurable subsets of $[-1,1]$ that have Lebesgue density one at 0.

We have $\mathcal{A}_{[-1,1]} \subset \mathcal{A}_{seg} \subset \mathcal{A}_d$.

Definition 1. We shall say that x is an $\mathcal{A}_d$-density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to χ_B. (In other words, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges on $[-1,1]$ in measure to χ_B.)

By analogy, we define a notion of $\mathcal{A}_{seg}$-density point and $\mathcal{A}_{[-1,1]}$-density point of $A \in S$. The family $\mathcal{A}_{[-1,1]}$ corresponds precisely to the definition of the Lebesgue density point. The set of all $\mathcal{A}_d$-density points, $\mathcal{A}_{seg}$-density points and Lebesgue density points of $A \in S$ are denoted by $\Phi_{\mathcal{A}_d}(A)$, $\Phi_{\mathcal{A}_{seg}}(A)$ and $\Phi(A)$, respectively.

Proposition 1. *For each $A \in S$, $\Phi(A) \subset \Phi_{\mathcal{A}_{seg}}(A) \subset \Phi_{\mathcal{A}_d}(A)$.*

It was proved in [22] that the definition of $\mathcal{A}_d$-density point leads to the above mentioned topology, defined as the family $\mathcal{T}_{\mathcal{A}_d} = \{A \in S : A \subset \Phi_{\mathcal{A}_d}(A)\}$. The $\mathcal{T}_{\mathcal{A}_d}$-topology is stronger than the density topology $\mathcal{T}$. However, it has similar properties, in particular:

- For an arbitrary set $A \subset \mathbb{R}$, $\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(A) = A \cap \Phi_{\mathcal{A}_d}(B)$, where B is a measurable kernel of A.
- A set $A \in \mathcal{T}_{\mathcal{A}_d}$ is $\mathcal{T}_{\mathcal{A}_d}$-regular open, if and only if $A = \Phi_{\mathcal{A}_d}(A)$.
- $\mathcal{I} = \{A \subset \mathbb{R} : A$ is $\mathcal{T}_{\mathcal{A}_d}$-nowhere dense set$\}$
  $= \{A \subset \mathbb{R} : A$ is $\mathcal{T}_{\mathcal{A}_d}$-first category set$\}$
  $= \{A \subset \mathbb{R} : A$ is $\mathcal{T}_{\mathcal{A}_d}$-closed $\mathcal{T}_{\mathcal{A}_d}$-discrete set$\}$.
- A σ-algebra of $\mathcal{T}_{\mathcal{A}_d}$-Borel sets coincides with S.
- If $E \subset \mathbb{R}$ is $\mathcal{T}_{\mathcal{A}_d}$-compact set, then E is finite.
- The space $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_d})$ is neither first countable, nor second countable, nor Lindelöf, nor separable.
- $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_d})$ is a Baire space.

We shall now investigate further properties of the $\mathcal{T}_{\mathcal{A}_d}$-topology. The following proposition is of great importance.

Proposition 2. *If 0 is an $\mathcal{A}_d$-density point of a set A, then*
a) $\liminf_{h\to 0^+} \frac{\lambda(A\cap[-h,0])}{h} > 0$ *and* $\liminf_{h\to 0^+} \frac{\lambda(A\cap[0,h])}{h} > 0$.

b) $\limsup_{h\to 0} \frac{\lambda(A\cap[-h,h])}{2h} = 1$.

PROOF. The proof of a) is given in [22]. We shall prove b). If 0 is an $\mathcal{A}_d$-density point of a set A, then there exists a decreasing to zero sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ such that a sequence $\left\{\chi_{\left(\frac{1}{t_n}\cdot A\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to χ_B for some $B\in\mathcal{A}_d$.

Since $\lim_{k\to\infty}\lambda\left((k\cdot B)\cap[-1,1]\right)=2$, we can find an increasing sequence of natural numbers $\{k_i\}_{i\in\mathbb{N}}$ such that $\lambda\left([(k_i\cdot B)\cap[-1,1]]\bigtriangleup[-1,1]\right)<\frac{1}{2i}$ for every $i\in\mathbb{N}$. The sequence $\left\{\chi_{\left(\frac{1}{t_n}\cdot A\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ converges in measure to χ_B on $[-1,1]$, thus given the sequence $\{k_i\}_{i\in\mathbb{N}}$, we can find an increasing sequence of natural numbers $\{n_i\}_{i\in\mathbb{N}}$ such that

$$\lambda\Big(\Big[\Big(\frac{1}{t_{n_i}}\cdot A\Big)\cap\Big[-\frac{1}{k_i},\frac{1}{k_i}\Big]\Big]\bigtriangleup\Big[B\cap\Big[-\frac{1}{k_i},\frac{1}{k_i}\Big]\Big]\Big)<\frac{1}{2ik_i}$$

for every $i\in\mathbb{N}$, or equivalently

$$\lambda\Big(\Big[\Big(\Big(k_i\cdot\frac{1}{t_{n_i}}\Big)\cdot A\Big)\cap\Big[-1,1\Big]\Big]\bigtriangleup\Big[\Big(k_i\cdot B\Big)\cap\Big[-1,1\Big]\Big]\Big)<\frac{1}{2i}$$

for every $i\in\mathbb{N}$.

