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ASYMPTOTIC STRUCTURE OF BANACH SPACES AND RIEMANN INTEGRATION

Abstract

In this paper we focus on the Lebesgue property of Banach spaces. A real Banach space X is said to have the Lebesgue property if any Riemann integrable function from $[0,1]$ into X is continuous almost everywhere on $[0,1]$. We obtain a partial characterization of the Lebesgue property, showing that it has connections with the asymptotic geometry of the space involved.

1 Introduction.

This section will give some historical background. In 1972, R. Redjouani et al. [15, 17] were the first to show that ℓ^1 has the Lebesgue property (or is a *Lebesgue space*, for short). On the other hand, it can easily be seen that the so-called classical Banach spaces including ℓ^p for $1 < p < \infty$, c_0, and L^p for $1 \leq p < \infty$ do not have the Lebesgue property. Moreover, all these spaces except L^1 do not contain any subspace having the Lebesgue property.

In 1984, R. Haydon [8] proved that if a *stable* Banach space with uniformly separable *types* has the Schur property, then it has the Lebesgue property. Recall that a real Banach space X is said to have the *Schur property* (or to be a *Schur space*, for short) if each weakly null sequence in X converges in norm. The reader should refer to [10, 7] for an extensive study of stable Banach spaces and types. In particular, it follows from this result that a Schur subspace of L^1 has the Lebesgue property. We ought to observe at this point

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that the same immediately follows from the *weak property of Lebesgue* of L^1 [20]. In the same paper Haydon went on to prove that a stable Lebesgue space is necessarily a Schur space. To this end he employed an unpublished result attributable to A. Pełczyński and G. C. da Rocha Filho which states that each *spreading model* of a Lebesgue space is equivalent to the standard unit vector basis of ℓ^1. We will present our proof of this important fact below. In the remainder of his paper Haydon provided a rather lengthy construction of a stable Schur space failing the Lebesgue property. We will give another example of a Schur space which does not have the Lebesgue property that is simpler than Haydon's. Nevertheless, we make note of the fact that both constructions are based on the *dyadic tree*.

In 1991, R. Gordon [6] published the first truly non-classical example of a Lebesgue space; the Tsirelson space T. T, being close to ℓ^1 in an asymptotic sense, is reflexive and does not contain an isomorphic copy of either ℓ^p for $1 \leq p < \infty$ or c_0. We extend Gordon's result to prove that an *asymptotic* ℓ^1 Banach space has the Lebesgue property.

2 Notation and Preliminaries.

In this section we set notation related to Banach spaces and the Riemann integral and prove some preliminary facts.

2.1 Banach Spaces.

In what follows X denotes a real Banach space and X^* its dual. c_{00} denotes the linear space of all real sequences that are finitely non-zero and $\{\mathbf{e}_i\}_{i=1}^{\infty}$ its standard unit vector basis.

Let $\{u_i\}$ be a sequence in a Banach space. $[u_i]$ denotes the closed linear span of $\{u_i\}$. $\{u_i\}$ is said to be *normalized* if $\|u_i\| = 1$ for each i.

$\{u_i\}$ is said to be a *basis* in X, if each $x \in X$ has a unique expansion of the form $\sum_i a_i u_i$. $\{u_i\}$ is called a *basic* sequence if it is a basis in $[u_i]$. A sequence of non-zero vectors $\{u_i\}$ is basic (C-*basic*) if and only if there exists $C \geq 1$ such that $\| \sum_{i=1}^m a_i u_i\| \leq C \| \sum_{i=1}^n a_i u_i\|$ for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ and for all $m \leq n$.

A basic sequence $\{u_i\}$ is said to be *unconditional* (C-*unconditional*) if there exists a constant $C \geq 1$ such that $\| \sum_{i=1}^n \epsilon_i a_i u_i\| \leq C \| \sum_{i=1}^n a_i u_i\|$ for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ and any sequence of signs $\{\epsilon_i = \pm 1\}_{i=1}^n$. It is useful to note that if $\{u_i\}$ is C-unconditional, then

$$\left\|\sum\nolimits_{i=1}^{n} \lambda_i a_i u_i\right\| \leq C \max\nolimits_i |\lambda_i| \cdot \left\|\sum\nolimits_{i=1}^{n} a_i u_i\right\|$$

for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ and for all $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}$ (see [11, Proposition 1.c.7]). In particular, a sequence of non-zero vectors $\{u_i\}$ is 1-unconditional if and only if $\|\sum_{i=1}^n \epsilon_i a_i u_i\| = \|\sum_{i=1}^n a_i u_i\|$ for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ and for any sequence of signs $\{\epsilon_i = \pm 1\}_{i=1}^n$.

A basic sequence $\{u_i\}$ is said to be *suppression-C-unconditional*, if

$$\left\|\sum\nolimits_{i\in I} a_i u_i\right\| \le C \left\|\sum\nolimits_{i=1}^n a_i u_i\right\|$$

for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ and for any $I \subset \{1,\dots,n\}$.

It is not hard to see that a suppression-C-unconditional sequence is $2C$-unconditional. Conversely, C-unconditional sequence is always suppression-C-unconditional.

