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# ORBITS OF DARBOUX-LIKE REAL FUNCTIONS

**Abstract**

We show that, with respect to the dynamics of iteration, Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$ can exhibit some strange properties which are impossible for continuous functions. To be precise, we show that (i) there is an extendable function from $\mathbb{R}$ to $\mathbb{R}$ which is 'universal for orbits' in the sense that it possesses every orbit of every function from $\mathbb{R}$ to $\mathbb{R}$ up to an arbitrary small translation, and which has orbits asymptotic to any real sequence, (ii) there is a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $n \in \mathbb{N}$, $f^n$ is almost continuous and the graph of $f^n$ is dense in $\mathbb{R}^2$, in spite of the fact that all $f$-orbits are finite. To prove (i) we assume the Continuum Hypothesis.

## 1 Introduction.

In the study of the iterative dynamics of functions $f : X \to X$ of a metric space $X$, two of the basic questions are the following:

(i) Which "types" of orbits can coexist in a system $(X, f)$?

(ii) If all orbits in a system $(X, f)$ are "simple", is the global dynamics "simple"?

First we make a peripheral investigation regarding these questions in the case of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Of course, (i) is a huge problem and we do not answer it in any non-trivial sense. Using a result from [6] we merely observe that there are uncountably many "different types" of possible orbits such that a continuous function can possess at most one of those "types". We also deduce that the answer to the second question is affirmative in a certain sense for continuous functions $f : \mathbb{R} \to \mathbb{R}$.

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After these preliminary observations, we examine the case of *Darboux-like functions*, functions satisfying generalized notions of continuity, $f : \mathbb{R} \to \mathbb{R}$. We remark that the dynamics of *Darboux-like functions* was considered before. For instance, it is known that there exists an *almost continuous* function $f : \mathbb{R} \to \mathbb{R}$ to which *Sarkovski's Theorem* cannot be extended [5]. For some positive results, see [2], [8], [9].

With respect to the two basic questions mentioned above, the results we obtain about *Darboux-like functions* are drastically different from those we get for continuous functions. We show that there is an *extendable function* from $\mathbb{R}$ to $\mathbb{R}$ which is 'universal for orbits' in the sense that it possesses every orbit of every function from $\mathbb{R}$ to $\mathbb{R}$ up to an arbitrary small translation, and which has orbits asymptotic to any real sequence. We also show that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $n \in \mathbb{N}$, $f^n$ is *almost continuous* and the graph of $f^n$ is dense in $\mathbb{R}^2$, in spite of the fact that all $f$-orbits are finite.

## 2 Preliminaries.

If $f : X \to X$ is a function of a metric space we call the pair $(X, f)$ a **dynamical system**. For $n \in \mathbb{N}$, by $f^n$ we mean the $n$-fold self-composition of $f$. For $x \in X$, the **$f$-orbit** of the point $x$ is $\{x, f(x), f^2(x), f^3(x), \ldots\}$, which we denote by $O_f(x)$. Let $G_f := \{(x, f(x)) : x \in X\}$ denote the graph of $f$. We say $f$ is **topologically transitive** if $\bigcup_{n=1}^{\infty} G_{f^n}$ is dense in $X^2$. One may refer to [3] to appreciate the role of topological transitivity in the study of *chaos*. The following fact is well-known, and can easily be deduced using the Baire Category Theorem.

**Proposition 1.** *Let $X$ be a complete, second countable metric space without isolated points (e.g. $X = \mathbb{R}$). Then for a continuous function $f : X \to X$ the following are equivalent:*

*(i) $f$ is topologically transitive.*
*(ii) There exists a point $x \in X$ whose $f$-orbit is dense in $X$.*
*(iii) $\{x \in X : O_f(x)$ is dense in $X\}$ is a dense $G_\delta$ subset of $X$.*

For a metric space $X$, let $\mathcal{O}_X$ be the collection of all sequences in $X$ which can be realized as orbits of functions $f : X \to X$ (need not be continuous). That is,

$$\mathcal{O}_X = \{(x_n)_{n=1}^{\infty} \in X^{\mathbb{N}} : \text{there is } f : X \to X \text{ with } f(x_n) = x_{n+1} \text{ for all } n \in \mathbb{N}\}.$$

We put an equivalence relation on $\mathcal{O}_X$ by defining $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$ if there is a homeomorphism $h : X \to X$ such that $h(x_n) = y_n$ for every $n \in \mathbb{N}$. Orbits in the same equivalence class are referred to as **orbits of the same type**. The proof of the following is straightforward.

