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ON DISCRETE LIMITS OF SEQUENCES OF PIECEWISE LINEAR FUNCTIONS

Abstract

In this article we characterize the discrete limits of sequences of piecewise linear functions. A function $f : [0,1] \to \mathbb{R}$ is the discrete limit of a sequence of piecewise linear functions iff there are closed sets A_n such that $[0,1] = \bigcup_n A_n$ and the restricted functions f/A_n are linear.

Let $I = [0,1]$ and let $\mathbb{R}$ denote the set of all reals. In [1] it is proved that every Baire class one function $f : I \to \mathbb{R}$ is the pointwise limit of a sequence of piecewise linear functions whose vertices belong to the graph $\mathrm{Gr}(f)$ of f. In [2] the following notion of discrete convergence is introduced and investigated.

A sequence of functions $f_n : I \to \mathbb{R}$ discretely converges to a function f if for each point $x \in I$ there is a positive integer $n(x)$ such that $f_n(x) = f(x)$ for all indices $n > n(x)$. Moreover in [2] it is proved that a function $f : I \to \mathbb{R}$ is the limit of a discretely convergent sequence of continuous functions $f_n : I \to \mathbb{R}$ if and only if there are nonempty closed sets A_n, $n \geq 1$, such that the restricted functions $f \restriction A_n$, $n \geq 1$, are continuous and $I = \bigcup_n A_n$.

In this article we characterize the class of the discrete limits of sequences of piecewise linear functions from I to $\mathbb{R}$.

Theorem 1. *Let $f : I \to \mathbb{R}$ be a function. Then the following conditions are equivalent:*

(1) there is a sequence of piecewise linear functions $f_n : I \to \mathbb{R}$ which discretely converges to f;

(2) there are nonempty closed sets A_n, $n \geq 1$, such that $I = \bigcup_n A_n$ and the restricted functions $f \restriction A_n$ are linear, i.e. for each positive integer n there are reals a_n and b_n with $f(x) = a_n x + b_n$ for $x \in A_n$;

(3) there is a sequence of piecewise linear functions $g_n : I \to \mathbb{R}$ with the vertices belonging to $\mathrm{Gr}(f)$ which discretely converges to f.

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PROOF. The implication (3) $\Longrightarrow$ (1) is obvious. For the proof of the implication (1) $\Longrightarrow$ (2) observe that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$ (compare [2]). Since each function f_n, $n \geq 1$, is piecewise linear, its graph $\mathrm{Gr}(f_n)$ is the union of closed segments $K_{n,i}$, $i = 1, 2, \ldots, k(n)$, with endpoints $(a_{n,i-1}, f(a_{n,i-1}))$ and $(a_{n,i}, f(a_{n,i}))$, being successive vertices of $\mathrm{Gr}(f_n)$. From the continuity of all f_n and from the discrete convergence of (f_n) to f it follows that there is a sequence of nonempty closed sets B_n, $n \geq 1$, such that $I = \bigcup_n B_n$ and all restrictions $f \restriction B_n$ are continuous. For positive integers i, j, r let $E_{i,j,r} = K_{i,j} \cap B_r$ and enumerate all nonempty sets $E_{i,j,r}$ in a sequence (A_k) such that $A_k \neq A_n$ for $k \neq n$. The sets $E_{i,j,r}$ are the projections of the compact sets $\mathrm{Gr}(f/B_r) \cap K_{i,j}$, so they are compact. Thus the sets A_n, $n \geq 1$, satisfy all requirements in (2).

For the proof of the implication (2) $\Longrightarrow$ (3) assume (2) and assume that $f(x) = a_n x + b_n$ for $x \in A_n$, $n \geq 1$. Let $\{(c_i, d_i)\}$ be the set (finite or infinite) of all terms of the sequence $((a_n, b_n))$ and let $N_i = \{n; (a_n, b_n) = (c_i, d_i)\}$. Put

$$G_i = \bigcup_{n \in N_i} A_n, \ H_1 = G_1 \text{ and for } i > 1, \ H_i = G_i \setminus \bigcup_{j<i} G_j.$$

Observe that the sets H_i are pairwise disjoint F_σ-sets and $I = \bigcup_i H_i$. For each set H_i there are closed sets $M_{i,k}$ such that

$$H_i = \bigcup_k M_{i,k} \text{ and } M_{i,k} \subset M_{i,k+1} \text{ for } k \geq 1.$$

Now we will define functions g_n, $n \geq 1$. Fix a positive integer n. Since the compact sets $M_{i,n}$, $i \leq n$, are pairwise disjoint, there are pairwise disjoint open set $U_{i,k} \supset M_{i,k}$, $i, k \leq n$. Moreover we can assume that each set $U_{i,n}$ has only finite many of components and each component of the set $U_{i,n}$ intersects the set $M_{i,n}$.