Basic sequences $\{u_i\}$ and $\{v_i\}$ are called *C-equivalent* for some $C \ge 1$, if

$$C^{-1}\left\|\sum\nolimits_{i=1}^n a_i v_i\right\| \le \left\|\sum\nolimits_{i=1}^n a_i u_i\right\| \le C\left\|\sum\nolimits_{i=1}^n a_i v_i\right\|$$

for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$. A basic sequence $\{u_i\}$ is *C-subsymmetric*, if $\{u_i\}$ is C-equivalent to any its subsequence $\{u_i^{'}\}$. Note that if $\{u_i\}$ is 1-subsymmetric, then $\|a_1u_1 + \cdots + a_nu_n\| = \|a_1u_{k_1} + \cdots + a_nu_{k_n}\|$ for all $k_1 < \cdots < k_n$ and for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$.

A non-zero vector x of the form $\sum_{i=m}^n a_i u_i$, $\{a_i\}_{i=m}^n \subset \mathbb{R}$, is called a *block vector* (or a *block*, in short) with respect to a fixed sequence $\{u_i\}$. Denote the set of all integers i for which $a_i \ne 0$ by $\operatorname{supp} x$. We write $x < y$ for two blocks x and y, if $\max\operatorname{supp} x < \min\operatorname{supp} y$. Blocks $x_1,\dots,x_n$ are called *successive* provided $x_1 < \cdots < x_n$. Note that successive blocks $\{x_i\}$ with respect to a C-basic sequence form a C-basic sequence.

Let x be a vector of the form $\sum_i a_i u_i$ and I, J be non-empty sets of integers. In this case, we write $Ix = \sum_{i\in I} a_i u_i$ and $I < J$, if $\max I < \min J$.

2.2 Riemann Integration.

In this section we will sharpen Theorems 3 and 5 of [6]. To begin with, we briefly recall the standard terminology related to the Riemann integral as presented in [6]. A *partition* $\mathscr{P}$ of $[a,b]$ is a finite set of *points* $\{a = t_0 < t_1 < \cdots < t_N = b\}$. A *tagged partition* $\mathscr{T}$ of $[a,b]$ consists of a partition $\{t_i\}_{i=0}^N$ of $[a,b]$ and a finite set of *tags* $\{\tau_i\}_{i=1}^N$ that satisfy $\tau_i \in [t_{i-1}, t_i]$ for each i. The points $\{t_i\}_{i=0}^N$ are called the *points* of $\mathscr{T}$, the intervals $\{[t_{i-1}, t_i]\}_{i=1}^N$ are called the *intervals* of $\mathscr{T}$, and the *norm* $|\mathscr{T}|$ of $\mathscr{T}$ is defined by $|\mathscr{T}| = \max_{1\le i\le N}(t_i - t_{i-1})$. If $f : [a,b] \to X$, then $f(\mathscr{T})$ denotes the Riemann sum $\sum_{i=1}^N f(\tau_i)(t_i - t_{i-1})$. A tagged partition $\mathscr{T}$ of $[a,b]$ *refines* a partition $\mathscr{P}$ of $[a,b]$ if each point of $\mathscr{P}$ is simultaneously a point of $\mathscr{T}$. Finally, we say that a tagged partition is an *interior* tagged partition if each tag of the tagged partition lies in the interior of its interval.

Definition 1. Let $f : [a, b] \to X$.

(i) f is R_δ *integrable (resp.* R^*_δ *integrable)* on $[a, b]$ if there exists a vector $z \in X$ such that for each $\epsilon > 0$ there is $\delta > 0$ so that $\|f(\mathscr{T}) - z\| < \epsilon$ whenever a tagged partition (resp. an interior tagged partition) $\mathscr{T}$ of $[a, b]$ satisfies $|\mathscr{T}| < \delta$.

(ii) f is R_Δ *integrable (resp.* R^*_Δ *integrable)* on $[a, b]$ if there exists a vector $z \in X$ such that for each $\epsilon > 0$ there is a partition $\mathscr{P}_\epsilon$ of $[a, b]$ so that $\|f(\mathscr{T}) - z\| < \epsilon$ whenever a tagged partition (resp. an interior tagged partition) $\mathscr{T}$ of $[a, b]$ refines $\mathscr{P}_\epsilon$.

A standard argument shows that a function integrable on $[a, b]$ in each of the above four senses must be bounded on $[a, b]$.

Theorem 1. *A function* $f : [a, b] \to X$ *is* R_Δ *integrable (resp.* R^*_Δ *integrable) on* $[a, b]$ *if and only if it is* R_δ *integrable (resp.* R^*_δ *integrable) on* $[a, b]$*.*

Proof. The proof of the theorem is completely analogous to that of Theorem 3 of [6] and is omitted. □

Lemma 1. *Suppose that* $f : [a, b] \to X$ *and* $\sup_{t \in [a,b]} \|f(t)\| = M < \infty$*. If positive numbers* ϵ *and* δ *satisfy* $\delta < \epsilon/4M$*, then for any tagged partition* $\mathscr{T}$ *of* $[a, b]$ *that satisfies* $|\mathscr{T}| < \delta/4$ *there exists an interior tagged partition* $\mathscr{T}^*$ *of* $[a, b]$ *such that* $\|f(\mathscr{T}) - f(\mathscr{T}^*)\| < \epsilon$ *and* $|\mathscr{T}^*| < \delta$*.*

Proof. Assume without loss of generality that $\mathscr{T} = \{(\tau_k, [t_{k-1}, t_k])\}_{k=1}^K$, $|\mathscr{T}| < \delta/2$, and $\tau_1 < \tau_2 < \cdots < \tau_K$. Set $K_0 = \{k \in \{2, \ldots, K-1\} : \tau_k = t_{k-1}\}$ and $K_1 = \{k \in \{2, \ldots, K-1\} : \tau_k = t_k\}$.