**Lemma 1.** *Let $f, g : X \to X$ be continuous, let $x, y \in X$ and let $h : X \to X$ be a homeomorphism such that $h(f^n(x)) = g^n(y)$ for $n = 0, 1, 2, \ldots$. If $O_f(x)$ is dense in $X$, then $h \circ f = g \circ h$.*

For $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in \mathcal{O}_{\mathbb{R}}$ we say $(x_n)_{n=1}^{\infty}$ is a **translate** of $(y_n)_{n=1}^{\infty}$ if there is $b \in \mathbb{R}$ such that $x_n = y_n + b$ for every $n \in \mathbb{N}$. Note that this is stronger than saying $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are of the same type.

A function $f : \mathbb{R} \to \mathbb{R}$ is called **Darboux** if $f(A)$ is connected for every connected subset $A \subset \mathbb{R}$. A classic Theorem of Darboux (c.f. [7]) says that if $g : \mathbb{R} \to \mathbb{R}$ is differentiable, then its derivative is Darboux. In search of a nice characterization of derivatives (which is still unavailable), many properties which are close to the Darboux property have been studied by various authors, see [1], [4] and the references therein. Functions satisfying these generalized continuity properties are collectively known as **Darboux-like functions**.

In this article, we will consider two classes of Darboux-like functions. A function $f : \mathbb{R} \to \mathbb{R}$ is called **almost continuous** if any open subset $U$ of $\mathbb{R}^2$ containing the graph of $f$ contains the graph of some continuous function $g : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an **extendable function** if there exists a function $F : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in \mathbb{R}$ and such that the graph of $F|_Z$ is a connected subset of $\mathbb{R} \times [0,1] \times \mathbb{R}$ for every connected subset $Z \subset \mathbb{R} \times [0,1]$. For $f : \mathbb{R} \to \mathbb{R}$, it is known that (c.f. [4])

continuous $\Longrightarrow$ extendable $\Longrightarrow$ almost continuous $\Longrightarrow$ Darboux,

where all the implications are strict. See [1], [4] for more information concerning Darboux-like functions.

As usual, $\mathfrak{c}$ will denote the cardinality of $\mathbb{R}$. A subset $A \subset \mathbb{R}$ is said to be **$\mathfrak{c}$-dense** if the cardinality of $A \cap J$ is $\mathfrak{c}$ for every nondegenerate interval $J \subset \mathbb{R}$.

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## 3 Continuous $f : \mathbb{R} \to \mathbb{R}$.

In the case of a continuous function $f : \mathbb{R} \to \mathbb{R}$, we have the following partial answers regarding the two basic questions mentioned in the beginning of the article.

**Proposition 2.** *There is an uncountable set $S \subset \mathcal{O}_{\mathbb{R}}$ such that*

*(i) each member of $S$ is an orbit of some continuous function from $\mathbb{R}$ to $\mathbb{R}$,*

*(ii) no two distinct members of $S$ are of the same type,*

*(iii) if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then at most one member of $S$ is of the same type as that of some orbit of $f$.*

