RESEARCH

Real Analysis Exchange
Vol. 33(1), 2007/2008, pp. 159–164

Annalisa Crannell, Department of Mathematics, Franklin & Marshall College, Lancaster, PA 17604, USA. email: annalisa.crannell@fandm.edu

M. Sohaib Alam, Department of Physics, University of Texas at Austin, Austin, Texas 78712, USA. email: malam@physics.utexas.edu

QUASICONTINUOUS FUNCTIONS WITH EVERYWHERE DISCONTINUOUS ITERATES

Abstract

This paper gives examples of two quasicontinuous functions whose second iterates are discontinuous everywhere. It is well-known that every quasicontinuousfunction has a dense—indeed, residual—set of points of continuity; our counter-examples show that this property does not hold for compositions of such functions.

Given topological spaces X and Y, a set $U \subset X$ is *quasi-open* if $\mathrm{cl}(\mathrm{int}(U)) \supset U$. A function $f : X \to Y$ is *quasicontinuous* if for any open V ⊂ Y, $f^{-1}(V)$ is quasi-open in X. We will use C_f and D_f to denote the points in X at which f is continuous or discontinuous, respectively, and $C_f^\infty = \{x \in X \mid f^k(x) \in C_f \quad \forall k \geq 0\}$. That is, if $x \in C_f^\infty$, then f is continuous at every point along the orbit of x, and, accordingly, f^k is continuous at x for every $k > 0$.

It is well known ([5], [6], [7], [4]) that if X and Y are "nice" (e.g., metric spaces), then C_f forms a residual subset of X. In [3], the authors showed that a similar statement can be made for compositions under the additional condition that f is *semi-open.* (For every non-empty open $U \subset X$, there is a non-empty open subset $V \subset f(U)$.) That is, they showed the following.

Theorem 1. *Let X be a compact metric space and let $f : X \to X$ be both quasicontinuous and semi-open. Then C_f^∞ is residual.*

Note that the theorem holds even though f^k might not be quasi-continuous for any $k > 1$. The purpose of this paper is to give two examples demonstrating that the "semi-open" condition in the theorem is necessary. Indeed, we provide

Key Words: iteration, quasicontinuity, quasicontinuous functions
Mathematical Reviews subject classification: 54C08, 54H20
Received by the editors July 20, 2007
Communicated by: Zbigniew Nitecki

two examples of maps which are quasicontinuous but whose second iterates are discontinuous everywhere in the domain. The first example is rather simple to describe and is defined on a cylinder. The second example is somewhat more complicated to construct and is defined on a closed interval of the real line.

Example 1. *A quasicontinuous map on the cylinder with everywhere discontinuous second iterate.*

Let $X = [-1, 1] \times \mathbf{S}^1$, where the model of $\mathbf{S}^1$ is $[0, 1]/(0 \sim 1)$. Define the function $g : X \to X$ such that

$$g(x, y) = \begin{cases} (x, 0) & \text{if } y \in (0, 1/2) \text{ or if } y \in \{0, 1/2\} \text{ and } x \in \mathbb{Q}, \\ (x, 1/2) & \text{otherwise.} \end{cases}$$

Note that g is not semi-open.

$$g(X) = \{\, (x, 0) \,|\, x \in [-1, 1] \,\} \cup \{\, (x, 1/2) \,|\, x \in [-1, 1] \,\},$$

so g sends the cylinder into the union of two line segments. Likewise, it is easy to see that g is quasi-continuous.

The second iterate of g is

$$g^2(x, y) = \begin{cases} (x, 0) & \text{if } x \in \mathbb{Q} \cap [-1, 1] \\ (x, 1/2) & \text{if } x \in \mathbb{Q}^c \cap [-1, 1] \end{cases}.$$

It follows that g^2 is discontinuous everywhere on the cylinder.

Example 2. *A quasicontinuous map on* $[0, 2]$ *with everywhere discontinuous second iterate.*

Define $h : [0, 2] \to [0, 2]$ as a periodic sum of two other functions s and q, which are defined below. We will make extensive use of the standard middle-thirds Cantor set, $\mathcal{C}$, and also of its connection to base-3 notation.