Let us consider a decreasing to zero sequence of real numbers $\left\{\frac{1}{k_i}\cdot t_{n_i}\right\}_{i\in\mathbb{N}}$. For every $i\in\mathbb{N}$ we have

$$\begin{aligned}
&\lambda\Big(\Big[\Big(\Big(\frac{1}{\frac{1}{k_i}\cdot t_{n_i}}\Big)\cdot A\Big)\cap\Big[-1,1\Big]\Big]\bigtriangleup\Big[-1,1\Big]\Big)\\
\le&\lambda\Big(\Big[\Big(\Big(\frac{1}{\frac{1}{k_i}\cdot t_{n_i}}\Big)\cdot A\Big)\cap\Big[-1,1\Big]\Big]\bigtriangleup\Big[\Big(k_i\cdot B\Big)\cap\Big[-1,1\Big]\Big]\Big)\\
&+\lambda\Big(\Big[\Big(k_i\cdot B\Big)\cap\Big[-1,1\Big]\Big]\bigtriangleup\Big[-1,1\Big]\Big)\le\frac{1}{2i}+\frac{1}{2i}=\frac{1}{i};
\end{aligned}$$

i.e., the sequence of characteristic functions $\left\{\chi_{\left(\left(\frac{1}{\frac{1}{k_i}\cdot t_{n_i}}\right)\cdot A\right)\cap[-1,1]}\right\}_{i\in\mathbb{N}}$ converges in measure to two. Hence, $\limsup_{h\to 0}\frac{\lambda(A\cap[-h,h])}{2h}=1$. □

In [22] we proved the following assertion.

Proposition 3. *There exists a set A such that $\Phi(A)\subsetneqq\Phi_{\mathcal{A}_d}(A)$.*

We may say more.

Proposition 4. *There exists a set A such that $\Phi(A)\subsetneqq\Phi_{\mathcal{A}_{seg}}(A)\subsetneqq\Phi_{\mathcal{A}_d}(A)$.*

PROOF. To prove that there exists a set B such that $\Phi_{\mathcal{A}_{seg}}(B) \subsetneqq \Phi_{\mathcal{A}_d}(B)$, we shall follow the proof of Proposition 2 in [22], defining the set B as described below.

Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of intervals such that $a_{n+1} < b_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$ and 0 is a right density point of the set $\cup_{n\in\mathbb{N}}(a_n, b_n)$. We put $D = \cup_{n\in\mathbb{N}}(a_n, b_n)$. Now, let $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, $c_1 < 1$, such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. We define the set $B \in S$ as

$$B = \Big(- \bigcup_{n=1}^{\infty} [(c_n \cdot D) \cap (c_{n+1}, c_n)] \Big) \cup \bigcup_{n=1}^{\infty} [(c_n \cdot D) \cap (c_{n+1}, c_n)].$$

To prove that there exists a set C such that $\Phi(A) \subsetneqq \Phi_{\mathcal{A}_{seg}}(A)$, we shall follow the proof of Proposition 2 in [22] again and define the set C as follows. Let $D = (0, \frac{1}{2})$, and $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, $c_1 < 1$, such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. We define a set $C \in S$ as

$$C = \Big(- \bigcup_{n=1}^{\infty} [(c_n \cdot D) \cap (c_{n+1}, c_n)] \Big) \cup \bigcup_{n=1}^{\infty} [(c_n \cdot D) \cap (c_{n+1}, c_n)].$$

Finally the set $A = B \cup (C + 2)$ satisfies $\Phi(A) \subsetneqq \Phi_{\mathcal{A}_{seg}}(A) \subsetneqq \Phi_{\mathcal{A}_d}(A)$, since $2 \in \Phi_{\mathcal{A}_{seg}}(A) \setminus \Phi(A)$ and $0 \in \Phi_{\mathcal{A}_d}(A) \setminus \Phi_{\mathcal{A}_{seg}}(A)$. □

We shall now discuss some properties of the continuity of real functions with respect to the $\mathcal{T}_{\mathcal{A}_d}$-topology.

Definition 2. We say that the real valued function f is $\mathcal{T}_{\mathcal{A}_d}$-topologically approximately continuous at a point x_0, if and only if for every number $\varepsilon > 0$, the set $\{x : |f(x) - f(x_0)| < \epsilon\}$ is a $\mathcal{T}_{\mathcal{A}_d}$-neighborhood of x_0; i.e., there exists a set $A_{x_0} \in S$, $A_{x_0} \subset \{x : |f(x) - f(x_0)| < \epsilon\}$ such that x_0 is a $\mathcal{T}_{\mathcal{A}_d}$-density point of A_{x_0}.

Definition 3. We say that the real valued function f is $\mathcal{T}_{\mathcal{A}_d}$-restrictively approximately continuous at a point x_0, if and only if there exists a set $E \in S$ such that $x_0 \in \Phi_{\mathcal{A}_d}(E)$ and $f(x_0) = \lim_{\substack{x\to x_o \\ x\in E}} f(x)$.

Remark 1. It is clear that every $\mathcal{T}_{\mathcal{A}_d}$-restrictively approximately continuous function at a point x_0 is $\mathcal{T}_{\mathcal{A}_d}$-topologically approximately continuous at the point x_0.