Proof. By [6], there exists an uncountable family $\{f_\alpha : \alpha \in \Lambda\}$ of continuous, topologically transitive functions from $\mathbb{R}$ to $\mathbb{R}$ such that for any homeomorphism $h : \mathbb{R} \to \mathbb{R}$ one has $h \circ f_\alpha \neq f_\beta \circ h$ for every $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. Now, by Proposition 1, choose $x_\alpha \in \mathbb{R}$ so that $O_{f_\alpha}(x_\alpha)$ is dense in $\mathbb{R}$. Put $S = \{O_{f_\alpha}(x_\alpha) : \alpha \in \Lambda\}$. Then (i) is clearly true, and (ii), (iii) are satisfied because of Lemma 1. □

**Proposition 3.** *Let $X$ be an infinite complete metric space without isolated points (e.g. $X = \mathbb{R}$). If $f : X \to X$ is a continuous function such that $O_f(x)$ is finite for each $x \in X$, then $\bigcup_{n=1}^{\infty} G_{f^n}$ is nowhere dense in $X^2$.*

Proof. Let $U, V \subset X$ be nonempty open sets. Our aim is to find nonempty open sets $U' \subset U$ and $V' \subset V$ such that $f^n(U') \cap V' = \emptyset$ for every $n \in \mathbb{N}$. For $m, k \in \mathbb{N}$, let $A(m,k) = \{x \in X : f^{m+k}(x) = f^m(x)\}$. Since $f$ is continuous, each $A(m,k)$ is closed. Also, by hypothesis $X$ is the union of $A(m,k)$'s. Therefore by Baire Category Theorem, there exist $m, k \in \mathbb{N}$ such that $A(m,k) \cap U$ has nonempty interior, say $W$. Fix $x \in W$ and choose $\epsilon > 0$ small enough so that $V \setminus \bigcup_{j=0}^{m+k} B(f^j(x), \epsilon)$ contains an open ball, say $V'$. Now using continuity, choose $\delta > 0$ so that $f^j(B(x,\delta)) \subset B(f^j(x), \epsilon)$ for $0 \leq j \leq m+k$. Put $U' = W \cap B(x, \delta)$. □

In the next two sections we show that the behavior of Darboux-like functions is very different in comparison with the last two Propositions.

## 4 An Extendable Function $f : \mathbb{R} \to \mathbb{R}$.

The following sufficient condition (c.f. [1]) for a function $f : \mathbb{R} \to \mathbb{R}$ to be an extendable function, will be useful.

**Proposition 4.** *Let $A \subset \mathbb{R}$ be a $\mathfrak{c}$-dense set which is $F_\sigma$ and of first category. Then there exists an extendable function $\phi : \mathbb{R} \to \mathbb{R}$ with the following property that if $f : \mathbb{R} \to \mathbb{R}$ is any function such that $f(a) = \phi(a)$ for every $a \in A$, then $f$ is also an extendable function.*

We show that all types of orbits can coexist in a strong sense for an extendable function. Moreover, the function can be chosen so that it has orbits asymptotic to any real sequence. This fact may be of interest since in the theory of iterative dynamics, one is concerned with the asymptotic behavior of orbits.

**Proposition 5.** *Assuming the Continuum Hypothesis, there is an extendable function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:*

*(i) For any function $g : \mathbb{R} \to \mathbb{R}$, any $x \in \mathbb{R}$ and any $\epsilon > 0$, there exist $y \in \mathbb{R}$ and $b \in (0, \epsilon)$ such that $f^n(y) = g^n(x) + b$ for $n = 0, 1, 2, \ldots$ .*

*(ii) For any real sequence $(r_n)_{n=0}^{\infty}$, and any decreasing sequence $(\epsilon_n)_{n=0}^{\infty}$ of positive reals converging to 0, there exists $s \in \mathbb{R}$ such that $|r_n - f^n(s)| < \epsilon_n$ for $n = 0, 1, 2, \ldots$ .*

Proof. Let $A \subset \mathbb{R}$ be a $\mathfrak{c}$-dense, $F_\sigma$ set of first category. Let $\phi : \mathbb{R} \to \mathbb{R}$ be as in Proposition 4. To get the required function $f : \mathbb{R} \to \mathbb{R}$, first we define $f$ on $A$ as $f(a) = \phi(a)$ for $a \in A$. Then by Proposition 4, $f : \mathbb{R} \to \mathbb{R}$ will be an extendable function irrespective of how we define $f$ on $\mathbb{R} \setminus A$. We will define $f$ on $\mathbb{R} \setminus A$ through a process of transfinite induction.