Now we construct the interior tagged partition $\mathscr{T}^* = \{(\tau^*_k, [t^*_{k-1}, t^*_k])\}_{k=1}^K$ of $[a, b]$ as follows. Choose $\tau^*_1 \in (a, t_1)$ and $\tau^*_K \in (t_{K-1}, b)$ freely. Let τ^*_k be equal to τ_k for $1 < k < K$. Next, if $k \in K_0$, then choose $t^*_{k-1} \in (\tau^*_{k-1}, t_{k-1})$ so that $t_{k-1} - t^*_{k-1} < \delta/K$. If $k \in K_1$, then choose $t^*_k \in (t_k, \tau^*_{k+1})$ so that $t^*_k - t_k < \delta/K$. The remaining points of $\mathscr{T}^*$ are equal to the corresponding points of $\mathscr{T}$. We have

$$\|f(\mathscr{T}) - f(\mathscr{T}^*)\| \le 4M \cdot \frac{\delta}{2} + 2M \sum\nolimits_{k=2}^{K-1} \frac{\delta}{K} < \epsilon$$

which is what we desired. □

Theorem 2. *If a function* $f : [a, b] \to X$ *is* R^*_δ *integrable on* $[a, b]$*, then* f *is* R_δ *integrable on* $[a, b]$*.*

Proof. Let z be the R_δ^* integral of f over $[a,b]$. Fix $\epsilon > 0$ and set $M = \sup_{t\in[a,b]} \|f(t)\| < \infty$. There exists $0 < \delta < \epsilon/8M$ such that $\|f(\mathscr{T}^*)-z\| < \epsilon/2$ whenever an interior tagged partition $\mathscr{T}^*$ of $[a,b]$ satisfies $|\mathscr{T}^*| < \delta$. Let $\mathscr{T}$ be a tagged partition of $[a,b]$ that satisfies $|\mathscr{T}| < \delta/4$. By Lemma 1 we have

$$\|f(\mathscr{T}) - z\| \le \|f(\mathscr{T}^*) - z\| + \|f(\mathscr{T}) - f(\mathscr{T}^*)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Thus, the above four notions of the Riemann integrability are equivalent and we make the following definition.

Definition 2. A function $f : [a,b] \to X$ is *Riemann integrable* on $[a,b]$ if f is either R_δ or R_δ^* or R_Δ or R_Δ^* integrable on $[a,b]$.

Theorem 3. *Let $f : [a,b] \to X$. If for each $\epsilon > 0$ there exists a partition $\mathscr{P}_\epsilon$ of $[a,b]$ such that $\|f(\mathscr{T}_1) - f(\mathscr{T}_2)\| < \epsilon$ for all interior tagged partitions $\mathscr{T}_1$ and $\mathscr{T}_2$ of $[a,b]$ that have the same points as $\mathscr{P}_\epsilon$, then f is Riemann integrable on $[a,b]$.*

Proof. Gordon's proof of (4) $\Rightarrow$ (3) in Theorem 5 of [6] can be adapted to prove the theorem. $\square$

3 Spreading Models of a Lebesgue Space.

The theory of *spreading models* is an important application of the Ramsey theory to Banach spaces. A spreading model arises from a normalized basic sequence and provides a way of studying the asymptotic nature of a Banach space. Some facts about spreading models are gathered in [16]. It is a well-known consequence of Rosenthal's ℓ^1 theorem [18] that each spreading model of a Schur space is equivalent to the standard unit vector basis of ℓ^1. In this section we demonstrate that the same property is fulfilled for a Lebesgue space. It has been widely noted that the Schur property is closely related to weak versions of the Riemann integral (see [6, 9, 21]).

We begin with two auxiliary facts concerning normalized 1-subsymmetric sequences. The first, Lemma 2, is actually Lemma I.1 of [3].

Lemma 2. *If $\{e_i\}_{i=1}^\infty$ is a normalized 1-subsymmetric sequence, then the sequence $\{e_{2i-1} - e_{2i}\}_{i=1}^\infty$ is suppression-1-unconditional.*

The second, Lemma 3, is a compilation of Lemmas III.2 and II.3 of [3]. However, we present complete proof for the reader's convenience.

Lemma 3. *Let $\{e_i\}_{i=1}^\infty$ be a normalized 1-subsymmetric sequence. If*

$$\|e_1 - e_2 + \cdots + e_{2n-1} - e_{2n}\| \ge n\delta$$

for some $\delta > 0$ and for all $n \in \mathbb{N}$, then $\{e_i\}_{i=1}^{\infty}$ is $4\delta^{-1}$-equivalent to the standard unit vector basis of ℓ^1.