Proposition 5. *There exists a function that is $\mathcal{T}_{\mathcal{A}_d}$-topologically but not $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous at zero.*

PROOF. We shall start with the continuity at zero from the right. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers decreasing to zero such that $c_{n+1} < \frac{1}{4^n}c_n$ and $c_1 = 1$. Let

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} \left(\sum_{i=0}^{\infty} \frac{1}{2^i} \chi_{(\frac{c_n}{2^{i+1}}, \frac{c_n}{2^i}]}(x) \right) \chi_{(c_{n+1}, c_n]}(x) & \text{for } x \in \ (0,1] \\ 0 & \text{for } x = 0. \end{cases}$$

Equivalently put

$$g(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \chi_{(\frac{1}{2^{i+1}}, \frac{1}{2^i}]}(x)$$

and let

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} g(\frac{1}{c_n}x) \chi_{(c_{n+1}, c_n]}(x) & \text{for } x \in \ (0,1] \\ 0 & \text{for } x = 0. \end{cases}$$

The function f is right $\mathcal{A}_d$-topologically continuous at zero. Indeed, consider the sequence

$$E_k = \left\{ x \in [0,1] : |f(x) - 0| \leq \frac{1}{2^k} \right\}, \ k \in \mathbb{N}.$$

By definition of f

$$E_k = \bigcup_{n\in\mathbb{N}} (c_{n+1}, \frac{c_n}{2^k}], \ k \in \mathbb{N}$$

and it is a simple observation that for every $k \in \mathbb{N}$, E_k has 0 as an $\mathcal{A}_d$-density point (even $\mathcal{A}_{seg}$-density point).

The function f is not right $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous at 0. Suppose, to the contrary, that there exists a set $E \in S$ such that $x_0 \in \Phi_{\mathcal{A}_d}(E)$ and $\lim_{x\in E,\ x\to 0} f(x) = 0$. Then, since $x_0 \in \Phi_{\mathcal{A}_d}(E)$, we can find $k \in \mathbb{N}$ such that $\liminf_{t\to 0} \frac{\lambda(E\cap[0,t])}{t} > \frac{1}{2^k}$.

On the other hand, since $\lim_{x\in E,\ x\to 0} f(x) = 0$, we can find $c > 0$ such that $f(x) < \frac{1}{2^k}$ for all $x \in E \cap (0,c)$. Hence, $E \cap (0,c) \subset \{x : f(x) < \frac{1}{2^k}\}$ and $\liminf_{t\to 0} \frac{\lambda(E\cap[0,t])}{t} < \liminf_{t\to 0} \frac{\lambda(\{x: f(x) < \frac{1}{2^k}\}\cap(0,t))}{t} < \frac{1}{2^k}$ from definition of f, a contradiction. Now the function

$$h(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ f(-x) & x < 0 \end{cases}$$

is $\mathcal{T}_{\mathcal{A}_d}$-topologically but not $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous at zero. $\square$

Remark 2. The existence of a function that is $\mathcal{T}_{\mathcal{A}_d}$-topologically but not $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous at zero, can be proved with the use of the Jędrzejewski condition (W) from Theorem 4 in [10]. By the theorem, the $\mathcal{T}_{\mathcal{A}_d}$-topological continuity from the right of a function at 0 is equivalent to its $\mathcal{T}_{\mathcal{A}_d}$-restrictive continuity from the right at the point 0, if and only if

(W) for every descending sequence of sets E_k such that each E_k is an $\mathcal{T}_{\mathcal{A}_d}$-neighborhood of 0 from the right, there exists a sequence of real numbers $\{a_k\}_{k\in\mathbb{N}}$, decreasing to 0 and such that $\cup_{k=1}^{\infty}\big((a_{k+1},a_k)\cap E_k\big)$ is also an $\mathcal{T}_{\mathcal{A}_d}$-neighborhood of 0 from the right.

We shall show that the condition (W) is not fulfilled in case of $\mathcal{T}_{\mathcal{A}_d}$-topology. Consider the sequence

$$E_k = \bigcup_{n\in\mathbb{N}} (c_{n+1}, \frac{c_n}{2^k}],\ k \in \mathbb{N},$$

defined in the proof of the above proposition. For every $k \in \mathbb{N}$, $E_k \in S$ is an $\mathcal{T}_{\mathcal{A}_d}$-neighborhood of 0.

Let $\{a_k\}_{k\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. We shall show that zero is not an $\mathcal{A}_d$-density point of $\cup_{k=1}^{\infty}\big((a_{k+1},a_k)\cap E_k\big)$ from the right and thus zero is not an $\mathcal{A}_d$-density point of

$$\left(\bigcup_{k=1}^{\infty}((a_{k+1},a_k)\cap E_k)\right)\cup\left(-\left(\bigcup_{k=1}^{\infty}((a_{k+1},a_k)\cap E_k)\right)\right).$$

Indeed, we have $E_{k+1}\subset E_k$. Let c_{n_k} be the first element of the sequence $\{c_n\}_{n\in\mathbb{N}}$ less or equal to a_k. Then

$$\begin{aligned}
&\liminf_{h\to 0}\frac{\lambda\left(\bigcup_{p=1}^{\infty}((a_{p+1},a_p)\cap E_p)\cap[0,h]\right)}{h}\\
\leq&\liminf_{k\to\infty}\frac{\lambda\left(\bigcup_{p=1}^{\infty}((a_{p+1},a_p)\cap E_p)\cap[0,c_{n_k}]\right)}{c_{n_k}}\\
=&\liminf_{k\to\infty}\frac{\lambda\left(\bigcup_{p=k}^{\infty}((a_{p+1},a_p)\cap E_p)\cap[0,c_{n_k}]\right)}{c_{n_k}}\\
\leq&\liminf_{k\to\infty}\frac{\lambda\left(\left(\bigcup_{p=k}^{\infty}(a_{p+1},a_p)\right)\cap E_k\cap[0,c_{n_k}]\right)}{c_{n_k}}\\
\leq&\liminf_{k\to\infty}\frac{\lambda(E_k\cap[0,c_{n_k}])}{c_{n_k}}\leq\liminf_{k\to\infty}\frac{\frac{c_{n_k}}{2^k}}{c_{n_k}}=\lim_{k\to\infty}\frac{1}{2^k}=0.
\end{aligned}$$

Hence, by Proposition 3 in [22], zero is not an $\mathcal{A}_d$-density point of the set $\bigcup_{k=1}^{\infty}\big((a_{k+1},a_k)\cap E_k\big)\in S$ from the right, i.e. $\bigcup_{k=1}^{\infty}\big((a_{k+1},a_k)\cap E_k\big)$ is not an $\mathcal{A}_d$-neighborhood of 0 from the right.