Recall that $\mathcal{O}_{\mathbb{R}} \subset \mathbb{R}^{\mathbb{N}}$ is the collection of orbits of all functions from $\mathbb{R} \to \mathbb{R}$. Let $\mathcal{E} \subset \mathbb{R}^{\mathbb{N}}$ be the collection of all decreasing sequences $(\epsilon_n)_{n=1}^{\infty}$ of positive reals converging to 0. Note that the sets $\mathbb{R}^{\mathbb{N}}$, $\mathcal{O}_{\mathbb{R}}$, $\mathcal{E}$ all have cardinality $\mathfrak{c}$. Let $\mathcal{A} = \mathbb{R}^{\mathbb{N}} \times \mathcal{O}_{\mathbb{R}} \times \mathcal{E}$. Then, $\mathcal{A}$ also has cardinality $\mathfrak{c}$. Since we assume the Continuum Hypothesis, we can index the elements of $\mathcal{A}$ using the first uncountable ordinal $\Omega$ as $\mathcal{A} = \{(R_\alpha, X_\alpha, E_\alpha) : \alpha < \Omega\}$, where we write $R_\alpha = (r_{\alpha,n})_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ , $X_\alpha = (x_{\alpha,n})_{n=1}^{\infty} \in \mathcal{O}_{\mathbb{R}}$ and $E_\alpha = (\epsilon_{\alpha,n})_{n=1}^{\infty} \in \mathcal{E}$.

Let $D_0 = A$, where $f$ is already defined. Suppose that for some $\alpha < \Omega$ and all $\beta < \alpha$ we have chosen $D_\beta \subset \mathbb{R}$ such that each $D_\beta$ is of first category and that $f$ is defined on $D_\beta$. Note that $\bigcup_{\beta<\alpha} D_\beta$ is also of first category. The $\alpha^{th}$ step is done as follows.

Consider $(R_\alpha, X_\alpha, E_\alpha) \in \mathcal{A}$. Using Baire Category Theorem, inductively choose $a_{\alpha,n} \in (0, \epsilon_{\alpha,n})$ such that all terms $s_{\alpha,n} := r_{\alpha,n} + a_{\alpha,n}$, $n \in \mathbb{N}$, are distinct and such that $s_{\alpha,n}$'s do not belong to the first category set $\bigcup_{\beta<\alpha} D_\beta$. Next, again by the help of Baire Category Theorem, choose a constant $b_\alpha \in (0, \epsilon_{\alpha,1})$ such that the elements $t_{\alpha,n} := x_{\alpha,n} + b_\alpha$, for $n \in \mathbb{N}$, do not belong to the first category set $\{s_{\alpha,n} : n \in \mathbb{N}\} \cup [\bigcup_{\beta<\alpha} D_\beta]$. Define $f(s_{\alpha,n}) = s_{\alpha,n+1}$ (possible since $s_{\alpha,n}$'s are distinct), and $f(t_{\alpha,n}) = t_{\alpha,n+1}$ (possible since $(t_{\alpha,n})_{n=1}^{\infty}$ is a translate of an orbit of some function). Put $D_\alpha = \{s_{\alpha,n} : n \in \mathbb{N}\} \cup \{t_{\alpha,n} : n \in \mathbb{N}\} \cup [\bigcup_{\beta<\alpha} D_\beta]$. To proceed with the transfinite induction, note that $D_\alpha$ is of first category. Finally, put $f(y) = 0$ for any possible $y \in \mathbb{R} \setminus \bigcup_{\alpha<\Omega} D_\alpha$. It is not difficult to verify that the function $f$ satisfies all the requirements. □

**Remark**: One of the referees pointed out that in the $\alpha^{th}$ step of the above proof, $s_{\alpha,n}$ can be chosen even if the complement of $\bigcup_{\beta<\alpha} D_\beta$ is only $\mathfrak{c}$-dense. Therefore, by some extra work involving the consideration of a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, etc., part (ii) might be proved without assuming the Continuum Hypothesis. However, the author does not know whether part (i) can be proved in ZFC.