Proof. Fix a sequence of signs $\{\epsilon_i = \pm 1\}_{i=1}^{n}$. It follows that

$$\|\epsilon_1 e_1 + \cdots + \epsilon_n e_n\| = \|\epsilon_1 e_1 + \cdots + \epsilon_n e_{2n-1}\| = \| - \epsilon_1 e_2 - \cdots - \epsilon_n e_{2n}\|.$$

By Lemma 2 we have

$$\begin{aligned} 2\|\epsilon_1 e_1 + \cdots + \epsilon_n e_n\| &\geq \|\epsilon_1(e_1 - e_2) + \cdots + \epsilon_n(e_{2n-1} - e_{2n})\| \\ &\geq \frac{1}{2} \cdot \|e_1 - e_2 + \cdots + e_{2n-1} - e_{2n}\| \geq \frac{n\delta}{2}. \end{aligned}$$

Now fix $\{p_i\}_{i=1}^{n} \subset \mathbb{N}$. Let $v_1 = \epsilon_2 p_2 e_2 + \cdots + \epsilon_n p_n e_n$ and $w_1 = \epsilon_2 p_2 e_{p_1+1} + \cdots + \epsilon_n p_n e_{p_1+\cdots+p_{n-1}+1}$. We have $\|\epsilon_1 e_1 + \frac{v_1}{p_1}\| = \|\epsilon_1 e_j + \frac{w_1}{p_1}\|$ for $j = 1, \ldots, p_1$. Summing up these equalities, we obtain $\|\epsilon_1 p_1 e_1 + v_1\| \geq \|\epsilon_1(e_1 + \cdots + e_{p_1}) + w_1\|$. Let $v_2 = \epsilon_1(e_1 + \cdots + e_{p_1}) + \epsilon_3 p_3 e_3 + \cdots + \epsilon_n p_n e_n$ and $w_2 = \epsilon_1(e_1 + \cdots + e_{p_1}) + \epsilon_3 p_3 e_{p_1+p_2+1} + \cdots + \epsilon_n p_n e_{p_1+\cdots+p_{n-1}+1}$. We have $\|\epsilon_2 e_2 + \frac{v_2}{p_2}\| = \|\epsilon_2 e_{p_1+j} + \frac{w_2}{p_2}\|$ for $j = 1, \ldots, p_2$. Summing up these equalities, we obtain $\|\epsilon_2 p_2 e_1 + v_2\| \geq \|\epsilon_2(e_{p_1+1} + \cdots + e_{p_1+p_2}) + w_2\|$. Continuing this process for n steps, we get

$$\begin{aligned} \|\epsilon_1 p_1 e_1 + \cdots + \epsilon_n p_n e_n\| &\geq \|\epsilon_1(e_1 + \cdots + e_{p_1}) + \epsilon_2(e_{p_1+1} + \cdots + e_{p_1+p_2}) \\ + \cdots + \epsilon_n(e_{p_1+\cdots+p_{n-1}+1} + \cdots + e_{p_1+\cdots+p_n})\| &\geq (p_1 + \cdots + p_n) \cdot \frac{\delta}{4} \end{aligned}$$

from which it follows that $\{e_i\}_{i=1}^{\infty}$ is $4\delta^{-1}$-equivalent to the standard unit vector basis of ℓ^1. □

Definition 3. Let $\{e_i\}_{i=1}^{\infty}$ be a normalized basic sequence. A basic sequence $\{s_i\}_{i=1}^{\infty}$ is said to be a *spreading model* of $\{e_i\}_{i=1}^{\infty}$ if for some sequence of positive numbers $\epsilon_n \downarrow 0$ and for all $\{a_i\}_{i=1}^{n} \subset [-1, 1]$ we have

$$\left| \left\| \sum_{i=1}^{n} a_i e_{k_i} \right\| - \left\| \sum_{i=1}^{n} a_i s_i \right\| \right| < \epsilon_n$$

whenever $n \leq k_1 < \cdots < k_n$.

A spreading model is necessarily 1-subsymmetric. It is well known that each normalized basic sequence has a subsequence with a spreading model. If $\{e_i\}$ is weakly null, then $\{s_i\}$ is suppression-1-unconditional. For the proofs of these results see, for example, [16, Theorem 2.2] and either [16, Proposition 2.3] or [3, Lemma I.2], respectively.

Let $D = \left\{d_{kj} = \frac{2j-1}{2^k}\right\}_{k=1,2,\ldots,j=1,\ldots,2^{k-1}}$ be the set of dyadic rational numbers in $(0, 1)$. Given $n \in \mathbb{N}$, there is a unique pair (k, j) such that $k \in \mathbb{N}$, $j \in \{1, \ldots, 2^{k-1}\}$ and $n = 2^{k-1} + j - 1$. Let $d_n = d_{kj}$. Then we have $D = \{d_1, d_2, \ldots\}$.

Theorem 4. *Let* $\{e_i\}_{i=1}^{\infty}$ *be a normalized* C*-basic sequence in* X *with a spreading model* $\{s_i\}_{i=1}^{\infty}$*. Then the following three are equivalent:*

(i) The function $f : [0,1] \to X$ *such that* $f(d_i) = u_i = e_{2i} - e_{2i+1}$ *and* $f(t) = 0$ *for* $t \notin D$ *is not Riemann integrable on* $[0,1]$.

(ii) $\overline{\lim}_{n\to\infty} \|s_1 - s_2 + \cdots - s_{2^n}\|/2^n > 0$.