Remark 3. Let us recall that for the $\mathcal{T}$-topology the notions of $\mathcal{T}$-topological continuity and $\mathcal{T}$-restrictive continuity coincide. Thus the notion of $\mathcal{T}$-continuity is used.

Remark 4. It is a simple observation that every $\mathcal{T}$-continuous function is $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous; the converse is not true. Indeed, the first part is a consequence of Remark 3 and $\mathcal{T}\subset\mathcal{T}_{\mathcal{A}_d}$. The characteristic function of the set $(-A\cup A)\cup\{0\}$, where A is defined in Proposition 2 in [22], is $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous, but not $\mathcal{T}$-continuous at 0.

Theorem 1. *For a real function f, defined on $\mathbb{R}$ the following conditions are equivalent:*

(i) f is measurable.
(ii) f is $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous almost everywhere on $\mathbb{R}$.
(iii) f is $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous almost everywhere on $\mathbb{R}$.

PROOF. $(i)\Rightarrow(ii)$ Suppose that f defined on $\mathbb{R}$ is measurable. Then, by the Denjoy-Stepanoff theorem, it is $\mathcal{T}$-continuous almost everywhere on $\mathbb{R}$; i.e., $\mathcal{T}$-topologically continuous almost everywhere on $\mathbb{R}$; hence $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous almost everywhere on $\mathbb{R}$, since $\mathcal{T}\subset\mathcal{T}_{\mathcal{A}_d}$.

$(ii)\Rightarrow(i)$ Suppose that f is $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous almost everywhere. Let $a,b\in\mathbb{R}$, and $B=\{x : a<f(x)<b\}$. We shall show that B is Lebesgue measurable. Let C be the set of $\mathcal{T}_{\mathcal{A}_d}$-continuity points of f. We have $B=(B\cap C)\cup(B-C)$ and $\lambda(B-C)=0$. The proof is completed by showing that $B\cap C$ is measurable. If $x\in B\cap C$, and $y=f(x)$, we take $\epsilon>0$, $\epsilon<\min(b-y,y-a)$. Then $\{x:|f(x)-y|<\epsilon\}$ is a $\mathcal{T}_{\mathcal{A}_d}$-neighborhood of x; i.e., there exist a set $A_x\in\mathcal{T}_{\mathcal{A}_d}$, $A_x\subset f^{-1}\{(f(x)-\epsilon,f(x)+\epsilon)\}$ such that x is a $\mathcal{A}_d$-density point of A_x. Of course, $A_x\subset B$, and we may assume $A_x\subset B\cap C$), by Theorem 1 (2) of [22], since $\lambda(B-C)=0$. Finally, we obtain $B\cap C=\bigcup_{x\in B\cap C}A_x\in\mathcal{T}_{\mathcal{A}_d}\subset S$.

$(i)\Rightarrow(iii)$ Suppose f defined on $\mathbb{R}$ is measurable. Then, by the Denjoy-Stepanoff theorem, it is $\mathcal{T}$-continuous almost everywhere on $\mathbb{R}$; i.e., $\mathcal{T}$-restrictively continuous almost everywhere on $\mathbb{R}$; hence $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous almost everywhere on $\mathbb{R}$ since $\mathcal{T}\subset\mathcal{T}_{\mathcal{A}_d}$.

$(iii)\Rightarrow(ii)$ Suppose that f is $\mathcal{T}_{\mathcal{A}_d}$-restrictively continuous almost everywhere. Then, by Remark 1 it is $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous almost everywhere. □

Corollary 1. *For every measurable real function f, the set of $\mathcal{T}_{\mathcal{A}_d}$-topological* *continuity points, the set of $\mathcal{T}_{\mathcal{A}_d}$-restrictive continuity points and the set of $\mathcal{T}$-continuity points differ by a null set.*

Remark 5. In the proof of part (i) of the above theorem we used a classical argument referring only to the Denjoy-Stepanoff theorem and to the inclusion $\mathcal{T} \subset \mathcal{T}_{\mathcal{A}_d}$. However, since $\mathcal{T}_{\mathcal{A}_d} \subset S$ and $\Phi_{\mathcal{A}_d}$ is a lower density operator, we could rely on Theorem 6.39 from [11] or use recent results of Bartoszewicz and Kotlicka given in more general settings (see [3] Theorem 2.2).