## 5 An Almost Continuous Function $f : \mathbb{R} \to \mathbb{R}$.

The aim of this section is to establish that an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ can exhibit complicated dynamical behavior stronger than topological transitivity even if all orbits are finite. First we obtain an auxiliary result.

**Lemma 2.** *Let $X$ be an infinite, second countable metric space without isolated points, and let $a \in X$. Then there exists a function $f : X \to X$ such that*
*(i) $f(a) = a$; and for every $x \in X$ there exists $n \in \mathbb{N}$ such that $f^n(x) = a$,*
*(ii) $G_{f^m}$ is dense in $X^2$ for every $m \in \mathbb{N}$.*

Proof. Let $\{B(j) : j \in \mathbb{N}\}$ be a countable base of nonempty open sets for $X$. Note that each $B(j)$ is infinite as $X$ has no isolated points. Let $\{(i_k, j_k) : k \in \mathbb{N}\}$ be an enumeration of $\mathbb{N}^2$. We define $f : X \to X$ in an inductive fashion. Define $f(a) = a$ and put $D_0 = \{a\}$. Next, choose two distinct points $x_{1,0}, x_{1,1}$ in $X \setminus D_0$ such that $x_{1,0} \in B(i_1)$ and $x_{1,1} \in B(j_1)$. Define $f(x_{1,0}) = x_{1,1}$, $f(x_{1,1}) = a$ and put $D_1 = D_0 \cup \{x_{1,0}, x_{1,1}\}$. At the $k^{th}$ step, choose $k+1$ distinct points $x_{k,0}, x_{k,1}, \ldots, x_{k,k}$ in $X \setminus D_{k-1}$ such that $x_{k,0} \in B(i_k)$ and $x_{k,r} \in B(j_k)$ for $1 \le r \le k$. Define $f(x_{k,r}) = x_{k,r+1}$ for $0 \le r < k$, $f(x_{k,k}) = a$ and put $D_k = D_{k-1} \cup \{x_{k,r} : 0 \le r \le k\}$. Having defined $f$ on $\bigcup_{k=0}^{\infty} D_k$, put $f(x) = a$ for $x \in X \setminus \bigcup_{k=0}^{\infty} D_k$. Clearly, statement (i) of the Lemma holds. Also, by construction we have that for any $k \in \mathbb{N}$, $f^r(B(i_k)) \cap B(j_k) \ne \emptyset$ for $1 \le r \le k$. Now, if $U, V$ are nonempty subsets of $X$ and $m \in \mathbb{N}$, we can find $k \ge m$ such that $B(i_k) \subset U$ and $B(j_k) \subset V$. Then, $f^m(U) \cap V \ne \emptyset$. □

A sufficient condition for a function $f : \mathbb{R} \to \mathbb{R}$ to be almost continuous, observed by K. R. Kellum in [5], is the following.

**Proposition 6.** *[5] Let $\mathcal{F}$ be the collection of all closed subsets $F$ of $\mathbb{R}^2$ such that $\pi(F)$ has cardinality $\mathfrak{c}$, where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the projection to the first coordinate. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose graph $G_f$ intersects every $F \in \mathcal{F}$. Then $f$ is almost continuous.*

This helps us to prove the following.