(iii) $\|s_1 - s_2 + \cdots + s_{2n-1} - s_{2n}\| \ge n\delta$ *for some* $\delta > 0$ *and for all* $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\|s_1 - s_2 + \cdots - s_{2^n}\|/2^n \to 0$ as $n \to \infty$. Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that

$$\left\|\frac{s_1 - s_2 + \cdots - s_{2^N}}{2^N}\right\| < \frac{\epsilon}{4C} \text{ and } \epsilon_{2^N} < \frac{\epsilon}{2C}.$$

Fix $\tau_j \in \left(\frac{j-1}{2^{N-1}}, \frac{j}{2^{N-1}}\right)$ for $j = 1, \ldots, 2^{N-1}$. Then

$$\{f(\tau_1), \ldots, f(\tau_{2^{N-1}})\} \setminus \{0\} = \{u_{k_1}, \ldots, u_{k_n}\}$$

for some $k_1 < \cdots < k_n$. We have $n \le 2^{N-1} \le k_1 < \cdots < k_n < k_{n+1} = k_n + 1 < \cdots < k_{2^{N-1}} = k_n + 2^{N-1} - n$. Hence

$$\begin{aligned}\|f(\tau_1) + \cdots + f(\tau_{2^{N-1}})\| =& \|u_{k_1} + \cdots + u_{k_n}\| \le C\|u_{k_1} + \cdots + u_{k_{2^{N-1}}}\| \\ \le& C\|s_1 - s_2 + \cdots - s_{2^N}\| + C\big|\|u_{k_1} + \cdots + u_{k_{2^{N-1}}}\| \\ & - \|s_1 - s_2 + \cdots - s_{2^N}\|\big| < C \cdot \frac{\epsilon}{4C} \cdot 2^N + C \cdot \epsilon_{2^N} \\ <& \frac{\epsilon}{2} \cdot 2^{N-1} + \frac{\epsilon}{2} \le \epsilon \cdot 2^{N-1},\end{aligned}$$

and, by Theorem 3, f is Riemann integrable on $[0,1]$.

(ii) $\Rightarrow$ (i). Suppose that $\overline{\lim}_{n\to\infty} \|s_1 - s_2 + \cdots - s_{2^n}\|/2^n > 0$. It follows that there exist $\delta > 0$ and a sequence of positive integers $N_k \nearrow \infty$ such that

$$\left\|\frac{s_1 - s_2 + \cdots - s_{2^{N_k}}}{2^{N_k}}\right\| > \delta \text{ for all } k.$$

Choose tags $\tau_j = d_{N_k j} = \frac{2j-1}{2^{N_k}} \in \left(\frac{j-1}{2^{N_k-1}}, \frac{j}{2^{N_k-1}}\right)$ for $j = 1, \ldots, 2^{N_k-1}$ and $K \in \mathbb{N}$ so that $\epsilon_{2^{N_K}} < \delta/2$. Then

$$\begin{aligned}2^{-N_k}\|f(\tau_1) + \cdots + f(\tau_{2^{N_k-1}})\| \ge& 2^{-N_k}\|s_1 - s_2 + \cdots - s_{2^{N_k}}\| \\ & - 2^{-N_k}\big|\|s_1 - s_2 + \cdots - s_{2^{N_k}}\| \\ & - \|f(\tau_1) + \cdots + f(\tau_{2^{N_k-1}})\|\big| > \delta/2\end{aligned}$$

whenever $k \ge K$. This contradicts the Riemann integrability of f on $[0,1]$.

(ii) $\Rightarrow$ (iii). Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$ for which $2n \cdot 2^{-N_k} < \delta/2$. Let $2^{N_k} = 2mn + r$, where $m \in \mathbb{N}$ and $0 \le r < 2n$. We have

$$\begin{aligned}\delta \cdot 2^{N_k} <& \|s_1 - s_2 + \cdots - s_{2^{N_k}}\| \\ \le& \|s_1 - s_2 + \cdots - s_{2mn}\| + \|s_{2mn+1} + \cdots - s_{2mn+r}\|.\end{aligned}$$

Hence

$$\frac{\delta}{2} \cdot 2mn \le \frac{\delta}{2} \cdot 2^{N_k} < \|s_1 - s_2 + \cdots - s_{2mn}\| \le m \cdot \|s_1 - s_2 + \cdots - s_{2n}\|.$$

(iii) $\Rightarrow$ (ii). The proof is clear. □

Theorem 5. *If X has the Lebesgue property, then each spreading model of X is equivalent to the standard unit vector basis of ℓ^1.*

Proof. Let $\{e\}_{i=1}^{\infty}$ be a normalized C-basic sequence in X with spreading model $\{s_i\}_{i=1}^{\infty}$. Consider the sequence $\{u_i = e_{2i} - e_{2i+1}\}_{i=1}^{\infty}$. We have

$$1 = \|e_{2i}\| \le C\|e_{2i} - e_{2i+1}\| = C\|u_i\|.$$

It follows that $2 \ge \|u_i\| \ge C^{-1}$ and the function f from item (i) of Theorem 4 is bounded and discontinuous everywhere on $[0, 1]$. Since X has the Lebesgue property, f is not Riemann integrable on $[0, 1]$. Thus $\{s_i\}_{i=1}^{\infty}$ satisfies the estimate of item (iii) of Theorem 4. Finally, it follows from Lemma 3 that $\{s_i\}_{i=1}^{\infty}$ is equivalent to the standard unit vector basis of ℓ^1. □

Remark. An application of Rosenthal's ℓ^1 theorem [18] together with some standard arguments show that a Banach space that satisfies the conclusion of Theorem 5 is necessarily ℓ^1-*convex* (see [14]). The converse is not true. For example, the Lorentz sequence space $d(w, 1)$ (see [11, Definition 4.e.1])is ℓ^1-convex and its standard unit vector basis is 1-subsymmetric and, clearly, not equivalent to that of ℓ^1.

The remainder of this section will give a construction of a Schur space E failing the Lebesgue property. The construction of the space E is borrowed from Talagrand's paper [19]. However, our notation is slightly different from that in [19].