Proposition 6. *There exists a set $A \subset [0,1]$ such that zero is an $\mathcal{A}_d$-density point of $A \cup (-A)$ and such that there are a sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ decreasing to zero and* **c** *different sets from $S \setminus I$ associated with (assigned to) different subsequences $\{\frac{1}{t_{n_m}}\}_{m\in\mathbb{N}}$ in Definition 1.*

PROOF. Let $\{w_i\}_{i\in\mathbb{N}}$ be a sequence of all rational numbers from interval $(\frac{1}{2},1)$ and $\{c_n\}_{n\in\mathbb{N}}$ an arbitrary sequence of real numbers decreasing to 0, $c_1 < 1$, such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. Put $D_i = [0,\frac{1}{2}] \cup (w_i,1]$. We define a set A by

$$A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} \left(c_{\frac{n(n-1)}{2}+i} \cdot D_i\right) \cap \left(c_{\frac{n(n-1)}{2}+i+1}, c_{\frac{n(n-1)}{2}+i}\right).$$

Every natural number k can be uniquely expressed as a sum $k = \frac{n(n-1)}{2} + i$, where $n \in \mathbb{N}$ and $i = 1,2,\ldots,\left(\frac{(n+1)n}{2} - \frac{n(n-1)}{2}\right) = n$. We shall write i as a function of k; i.e., $i(k)$. We have in particular $\frac{(n+1)n}{2} = \frac{n(n-1)}{2} + n$ and $i\left(\frac{(n+1)n}{2}\right) = n$. We may rewrite the definition of set A as

$$A = \bigcup_{k=1}^{\infty} \left(c_k \cdot D_{i(k)}\right) \cap \left(c_{k+1}, c_k\right).$$

The $i(k)$, as a function of k, takes the consecutively values, $1,1,2,1,2,3,$ $1,2,3,4,\ldots$ .We shall show that zero is an A_d-density point of $A \cup (-A)$.

Suppose that $\{t_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence of real numbers decreasing to zero. As in the proof of Proposition 2 in [22], we choose two subsequences $\{t_{n_r}\}_{r\in\mathbb{N}}$ and $\{c_{m_r}\}_{r\in\mathbb{N}}$ such that $c_{m_r} \le t_{n_r}$, $r \in \mathbb{N}$ and there are no elements of $\{c_m\}_{m\in\mathbb{N}}$ nor of $\{t_n\}_{n\in\mathbb{N}}$ between c_{m_r} and t_{n_r}. Again, we consider the sequence $\left\{\frac{c_{m_r}}{t_{n_r}}\right\}_{r\in\mathbb{N}}$ and find a subsequence $\left\{\frac{c_{m_{r_k}}}{t_{n_{r_k}}}\right\}_{k\in\mathbb{N}}$ convergent to some $a \in [0,1]$.

There are two possible situations:

a) $\lim_{k\to\infty}\left(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\right) = a \neq 0$; i.e., $\lim_{k\to\infty}\left(c_{m_{r_k}} \cdot \frac{1}{a t_{n_{r_k}}}\right) = 1$. In this case we consider the behavior of the sequence $c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}} w_{i(m_{r_k})}$. Since it is bounded, it contains a subsequence $c_{m_{r_{k_p}}} \cdot \frac{1}{t_{n_{r_{k_p}}}} w_{i(m_{r_{k_p}})}$ convergent to some $c \in a \cdot \left[\frac{1}{2}, 1\right]$, and $\chi_{\left(\frac{1}{t_{n_{r_{k_p}}}} \cdot A\right) \cap [0,a]}$ converges a.e. to $\chi_{a \cdot \left(\left[0, \frac{1}{2}\right] \cup [c,1]\right)}$. Thus, we obtain B on $[0, a]$, as

$$B \cap [0, a] = a \cdot \left(\left[0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right]\right).$$

If

a1) $a = 1$ we are done; $B \in \mathcal{A}_{seg} \subset \mathcal{A}_d$.

If

a2) $a < 1$, as in the proof of Proposition 2 we obtain

$$B \cap [0, a] \cdot \left(\left[0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right]\right) \text{ and } B \cap (a, 1] = (a, 1].$$

And, again, $B \in \mathcal{A}_{seg} \subset \mathcal{A}_d$.

b) $\lim_{k \to \infty} \left(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\right) = 0$. In this case we have two possible situations again:

b1) The sequence $\left\{\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}\right\}_{k \in \mathbb{N}}$ is bounded from above. We take a subsequence $\left\{\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}\right\}_{p \in \mathbb{N}}$ such that $\lim_{p \to \infty} \frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}} = b < \infty$, and proceed similarly as in a). We find a subsequence $c_{m_{r_{k_{p_s}}}-1} \cdot \frac{1}{t_{n_{r_{k_{p_s}}}}} w_{i\left(m_{r_{k_{p_s}}}-1\right)}$ of $c_{m_{r_{k_p}}-1} \cdot \frac{1}{t_{n_{r_{k_p}}}} w_{i\left(m_{r_{k_p}}-1\right)}$ convergent to some $c \leq b$ and obtain convergence a.e. of $\chi_{\left(\frac{1}{t_{n_{r_{k_{p_s}}}}} \cdot A\right) \cap [0,b]}$ to $\chi_{b \cdot ([0, \frac{1}{2}] \cup [\frac{c}{b}, 1])}$. Thus we obtain as the set B

$$B = \left[b \cdot \left(\left[0, \frac{1}{2}\right] \cup \left[\frac{c}{b}, 1\right]\right)\right] \cap [0, 1].$$

And, again, $B \in \mathcal{A}_{seg} \subset \mathcal{A}_d$.

b2) The sequence $\left\{\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}\right\}_{k \in \mathbb{N}}$ is not bounded from above. We take a subsequence $\left\{\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}\right\}_{p \in \mathbb{N}}$ such that $\lim_{p \to \infty} \frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}} = \infty$. As every D_i contains the interval $[0, \frac{1}{2}]$, we have $[0, 1] \subset \frac{1}{t_{n_{r_{k_p}}}} \cdot A$, for p appropriately large, and the sequence $\chi_{\left(\frac{1}{t_{n_{r_{k_p}}}} \cdot A\right)}$ converges to $\chi_{[0,1]}$ a.e. on $[0, 1]$ and we obtain B on $[0, 1]$, as $B \cap [0, 1] = [0, 1]$. And, again, $B \in \mathcal{A}_{seg} \subset \mathcal{A}_d$. Finally, zero is an $\mathcal{A}_d$-density point of $(-A \cup A)$.