**Proposition 7.** *There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following:*
*(i) $f^k$ is almost continuous for every $k \in \mathbb{N}$.*
*(ii) $f(0) = 0$; and for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $f^n(x) = 0$.*
*(iii) $G_{f^m}$ is dense in $\mathbb{R}^2$ for every $m \in \mathbb{N}$.*

Proof. Applying Lemma 2 with $X = \mathbb{Q}$ and $a = 0$, we get a function $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(0) = 0$, that for every $x \in \mathbb{Q}$ there is $n \in \mathbb{N}$ with $f^n(x) = 0$, and such that $G_{f^m}$ is dense in $\mathbb{Q}^2$ for every $m \in \mathbb{N}$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, it is clear that *any* extension of $f$ to a function (for which we use the same notation) $f : \mathbb{R} \to \mathbb{R}$ will satisfy the statement (iii) of the Proposition. Hence it suffices to take care of statements (i) and (ii). We use transfinite induction to extend $f$ from $\mathbb{Q}$ to $\mathbb{R}$.

Let $\mathcal{F}$ be the special collection of closed subsets of $\mathbb{R}^2$ mentioned in Proposition 6. We write $\mathcal{F} \times \mathbb{N} = \{(F_\alpha, k_\alpha) : \alpha < \mathfrak{c}\}$. To start the induction procedure, let $D_0 = \mathbb{Q}$. Suppose that for some $\alpha < \mathfrak{c}$ and for all $\beta < \alpha$ we have chosen $D_\beta \subset \mathbb{R}$ such that $D_\beta$ has cardinality less than $\mathfrak{c}$ and that $f$ is defined on $D_\beta$. At the $\alpha^{th}$ step we do the following.

Consider $(F_\alpha, k_\alpha)$. Since $\pi(F_\alpha)$ has cardinality $\mathfrak{c}$ and since $\bigcup_{\beta<\alpha} D_\beta$ has cardinality less than $\mathfrak{c}$, we may choose $k_\alpha+1$ distinct points $\{x_{\alpha,j} : 0 \le j \le k_\alpha\}$ such that $x_{\alpha,j} \notin \bigcup_{\beta<\alpha} D_\beta$ for $0 \le j < k_\alpha$ and such that $(x_{\alpha,0}, x_{\alpha,k_\alpha}) \in F_\alpha$. Define $f(x_{\alpha,j}) = x_{\alpha,j+1}$ for $0 \le j < k_\alpha$. This ensures that $G_{f^j} \cap F_\alpha \ne \emptyset$ for $1 \le j \le k_\alpha$ so that Proposition 6 may be invoked later. If $x_{\alpha,k_\alpha}$ does not belong to $\bigcup_{\beta<\alpha} D_\beta$, also define $f(x_{\alpha,k_\alpha}) = 0$. Put $D_\alpha = [\bigcup_{\beta<\alpha} D_\beta] \cup \{x_{\alpha,j} : 0 \le j \le k_\alpha\}$, which is again a set of cardinality less than $\mathfrak{c}$, thereby allowing the passage to the next step. Finally, we define $f(x) = 0$ for $x \in \mathbb{R} \setminus \bigcup_{\alpha<\mathfrak{c}} D_\alpha$. That the resulting $f$ works for us is easy verification. □

With slight modifications of the above proof one can get variants of Proposition 7. For instance, we can prove the following.

**Proposition 8.** *There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following:*

*(i) $f^k$ is almost continuous for every $k \in \mathbb{N}$.*

*(ii) the set $Fix(f)$ of fixed points of $f$ is dense in $\mathbb{R}$.*

*(iii) for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $f^n(x) \in Fix(f)$. (Thus every $f$-orbit is finite.)*

*(iv) $G_{f^m}$ is dense in $\mathbb{R}^2$ for every $m \in \mathbb{N}$.*

Hint for Proof. Using Lemma 2, first obtain a function $f : \mathbb{Q} \to \mathbb{Q}$. Next, define $f(x) = x$ for $x \in \mathbb{Q} + \sqrt{2}$. Put $D_0 = \mathbb{Q} \cup [\mathbb{Q} + \sqrt{2}]$ and proceed by transfinite induction as in the proof of the previous Proposition. □

**Question:** Can the functions in the last two Propositions be extendable?

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