Consider the set $T = \bigcup_{n=0}^{\infty} T_n$, $T_n = \{0, 1\}^n$. Given $s = (s_0, \ldots, s_{n-1}) \in T_n$ and $t = (t_0, \ldots, t_{m-1}) \in T_m$. Let $s \centerdot t = (s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1}) \in T_{n+m}$. We write $s \prec t$, if $t = s \centerdot q$ for some $q \in T$. Then $(T, \prec)$ is the usual *dyadic tree*. For each $t \in T$ there exists a unique $|t| \ge 0$ such that $t \in T_{|t|}$. If A is a finite non-empty subset of T, then the *stem* $s(A)$ of A is the maximal $s \in T$ that satisfies $s \prec t$ for all $t \in A$ and $|A| = \max\{|t| : t \in A\}$.

Let $\mathbb{R}^{(T)}$ denote the set of real functions g on T that have finite support $\operatorname{supp} g$ and let $\{\mathbf{e}_t\}_{t\in T}$ be the standard unit vector basis in $\mathbb{R}^{(T)}$. Let $s(g) = s(\operatorname{supp} g)$, $|g| = |\operatorname{supp} g|$, and $\langle\, g\,,\, h\,\rangle = \sum_{t\in T} g_t \cdot h_t$ for $g, h \in \mathbb{R}^{(T)}$.

We start with $H_0 = \{\mathbf{e}_t\}_{t\in T}$ and write $H_1 = \bigcup_{n=1}^{\infty} H_1^n$, where

$$H_1^n \;=\; \Big\{ 4^{-1} \sum_{t\in A=\{t_1,\dots,t_p\}} \mathbf{e}_t \;:\; |s(A)| \;\geq\; n, s(\{t_{i+1},\dots,t_p\}) \;\geq\; |t_i| \Big\}.$$

Next, write $H_2 = \bigcup_{n=1}^{\infty} H_2^n$, where

$$H_2^n = \Big\{ g = 4^{-1} \sum\nolimits_{i=1}^{p} g_i : g_i \in H_1^{k_i}, |g_i| < k_{i+1} \text{ for some } k_1 = n < \dots \\ < k_p, |s(g)| \geq n, |s(g_{i+1} + \dots + g_p)| \geq |g_i| \Big\}.$$

We continue this process by induction. For each $g \in H_m$ we have $g = 4^{-m} \sum_{t\in A} \mathbf{e}_t$. Moreover, if $B \subset A$, then $4^{-m}\sum_{t\in B} \mathbf{e}_t \in H_m$. Let $H^{'}$ be the set of finite sums of the form $\sum_{m\geq 2} g_m$ where $g_m \in H_m$. Finally, $H = H_0 \bigcup H_1 \bigcup H^{'}$.

Denote, by E, the completion of $\mathbb{R}^{(T)}$ with respect to the norm $\|\cdot\| = \sup_{g\in H} |\langle\, g\,,\,\cdot\,\rangle|$. Clearly, we have $\|\mathbf{e}_t\| = 1$. In spite of the fact that E is a Schur space [19], E actually contains "very few" sequences which are equivalent to the standard unit vector basis of ℓ^1 and does *not* have the Lebesgue property.

Dyadic rational numbers can in a natural way be indexed by T. Each $d \in D$ has a unique expression of the form

$$d = \frac{1}{2} + \sum_{k=0}^{n-1} \frac{s_k - \frac{1}{2}}{2^{k+1}}$$

for some $n \geq 0$. So, to a fixed $d \in D$ assign $\varphi(d) = (s_0, \dots, s_{n-1}) \in T_n$. Let $f : [0,1] \to E$ be a function such that $f(d) = \mathbf{e}_{\varphi(d)}$ for $d \in D$ and $f(t) = 0$ for $t \notin D$. Evidently, f is discontinuous everywhere on $[0,1]$.

Lemma 4. *Suppose that $n \in \mathbb{N}$ and $\tau_j \in \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right)$ for $j = 1, \dots, 2^n$. Then $0 \leq \langle\, g\,,\, f(\tau_1) + \dots + f(\tau_{2^n})\,\rangle \leq 1$ for all $g \in H$.*

Proof. Let $\{f(\tau_1), \dots, f(\tau_{2^n})\} \setminus \{0\} = \{\mathbf{e}_{t_1}, \dots, \mathbf{e}_{t_p}\}$. Note that $|t_i| \geq n$ for $i = 1, \dots, p$ and $|s(\{t_i, t_j\})| < n$ for $j \neq i$. Fix $g \in H$. If $g \in H_0$, then the inequality is obvious. If $g \in H_1$, then it follows from the definition of H_1 that the set $A = \operatorname{supp} g \cap \{t_1, \dots, t_p\}$ has at most two elements and, hence, the inequality is valid. By induction we obtain that A has at most two elements for $g \in H_m$ when $m \geq 2$. Thus for $g \in H^{'}$ we have

$$\langle\, g\,,\, f(\tau_1) + \dots + f(\tau_{2^n})\,\rangle \leq 2 \sum\nolimits_{m=2}^{\infty} 4^{-m} = 1/6$$

and our lemma is proved. □

Now it follows from the definition of the norm that $\|f(\tau_1)+\cdots+f(\tau_{2^n})\| \leq 1$. And, by Theorem 3, the function f is Riemann integrable on $[0,1]$. This in turn means that E does not have the Lebesgue property.