First, we choose a subset $S \subset \mathcal{C}$ to be the set of all numbers whose base-3 expansion ends in '$\overline{022}_3$'. Note that S has the following properties:

(1) S is dense in $\mathcal{C}$;
(2) $\mathcal{C} \setminus S$ is dense in $\mathcal{C}$; and,
(3) S contains no points of the form $k/2^n$, for $k, n \in \mathbb{N}$.

(The last property follows from geometric series; the reader can verify that every point in S can be written as $m/(19 \cdot 3^n)$ for some $m, n \in \mathbb{N}$.)

For $x \in [0, 1] \setminus \mathcal{C}$, represent the location of the first '1' in the base-3 expansion by $o(x) = \min\{k \mid x = 0.x_1x_2x_3\ldots_3 \text{ and } x_k = 1\}$.

Now we are ready to define $s : [0,1] \to [0,1]$ by

$$s(x) = \begin{cases} 0 & x \in \mathcal{C} \setminus S \text{ or } x \in [0,1] \setminus \mathcal{C} \text{ with } o(x) \text{ even} \\ 1 & x \in S \text{ or } x \in [0,1] \setminus \mathcal{C} \text{ with } o(x) \text{ odd} \end{cases}.$$

By definition, s is locally constant on $[0,1] \setminus \mathcal{C}$. It follows that its discontinuity set is $D_s = \mathcal{C}$. Note that s is not semi-open. It is, however, quasicontinuous; i.e., $[0,1] \setminus \mathcal{C} \subset s^{-1}(0) \subset [0,1]$, and so the inverse image of any open set containing 0 is quasi-open. The same is true for $s^{-1}(1)$.

We now define the function $q : [0,1] \to [0,1]$. We begin by writing points in base-2 notation. If $x \in [0,1]$ has two base-2 expansions, choose the one that ends $\overline{0}_2$, not $\overline{1}_2$. Then let $q(0.x_1x_2x_3\ldots_2) = 0.(2x_1)(2x_2)(2x_3)\ldots_3$.

It is easy to see that q is not semi-open: $q([0,1]) \subset \mathcal{C}$. (Indeed, although q^{-1} is not defined on all of $\mathcal{C}$, q^{-1} can be continuously extended to all of $[0,1]$ to form the "Devil's staircase", a common example from introductory analysis and dynamics [1].)

It is clear that q is not a continuous function. However, let us now show that q is a quasi-continuous function. It suffices [7] to show that for all $x \in [0,1]$ and for all $\epsilon > 0$, there exists an open set U, where $x \in \mathrm{cl}(U)$, such that $q(U) \subset B_\epsilon(q(x))$. Choose x and ϵ as above; choose $m \in \mathbb{N}$ such that $\left(\frac{1}{3}\right)^m < \epsilon$. Let $N > m$ be the first place value after m in the base-2 expansion of x for which $x_N = 0$, and let $\delta = \left(\frac{1}{2}\right)^N$. Pick $y \in (x, x+\delta)$. It follows that $x = 0.x_1x_2\ldots x_{N-1}0x_{N+1}\ldots_2$ and $y = 0.x_1x_2\ldots x_{N-1}0y_{N+1}\ldots_2$. Accordingly,

$$\begin{aligned} |q(y) - q(x)| &= 0.(2x_1)(2x_2)\ldots(2x_{N-1})0(2y_{N+1})(2y_{N+2})\ldots_3 \\ &\qquad - 0.(2x_1)(2x_2)\ldots(2x_{N-1})0(2x_{N+1})(2x_{N+2})\ldots_3 \\ &\le 0.00\ldots 001\bar{0}_3 \\ &= \left(\frac{1}{3}\right)^N < \epsilon, \end{aligned}$$

where the '1' on the second line of the inequality appears at the N^{th} place value. Thus, q is quasi-continuous at every point $x \in [0,1]$.

A similar argument shows that q is continuous at every point not of the form $k/2^n$; that is, the set of discontinuity points of q is exactly $D_q = \left\{\frac{k}{2^n} \,|\, k, n \in \mathbb{N} \text{ such that } 0 < k < 2^n\right\}$.

Let $\tilde{s}$ and $\tilde{q}$ be the periodic extensions of s and q to $[0,2]$; i.e., $\tilde{q}(x) = q(x)$ for $x \in [0,1]$ and $\tilde{q}(x) = q(x-1)$ for $x \in (1,2]$. We are now in a position to describe our main example, $h : [0,2] \to [0,2]$, defined by $h(x) = \tilde{q}(x) + \tilde{s}(x)$.