Now, let $d \in \left[\frac{1}{2}, 1\right]$ and $\{w_{n_i}\}_{i \in \mathbb{N}}$ be a subsequence of $\{w_n\}_{n \in \mathbb{N}}$ convergent to d. As a sequence $\{t_n\}_{n \in \mathbb{N}}$, we take $\{c_n\}_{n \in \mathbb{N}}$. The set $\left[0, \frac{1}{2}\right] \cup (d, 1] \in$

$\mathcal{A}_d$ is associated with the subsequence $\{c_{\frac{(n_i+1)n_i}{2}}\}_{i\in\mathbb{N}}$ and we obtain the sequence of characteristic functions $\chi_{\left(\left(\left(\frac{1}{c_{\frac{(n_i+1)n_i}{2}}}\right)\cdot A\right)\cap[0,1]\right)}$ convergent *a.e.* to $\chi_{\left[0,\frac{1}{2}\right]\cup[d,1]}$ on $[0,1]$. □

We shall prove now that the $\mathcal{A}_d$-density topology is completely regular. Following the approach from [2] and [18] we start with the Lusin-Menchoff condition for the $\mathcal{A}_d$-density topology.

Theorem 2. *Let $E \in S$ and F be a closed subset of E such that $F \subset \Phi_{\mathcal{A}_d}(E)$. Then, there exists a closed set P such that $F \subset P \cap \Phi_{\mathcal{A}_d}(P) \subset E \cap \Phi_{\mathcal{A}_d}(E)$ (i.e., $F \subset \mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(P) \subset \mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(E)$).*

Proof. Let

$$R_n = \left\{x \in E \cap \Phi_{\mathcal{A}_d}(E);\ \frac{1}{n+1} < \mathrm{dist}(x, F) \leq \frac{1}{n}\right\} \text{ and } E_0 = F \cup \bigcup_{n=1}^{\infty} R_n.$$

For any $n \in \mathbb{N}$ there is a closed set $P_n \subset R_n$ such that $\lambda\left(R_n \setminus P_n\right) < \frac{1}{2^n}$. Put $P = F \cup \bigcup_{n=1}^{\infty} P_n$. Obviously, P is a closed set and $F \subset P \subset E_0 \subset E \cap \Phi_{\mathcal{A}_d}(E)$. We will show that $F \subset \Phi_{\mathcal{A}_d}(P)$. It's enough to show that for every $x \in F$, x is an $\mathcal{A}_d$-density point of P from the right. (The proof that x is an $\mathcal{A}_d$-density point of P from the left is analogous.)

Let $x \in F$. For every $n \in \mathbb{N}$, $[x, x+\frac{1}{n}] \cap \bigcup_{k=1}^{n-1} R_k = \emptyset$. Since $x \in \Phi_{\mathcal{A}_d}(E)$, for any sequence of real numbers $\{t_p\}_{p\in\mathbb{N}}$ decreasing to zero, there exists its subsequence $\{t_{p_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{p_m}}\cdot(E-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to χ_B. Equivalently, for any sequence of real numbers $\{t_p\}_{p\in\mathbb{N}}$ decreasing to zero, there is its subsequence $\{t_{p_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{p_m}}\cdot(E-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges on $[-1,1]$ in measure to χ_B.

Consider $t_{p_m} \in (\frac{1}{n+1}, \frac{1}{n}]$. Then

$$\begin{aligned}
&\lambda\Big(\Big(\Big(\frac{1}{t_{p_m}} \cdot (E-x)\Big) \cap [-1,1]\Big) \triangle \Big(\Big(\frac{1}{t_{p_m}} \cdot (P-x)\Big) \cap [-1,1]\Big)\Big) \\
=&\lambda\Big(\Big(\frac{1}{t_{p_m}} \cdot (E-x)\Big) \cap [-1,1] \setminus \Big(\frac{1}{t_{p_m}} \cdot (P-x)\Big) \cap [-1,1]\Big) \\
=&\lambda\Big(\Big(\frac{1}{t_{p_m}} \cdot ((E \setminus P)-x)\Big) \cap [-1,1]\Big) \\
=&\lambda\Big(\Big(\frac{1}{t_{p_m}} \cdot ((E_0 \setminus P)-x)\Big) \cap [-1,1]\Big) \\
<&\lambda\Big(\Big(\frac{1}{t_{p_m}} \cdot ((\bigcup_{k=n}^{\infty}(R_k \setminus P_k)) - x\Big)\Big) \cap [-1,1]\Big) \\
<&\lambda\Big(\Big((n+1) \cdot ((\bigcup_{k=n}^{\infty}(R_k \setminus P_k)) - x\Big)\Big) \cap [-1,1]\Big) \\
=&(n+1)\lambda\Big(\Big((\bigcup_{k=n}^{\infty}(R_k \setminus P_k)) - x\Big)\Big) < (n+1)\sum_{k=n}^{\infty}\frac{1}{2^k} = \frac{n+1}{2^{n-1}}.
\end{aligned}$$