If $J = (\alpha, \beta)$ is a component of a set $U_{i,n}$ $(i \leq n)$, then we put $c(J) = \inf(M_{i,n} \cap J)$, $d(J) = \sup(M_{i,n} \cap J)$, $g_{n,J}(x) = a_i x + b_i$ for $x \in [c(J), d(J)]$, $g_{n,J}(\alpha) = f(\alpha)$, $g_{n,J}(\beta) = f(\beta)$ and $g_{n,J}$ is linear on the intervals $[\alpha, c(J)]$ and $[d(J), \beta]$. Similarly we define $g_{n,J}$ in the case of the interval $J = [0, \beta)$ or $J = [\alpha, 1)$ whenever there is such a component of some set $U_{i,n}$ Let

$$g_n(x) = g_{n,J}(x) \text{ for } x \in J \text{ where } J \text{ is a component of a set } U_{i,n}, \ i \leq n,$$

and assume that g_n is a continuous linear function on the components of the set $I \setminus \bigcup_{i \leq n} U_{i,n}$. Then g_n is a continuous piecewise linear function with the vertices belonging to $\mathrm{Gr}(f)$ such that

$$g_n \restriction (\bigcup_{i \leq n} M_{i,n}) = f \restriction (\bigcup_{i \leq n} M_{i,n}). \tag{1}$$

Since $I = \bigcup_n \bigcup_{i \le n} M_{i,n}$, it follows by (1) that the sequence (g_n) discretely converges to f. □

Remark 1. *A function $f : I \to \mathbb{R}$ is the discrete limit of a sequence of piecewise linear functions if and only if for each nonempty closed set $A \subset I$ there is an open interval J such that $J \cap A \neq \emptyset$ and the restricted function $f \restriction (J \cap A)$ is linear.*

Proof. Necessity. If $f : I \to \mathbb{R}$ is the discrete limit of a sequence of piecewise linear functions, then by (2) from Theorem 1 there are nonempty closed sets A_n, $n \ge 1$, such that $I = \bigcup_n A_n$ and the restricted functions $f \restriction A_n$ are linear. Since $A = \bigcup_n (A_n \cap A)$, by Baire category theorem there are an index k and an open interval J such that $\emptyset \neq J \cap A \subset A_k$. Obviously, the restricted function $f \restriction (J \cap A)$ is linear.

Sufficiency. Let (J_n) be a sequence of all open subintervals of $[0,1]$ with rational endpoints. From our hypothesis it follows that there is the first index n_1 such that $f \restriction I_{n_1}$ is linear. If $I \setminus I_{n_1} \neq \emptyset$, then there is the first index n_2 such that $I_{n_2} \setminus I_{n_1} \neq \emptyset$ and the restricted function $f \restriction (I_{n_2} \setminus I_{n_1})$ is linear. Similarly for $k > 2$ if $I \setminus \bigcup_{i<k} I_{n_i} \neq \emptyset$, then there exists the first index n_k such that $I_{n_k} \setminus \bigcup_{i<k} I_{n_i} \neq \emptyset$ and the restricted function $f \restriction (I_{n_k} \setminus \bigcup_{i<k} I_{n_i})$ is linear. Putting $B_1 = I_{n_1}$ and $B_k = I_{n_k} \setminus \bigcup_{i<k} I_{n_i}$ we obtain a sequence of pairwise disjoint F_σ-sets such that $I = \bigcup_n B_n$ and the restricted functions $f \restriction B_n$ are linear. Each set B_n is the union of a sequence of closed sets $M_{n,k}$, $k \ge 1$ and every enumeration (A_j) of all sets $M_{n,k}$, $n, k \ge 1$, satisfies all requirements. This completes the proof. □

In a similar manner we can prove the following Remark 2.

Remark 2. *Let $f : I \to \mathbb{R}$ be a function. Then the following conditions are equivalent:*

(1) There is a sequence of spline functions of order $\le m$ which discretely converges to f;

(2) There is a sequence of closed sets A_n, $n \ge 1$, such that $I = \bigcup_n A_n$ and the restrictions f/A_n are polynomials of degree $\le m$;

(3) For each nonempty closed set $A \subset I$ there is an open interval J such that $J \cap A \neq \emptyset$ and the restriction $f/(J \cap A)$ is a polynomial of degree $\le m$;

(4) There is a sequence of spline functions of order $\le m$ with the vertices belonging to $\mathrm{Gr}(f)$ which discretely converges to f, where $g : I \to \mathbb{R}$ is a spline function of order $\le m$ if there are points $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ such that each restricted function $g/[x_{i-1}, x_i]$ is a polynomial of degree $\le m$ for $i = 1, \ldots, n$ and $g \in C^{m-1}([0,1])$.

References

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