Let $\mathbf{C} = \mathcal{C} \cup \{x \in [1,2] \mid x - 1 \in \mathcal{C}\}$. It follows from the definitions of $\tilde{s}$ and $\tilde{q}$ that the set of discontinuities of h is $D_h = \mathbf{C} \cup \{k/2^n\}$. Because the sum of a continuous function with a quasicontinuous function is quasicontinuous, it follows that h is quasicontinuous on $C_{\tilde{q}} \cup C_{\tilde{s}}$—that is, on the complement of $\mathbf{C} \cap \{k/2^n\}$. We will now show that h is also quasicontinuous at points in $\mathbf{C} \cap \{k/2^n\}$ as well.

As in the proof for the quasicontinuity of q, we choose $x \in \mathbf{C} \cap \{k/2^n\}$, $\epsilon > 0$ and then fix m, δ such that $(1/3)^m < \epsilon$ and $\delta = (1/2)^N$, where N is the first place value after m in the base-2 expansion of x for which $x_N = 0$. Property (3) of the set S tells us that neither x nor $x - 1$ is in S, so $\tilde{s}(x) = 0$. Let $U = \{y \in (x, x+\delta) \mid y \notin \mathbf{C}; o(y) \text{ even}\}$. The argument above shows that $h(U) \subset B_\epsilon(x)$; it remains to show $x \in \text{cl}(U)$. Because $x = k/2^n$ is not an endpoint of the Cantor set, we know the base-3 expansion $x = x_0.x_1x_2\ldots_3$ does not end in '$\overline{2}_3$'. Therefore, we may find points arbitrarily close to x with their first '1' in an even position. Either replace a '0' in an even position with a '1', or if no such '0' exists, replace '02' with '21'. This shows $x \in \text{cl}(U)$, and h is quasicontinuous at x.

We now show that the second iterate of h is discontinuous everywhere on $[0, 2]$. To do so, let R denote the set of points whose base-2 expansions terminate in $\overline{011}$. We will use the fact that both R and $[0, 2] \setminus R$ form dense subsets of the interval $[0, 2]$. (Compare this to properties (1) and (2) of the set $S \subset \mathcal{C}$.) Now look at the effect that the second iterate of h has on both of these dense subsets.

First, take a point $x = x_0.x_1x_2\ldots x_n\overline{011}_2 \in R$. We see that $h(x) = y_0.(2x_1)(2x_2)\ldots(2x_n)\overline{022}_3$, where y_0 might be either 0 or 1. It follows from the definition of $\tilde{s}$ that $h^2(x) > 1$. Similarly, if $x \in [0,2] \setminus R$, we have $h^2(x) \leq 1$, with equality only in the case $x = 1$. Because both R and $[0, 2] \setminus R$ are dense, and because q is strictly monotone, it follows that at any point $x \in [0, 2]$ and for any $\alpha < 1$, we can find another point y arbitrarily close to x with $|h(x) - h(y)| > \alpha$. Therefore, h^2 is discontinuous on all of $[0, 2]$.

References

[1] Gregory L. Baker, Jerry P. Gollub, *Chaotic Dynamics: An Introduction*, Cambridge University Press, Cambridge, 1996.

[2] J. Borsík, J. Doboš, M. Repický, *Sums of Quasicontinuous Functions with Closed Graphs*, Real Anal. Exchange **25(2)** (1999/00), 679–690.

[3] Annalisa Crannell, Marc Frantz, Michelle LeMasurier, *Closed Relations and Equivalence Classes of Quasicontinuous Functions.* Real Anal. Exchange, **31(2)** (2005/06), 409–423.

[4] A. Crannell, M. Martelli, *Dynamics of Quasicontinuous Systems*, J. Difference Eqn. Appl., **6** (2000) 351–361.

[5] J. Ewert, J. S. Lipiński, *On Points of Continuity, Quasicontinuity, and Cliquishness of Real Functions*, Real Anal. Exchange, **8(2)** (1982-83), 473–478.

[6] M. Matejdes, *Quelques remarques sur la quasi-continuit des multifonctions*, Math. Slovaca, **37(3)** (1987), 267–271.

[7] T. Neubrunn, *Quasi-continuity*, Real Anal. Exchange, **14(2)** (1988–89), 259-306.