4 Asymptotic ℓ^1 Banach Spaces.

The Tsirelson space T has the Lebesgue property. The proof of this fact was first published in [6]. We will refine this argument to prove that an *asymptotic* ℓ^1 space has the Lebesgue property.

First of all, recall Tsirelson's definition [4] of a norm on c_{00}. For a fixed $0<\theta<1$ there is a unique norm on c_{00} that satisfies the equation

$$\|x\|_{T_\theta} = \max\Big\{\|x\|_\infty, \theta \sup \sum\nolimits_{i=1}^{n} \|I_i x\|_{T_\theta}\Big\},$$

where $I_1,\ldots,I_n$ run over finite sets of integers for which $n \leq I_1 < \cdots < I_n$. T_θ ($T_{1/2}$ is the original Tsirelson space T) is the completion of c_{00} with respect to $\|\cdot\|_{T_\theta}$. T_θ has normalized 1-unconditional basis $\{\mathbf{e}_i\}_{i=1}^{\infty}$ and its norm is asymptotically close to the norm of ℓ^1. However, it does not contain an isomorphic copy of either ℓ^p for $1 \leq p < \infty$ or c_0.

At the beginning of the 90s, asymptotic ℓ^1 spaces were introduced by Banach space theorists [13, 12] to obtain a generalization of Tsirelson's spaces. In this paper we are concerned with Banach spaces which are asymptotic ℓ^1 with respect to a basis, exclusively. So, we make the following definition.

Definition 4 ([5])**.** A Banach space is said to be *asymptotic* ℓ^1 space (C-*asymptotic* ℓ^1 space) with respect to its normalized basis $\{e_i\}$, if there is $C \geq 1$ such that for each $n \in \mathbb{N}$ there exists a function $F_n : \mathbb{N}_0 \to \mathbb{N}$ (with $F_n(k) \geq k$ for all k) so that

$$C^{-1} \sum\nolimits_{i=1}^{n} |a_i| \leq \Big\|\sum\nolimits_{i=1}^{n} a_i x_i\Big\|$$

for all normalized successive blocks $\{x_i\}_{i=1}^n$ with respect to $\{e_i\}$ that satisfy $F_n(0) \leq \operatorname{supp} x_1$ and $F_n(\max\operatorname{supp} x_i) < \min\operatorname{supp} x_{i+1}$, $i=1,\ldots,n-1$, and for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$. In other words, $\{x_i\}_{i=1}^n$ is C-equivalent to the unit vector basis of ℓ_1^n. If it is possible to choose $F_n(k)=k$ for all $n,k\in\mathbb{N}$, then X is called *stabilized asymptotic* ℓ^1 space with respect to $\{e_i\}$.

Let E_n be an arbitrary finite dimensional normed space. Then the direct ℓ^1-sum $\big(\sum_{n=1}^{\infty} \oplus E_n\big)_1$ gives a trivial example of a 1-asymptotic ℓ^1 space with respect to its natural unit vector basis. Such a sum is not always isomorphic to ℓ^1. It follows from the definition that the Tsirelson space T_θ is a stabilized θ^{-1}-asymptotic ℓ^1 space with respect to $\{\mathbf{e}_i\}_{i=1}^\infty$. Many other examples of asymptotic ℓ^1 spaces have been constructed in connection with various problems in Banach space geometry [1, 2]. These examples show that asymptotic ℓ^1 spaces form a large class. As an illustration, an asymptotic ℓ^1 space can contain no unconditional basic sequence [1].

Theorem 6. *Let X be C-asymptotic ℓ^1 space with respect to its normalized basis $\{e_i\}$. Then X has the Lebesgue property.*

Proof. Suppose on the contrary that there exits a Riemann integrable function $f : [0,1] \to X$ which is not continuous almost everywhere on $[0,1]$. Then, writing μ for Lebesgue measure and $\omega(f)(t)$ for the oscillation of f at a point t, there are positive numbers α and β such that $H = \{t \in [0,1] : \omega(f)(t) > \beta\}$ and $\mu(H) = \alpha$.

Consider the functions $f_j(t) = e_j^*(f)(t)$, where $\{e_i^*\} \subset X^*$ are the coefficient functionals of $\{e_i\}$. Denote, by G_j, the set of discontinuities of f_j on $[0,1]$ and note that $\mu(G_j) = 0$. If $G = \bigcup_{j=1}^\infty G_j$, then $\mu(G) = 0$ and each f_j is continuous on $[0,1] \setminus G$. Clearly, $f(t) = \sum_{j=1}^\infty f_j(t) e_j$ for all $t \in [0,1]$.

Fix $\delta > 0$ and $N \in \mathbb{N}$ such that $N^{-1} < \delta$. Set $\mathscr{P}_N = \{k/N\}_{k=0}^N$. List all the intervals $\{[c_i,d_i]\}_{i=1}^p$ of $\mathscr{P}_N$ for which $\mu(H \cap [c_i,d_i]) > 0$ in the increasing order. Then it is evident that $p/N \ge \alpha$.