Clearly, the sequence

$$\lambda\Big(((\frac{1}{t_{p_m}} \cdot (E-x)) \cap [-1,1]) \triangle ((\frac{1}{t_{p_m}} \cdot (P-x)) \cap [-1,1])\Big)$$

converges to zero as $m \to \infty$, and, consequently, the sequence of characteristic functions $\Big\{\chi_{\frac{1}{t_{p_m}} \cdot (P-x) \cap [-1,1]}\Big\}_{m \in \mathbb{N}}$ converges in measure to χ_B on $[-1,1]$. This means that $x \in \Phi_{\mathcal{A}_d}(P)$. Since x was an arbitrary point of F, we have $F \subset \Phi_{\mathcal{A}_d}(P)$. □

Theorem 3. *Let E be a $\mathcal{T}_{\mathcal{A}_d}$-open set of type F_σ. There exists a $\mathcal{T}_{\mathcal{A}_d}$-continuous and upper semi-continuous function g such that*

$$0 < g(x) \le 1 \text{ for } x \in E, \text{ and } g(x) = 0 \text{ for } x \notin E.$$

PROOF. We may adapt here the proof from [[2] Theorem 6.5]. □

Theorem 4. *$\mathcal{T}_{\mathcal{A}_d}$ topology is completely regular.*

PROOF. Since $\mathcal{T}_{\mathcal{A}_d}$ includes the natural topology, it is a T_1-topology. Let F be a $\mathcal{T}_{\mathcal{A}_d}$-closed set and $x_0 \notin F$. There is a G_δ-set P such that $F \subset P$, $\lambda(P \setminus F) = 0$ and $x_0 \notin P$. The complements of P and $\{x_0\}$ are $\mathcal{T}_{\mathcal{A}_d}$-open and of type F_σ. From the above theorem it follows that there are $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous functions g_1 and g_2 such that

$$0 < g_1(x) \le 1 \text{ for } x \notin P,\ g_1(x) = 0 \text{ for } x \in P,$$

and

$$0 < g_2(x) \le 1 \text{ for } x \neq x_0,\ g_2(x) = 0 \text{ for } x = x_0.$$

Put

$$g(x) = \frac{g_1(x)}{g_1(x) + g_2(x)}.$$

The function $g(x)$ is clearly $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous, $g(x_0) = 1$ and $g(x) = 0$ for $x \in F$. Thus the topology $\mathcal{T}_{\mathcal{A}_d}$ is completely regular. □

Theorem 5. *The $\mathcal{T}_{\mathcal{A}_d}$-topology is not normal.*

PROOF. We shall adapt the argument given by Foran in [6] for the density topology. Let X and Y be two disjoint, countable, dense sets. Suppose that there are $\mathcal{T}_{\mathcal{A}_d}$-open sets U and V such that $X \subset U$, $Y \subset V$ and $U \cap V = \emptyset$. We will show first that U and V cannot have disjoint $\mathcal{T}_{\mathcal{A}_d}$-closures. Take $x_1 \in X$. Since $x_1 \in U = \mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(U) = U \cap \Phi_{\mathcal{A}_d}(U)$, by Remark 1 in [22] and Proposition 2 we have $\limsup_{h\to 0} \dfrac{\lambda(U \cap [x_1 - h, x_1 + h])}{2h} = 1$ and we can find a closed interval I_1 with $\lambda(I_1) < 1$, such that $\lambda(I_1 \cap U) > \frac{1}{2}\lambda(I_1)$. Since Y is dense and $V = \mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(V) = V \cap \Phi_{\mathcal{A}_d}(V)$, we can by analogy find a closed interval $I_2 \subset I_1$ with $\lambda(I_2) < \frac{1}{2}$ such that $\lambda(I_2 \cap V) > \frac{2}{3}\ \lambda(I_2)$. Then, again, since X is dense we can find a closed interval $I_3 \subset I_2$ with $\lambda(I_3) < \frac{1}{3}$ and $\lambda(I_3 \cap U) > \frac{3}{4}\lambda(I_3)$. Consecutively, by induction we can select a sequence $\{I_n\}_{n\in\mathbb{N}}$ of closed intervals with $I_{n+1} \subset I_n$, $\lambda(I_n) < \frac{1}{n}$ and, if n is even, $\lambda(I_n \cap U) > \frac{n}{n+1}\lambda(I_n)$ and, if n is odd, $\lambda(I_n \cap V) > \frac{n}{n+1}\lambda(I_n)$. Then $\bigcap_{n=1}^{\infty} I_n$ contains a single point x_0 and, clearly, the upper Lebesgue density of U at x_0 from the right is 1 or the upper Lebesgue density of U at x_0 from the left is 1, and the upper Lebesgue density of V at x_0 from the right is 1 or the upper Lebesgue density of V at x_0 from the left is 1. This implies that x_0 belongs to both $\mathrm{cl}_{\mathcal{T}_{\mathcal{A}_d}}(U)$ and to $\mathrm{cl}_{\mathcal{T}_{\mathcal{A}_d}}(V)$, since by Proposition 2 we have $\liminf_{h\to 0^+} \frac{\lambda\left((\mathbb{R}\setminus U)\cap[x,x+h]\right)}{h} > 0$ and $\liminf_{h\to 0^+} \frac{\lambda\left((\mathbb{R}\setminus U)\cap[x-h,x]\right)}{h} > 0$ and $\liminf_{h\to 0^+} \frac{\lambda\left((\mathbb{R}\setminus V)\cap[x,x+h]\right)}{h} > 0$ and $\liminf_{h\to 0^+} \frac{\lambda\left((\mathbb{R}\setminus V)\cap[x-h,x]\right)}{h} > 0$ for points x from $\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(\mathbb{R} \setminus U)$ and from $\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_d}}(\mathbb{R} \setminus V)$, respectively.

Let us consider now the two $\mathcal{T}_{\mathcal{A}_d}$-closed sets $\mathbb{R} \setminus V$ and Y. We shall show that they cannot be contained in two disjoint $\mathcal{T}_{\mathcal{A}_d}$-open sets. To see this, suppose that $\mathbb{R} \setminus V \subset U_1$ and $Y \subset V_1$ with U_1, V_1 $\mathcal{T}_{\mathcal{A}_d}$-open. Then $X \subset U \subset$

$\mathrm{cl}_{\mathcal{T}_{\mathcal{A}_d}}(U) \subset \mathbb{R} \setminus V$ and $Y \subset V_1 \subset \mathrm{cl}_{\mathcal{T}_{\mathcal{A}_d}}(V_1) \subset \mathbb{R} \setminus U_1 \subset V$ would imply that both $\mathcal{T}_{\mathcal{A}_d}$-open U and V_1 contain X and Y, respectively, and have disjoint $\mathcal{T}_{\mathcal{A}_d}$-closures, which cannot happen in accordance with the first part of the proof. Thus, the $\mathcal{T}_{\mathcal{A}_d}$-topology is not normal. □

Theorem 6. *Every $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous function is of the first class of Baire.*

PROOF. (communicated by W. Wilczyński) Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous function. Suppose the theorem is false. Then by D. Preiss [15] there exist a perfect set F and two real numbers a, b ($a < b$), such that the sets $T_1 = \{x : f(x) < a\}$ and $T_2 = \{x : f(x) > b\}$ are dense (in natural topology) in F (i.e., $\overline{T_1 \cap F} = F$ and $\overline{T_2 \cap F} = F$) (compare [14]). Observe that T_1 and T_2 are $\mathcal{A}_d$-open. We shall show (following Foran [6] pp 283–284), that they cannot have disjoint $\mathcal{A}_d$-closures. (T_1 and T_2 replace Foran's sets U and V, and F the real line $\mathbb{R}$.) We define a descending sequence $\{I_n\}_{n\in\mathbb{N}}$ of closed intervals, such that if n is odd, $\lambda(I_n \cap T_1) > \lambda(T_1) \cdot \frac{n}{n+1}$ and, if n is even $\lambda(I_n \cap T_2) > \lambda(T_1) \cdot \frac{n}{n+1}$. It is possible, since T_1 and T_2 are dense in F and T_1, as well as T_2 by Proposition 2, as $\mathcal{A}_d$-open set, has upper density 1 at each of its points. Let $\{x_0\} = \cap_{n=1}^{\infty} I_n$. Then, the upper $\mathcal{A}_d$-density of T_1 at x_0 from the right or from the left is 1, and the upper density of T_2 at x_0 from the right or from the left is 1. Also $x_0 \in F$, since F is closed and $I_n \cap F \neq \emptyset$, for $n \in \mathbb{N}$.

We shall show that $x_0 \in \mathrm{cl}_{\mathcal{A}_d} T_1$ and $x_0 \in \mathrm{cl}_{\mathcal{A}_d} T_2$. Let $G \ni x_0$ be an $\mathcal{A}_d$-open set. Then, by Proposition 2, both unilateral lower densities of G at x_0 are greater than zero. Thus, $G \cap T_1 \neq \emptyset$ and $x_0 \in \mathrm{cl}_{\mathcal{A}_d} T_1$. Similarly, $x_0 \in \mathrm{cl}_{\mathcal{A}_d} T_2$. Hence, $\mathrm{cl}_{\mathcal{A}_d} T_1 \cap \mathrm{cl}_{\mathcal{A}_d} T_2 \neq \emptyset$. Now the $\mathcal{A}_d$-continuity of f is equivalent to the property that for every $B \subset \mathbb{R}$, $\mathrm{cl}_{\mathcal{A}_d}(f^{-1}(B)) \subset f^{-1}(\bar{B})$. Hence

$$\mathrm{cl}_{\mathcal{A}_d}(T_1) = \mathrm{cl}_{\mathcal{A}_d}(f^{-1}((-\infty, a))) \subset f^{-1}((-\infty, a])$$

and

$$\mathrm{cl}_{\mathcal{A}_d}(T_2) = \mathrm{cl}_{\mathcal{A}_d}(f^{-1}((b, \infty))) \subset f^{-1}([b, \infty))$$

and therefore $f^{-1}((-\infty, a]) \cap f^{-1}([b, \infty)) \neq \emptyset$, a contradiction. Finally f is of the first class of Baire. □

Theorem 7. *Every $\mathcal{T}_{\mathcal{A}_d}$-topologically continuous function is a Darboux function.*

PROOF. It is a simple consequence of Theorem 6 and Theorem 1.1 of [2]. □

Theorem 8. *The family of $\mathcal{T}_{\mathcal{A}_d}$-connected sets coincides with the family of sets connected in the natural topology.*

Proof. We can follow here the proof of Theorem 3,7 from [20]. □

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