Let F_p be the function as in Definition 4. Choose $\epsilon = \alpha\beta/16C$. Construct by induction the following sets: $\{u_i\}_{i=1}^p$, $u_i \in (H \setminus G) \cap (c_i,d_i)$, $\{v_i\}_{i=1}^p$, $v_i \in (c_i,d_i)$, $\{n_i\}_{i=0}^p$, $F_p(0) = n_0 < n_1$, $\max(F_p(n_{i-1}+1), \dots, F_p(n_i)) < n_{i+1}$ for $i = 1, \dots, p-1$ so that

$$\begin{aligned} z_i =& f(u_i) - f(v_i) = \sum\nolimits_{i=1}^\infty a_j^i e_j = w_i + x_i + y_i, \\ w_i =& \sum\nolimits_{j=1}^{n_{i-1}} a_j^i e_j, x_i = \sum\nolimits_{j=n_{i-1}+1}^{n_i} a_j^i e_j, y_i = \sum\nolimits_{j=n_i+1}^\infty a_j^i e_j, \\ \|z_i\| \ge& \beta/2,\ \|w_i\| \le \epsilon 2^{-i},\ \|y_i\| \le \epsilon 2^{-i}. \end{aligned}$$

Note that

$$\|x_i\| \ge \|z_i\| - \|w_i\| - \|y_i\| \ge \frac{\beta}{2} - 2\epsilon \cdot 2^{-i} \ge \frac{\beta}{2} - \epsilon = \frac{\beta}{2}\Big(1 - \frac{\alpha}{8C}\Big) \ge \frac{7\beta}{16}.$$

Choose $n_0 = F_p(0)$ and $u_1 \in (H \setminus G) \cap (c_1,d_1)$ so that $\omega(f)(u_1) > \beta$. Since f_j is continuous at u_1 for $j = 1, \dots, n_0$, there exists $v_1 \in (c_1,d_1)$ such that $\|f(u_1) - f(v_1)\| \ge \beta/2$ and $\sum_{j=1}^{n_0} |f_j(u_1) - f_j(v_1)| < \epsilon 2^{-1}$. Let $z_1 = f(u_1) - f(v_1) = \sum_{j=1}^\infty a_j^1 e_j$. Then $\|\sum_{j=1}^{n_0} a_j^1 e_j\| \le \sum_{j=1}^{n_0} |a_j^1| < \epsilon 2^{-1}$. Now choose $n_1 > n_0$ and $u_2 \in (H \setminus G) \cap (c_2,d_2)$ so that $\|\sum_{j=n_1+1}^\infty a_j^1 e_j\| < \epsilon 2^{-1}$ and

$\omega(f)(u_2) > \beta$. Since f_j is continuous at u_2 for $j = 1, \dots, n_1$, there exists $v_2 \in (c_2, d_2)$ such that $\|f(u_2) - f(v_2)\| \geq \beta/2$ and $\sum_{j=1}^{n_1} |f_j(u_2) - f_j(v_2)| < \epsilon 2^{-2}$. Let $z_2 = f(u_2) - f(v_2) = \sum_{j=1}^{\infty} a_j^2 e_j$. Then $\| \sum_{j=1}^{n_1} a_j^2 e_j \| \leq \sum_{j=1}^{n_1} |a_j^2| < \epsilon 2^{-2}$. Next choose $n_2 > \max(F_p(n_0 + 1), \dots, F_p(n_1))$ so that $\| \sum_{j=n_2+1}^{\infty} a_j^2 e_j \| < \epsilon 2^{-2}$. We continue this process for p steps and obtain the desired sets.

By Definition 4 we have

$$\Big\| \sum_{i=1}^{p} x_i \Big\| = \Big\| \sum_{i=1}^{p} \|x_i\| \cdot \frac{x_i}{\|x_i\|} \Big\| \geq \frac{1}{C} \sum_{i=1}^{p} \|x_i\|.$$

Consequently,

$$\begin{aligned} \Big\| \sum_{i=1}^{p} z_i \Big\| &\geq \Big\| \sum_{i=1}^{p} x_i \Big\| - \Big\| \sum_{i=1}^{p} (w_i + y_i) \Big\| \\ &\geq \frac{1}{C} \sum_{i=1}^{p} \|x_i\| - \sum_{i=1}^{p} (\|w_i\| + \|y_i\|) \\ &\geq \frac{1}{C} \sum_{i=1}^{p} \|z_i\| - \Big(1 + \frac{1}{C}\Big) \sum_{i=1}^{p} (\|w_i\| + \|y_i\|) \\ &\geq \frac{p}{C} \cdot \frac{\beta}{2} - 2\epsilon\Big(1 + \frac{1}{C}\Big) \cdot \sum_{i=1}^{p} 2^{-i} > \frac{p}{C} \cdot \frac{\beta}{2} - 2\epsilon\Big(1 + \frac{1}{C}\Big). \end{aligned}$$

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be tagged partitions of $[0,1]$ that have the same intervals as $\mathscr{P}_N$, where $(u_i, [c_i, d_i]) \in \mathscr{T}_1$ and $(v_i, [c_i, d_i]) \in \mathscr{T}_2$ for $i = 1, \dots, p$. The tags of $\mathscr{T}_1$ and $\mathscr{T}_2$ in the remaining intervals are the same. Thus we obtain

$$\begin{aligned} \|f(\mathscr{T}_1) - f(\mathscr{T}_2)\| &= \Big\| \sum_{i=1}^{p} \frac{1}{N} \cdot z_i \Big\| \geq \frac{p}{N} \cdot \frac{\beta}{2C} - \frac{2\epsilon}{N} \cdot \Big(1 + \frac{1}{C}\Big) \\ &\geq \frac{\alpha\beta}{2C} - 4\epsilon = \frac{\alpha\beta}{4C} \end{aligned}$$

that contradicts the Riemann integrability of f on $[0,1]$. □

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