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SOBCZYK-HAMMER DECOMPOSITIONS AND CONVERGENCE THEOREMS FOR MEASURES WITH VALUES IN l-GROUPS

Abstract

We find a decomposition of the type of Sobczyk-Hammer for measures with values in l-groups, and also deduce some convergence theorems for such decompositions. Our procedure is based on some theorems of the type of Vitali-Hahn-Saks, and on the so-called Stone extension method.

1 Introduction.

In [3] and [4], we obtained some versions of the Lebesgue decomposition theorem, and of the Vitali-Hahn-Saks theorem for finitely additive measures with values in (super) Dedekind complete l-groups. In the quoted papers, the notion of convergence was related to (D)-sequences and therefore many of the concepts and proofs appear somewhat complicated. In this paper we investigate a different kind of decomposition, and require that the involved l-group is super Dedekind complete and weakly σ-distributive. This allows us to avoid the machinery of (D)-convergence [2], thanks to a powerful result concerning suitable subsequences of an (O)-sequence 2.7. Hence all the relevant convergence properties can be formulated and proved only by means of (O)-sequences, which are a more natural tool.

Key Words: l-groups, (RO)-convergence, uniform s-boundedness, uniform continuity, Vitali-Hahn-Saks theorems, Sobczyk-Hammer decomposition

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The core of our research here concerns continuous and atomic l-group-valued measures. We obtain sufficient conditions for continuity and uniform continuity. By means of these results, we deduce a decomposition of a measure, in the sense of Sobczyk-Hammer, that is into its continuous and atomic part. Finally, we give a convergence theorem for such decompositions.

In Section 2 we give the definitions and state some preliminary results about the techniques to be used later. In Section 3 we introduce and study continuity properties of measures, obtaining a sufficient condition for a sequence of finitely additive measures to be uniformly continuous, and also some useful relations between continuity and absolute continuity. Finally, in Section 4 we introduce the sectional decompositions, and, thanks also to the previous theorems, we find some results concerning existence and convergence for Sobczyk-Hammer decompositions.

2 Preliminary Definitions and Results.

We shall introduce now the main definitions we need, together with some results.

Definition 2.1. An Abelian group $(R,+)$ is called an *l-group*, if it is endowed with a compatible ordering $\leq$, and is a lattice with respect to it. An l-group R is said to be *Dedekind complete,* if every nonempty subset of R, bounded from above, has supremum in R.

One important consequence of this definition is that convergence of series can be defined, at least when the terms are in $R_0^+ = \{r \in R : r \geq 0\}$.

Definition 2.2. Given any sequence $(a_n)_n$ in R_0^+, we say that the *series* $\sum_{n=1}^\infty a_n$ is *convergent* if the set of all partial sums $\{s_n : n \in \mathbb{N}\}$ is bounded in R, where $s_n = \sum_{i=1}^n a_i$ for all n. If this is the case, we set $\sum_{n=1}^\infty a_n = \sup\{s_n : n \in \mathbb{N}\}$.

Convergence of series is also related to the so-called (O)-convergence, according with the following definition.

Definition 2.3. Given a sequence $(r_n)_n$ in R, we say that $(r_n)_n$ *(O)-converges* to an element $r \in R$ if there exists a sequence $(p_n)_n$ in R, such that $p_n \downarrow 0$ (Such a sequence will be called an *(O)-sequence.*), satisfying $|r_n - r| \leq p_n \; \forall\, n \in \mathbb{N}$.

It is not difficult to see that a series $\sum_{n=1}^\infty a_n$, $a_n \geq 0$, is convergent to some element s if and only if the sequence $(s_n)_n$ (O)-converges to s.

We now introduce a first concept of σ-additivity, similar to the classical one. (In the sequel we will slightly sharpen this concept.)

Definition 2.4. Let R be a Dedekind complete l-group, $\mathcal{F}$ be an algebra of subsets of a nonempty set X and $m : \mathcal{F} \to R_0^+$ be a finitely additive measure. We say that m is *order σ-additive* if $m\left(\bigcup_{n=1}^{\infty} H_n\right) = \sum_{n=1}^{\infty} m(H_n)$ whenever $(H_n)_n$ is a disjoint sequence of elements of $\mathcal{F}$, such that $\bigcup_{n=1}^{\infty} H_n \in \mathcal{F}$.

In the sequel we shall assume further properties in our l-group, so we introduce now some definitions.

Definitions 2.5. A bounded double sequence $(a_{i,j})_{i,j}$ in R, such that $a_{i,j} \downarrow 0$ for each $i \in \mathbb{N}$, is called a *regulator* or (D)*-sequence*.

For every (D)-sequence $(a_{i,j})_{i,j}$, and every mapping $\phi : \mathbb{N} \to \mathbb{N}$, the element $\bigvee_{i=1}^{\infty} a_{i,\phi(i)}$ is called a *domination* of the (D)-sequence.

From now on, we shall denote by Φ the set of all mappings $\phi : \mathbb{N} \to \mathbb{N}$.

Now we can introduce the conditions we shall impose on R.

Definitions 2.6. We say that R is *weakly σ-distributive* if, for every (D)-sequence $(a_{i,j})_{i,j}$, the greatest lower bound of its dominations is 0; i.e.,

$$\bigwedge_{\phi \in \Phi} \left(\bigvee_{i=1}^{\infty} a_{i,\phi(i)} \right) = 0.$$

A Dedekind complete l-group R is said to be *super Dedekind complete*, if for any nonempty set $A \subset R$, bounded from above, there exists a countable subset $A^* \subset A$, such that $\sup A = \sup A^*$.

From now on, we shall always assume that R is a super Dedekind complete and weakly σ-distributive l-group.

The following lemma is a version of the *Fremlin Lemma* in the context of (O)-sequences. Though it is possible to prove it as a consequence of the Fremlin-type [6, Theorem 3.2.3, page 42], we give here a direct proof, because it looks somewhat easier.

Lemma 2.7. *Let $(r_n)_n$ be any (O)-sequence in R_0^+. For every $U \in R_0^+$ there exists an element $\omega \in \Phi$ such that the mapping $N \mapsto U \wedge \sum_{n=N}^{\infty} r_{\omega(n)}$ is an (O)-sequence.*

Proof. For any couple (i, k) of positive integers, set $A_{i,k} = U \wedge \left(\sum_{n=k}^{k+2^i-1} r_n \right)$. Clearly, $(A_{i,k})_{i,k}$ is a (D)-sequence. Next, for every element $\phi \in \Phi$, define $d_\phi := \bigvee_{i=1}^{\infty} A_{i,\phi(i)}$. Since R is weakly σ-distributive and super Dedekind complete, there exists a sequence $(\phi_h)_h$ in Φ, such that $\inf_h d_{\phi_h} = 0$. Without loss of generality, we shall assume that $\phi_h(n) < \phi_h(n+1)$ and $\phi_h(n) < \phi_{h+1}(n)$ for every h and n in $\mathbb{N}$, so that $N \mapsto g_N := d_{\phi_N}$ defines an (O)-sequence. Thus we have $g_1 \geq U \wedge \{r_{\phi_1(1)} + r_{\phi_1(1)+1}\}$ and also $g_1 \geq U \wedge \{r_{\phi_1(2)} + r_{\phi_1(2)+1} + r_{\phi_1(2)+2} + r_{\phi_1(2)+3}\}$, and so on. Now we observe that

$$U \wedge 2r_{\phi_1(1)+1} \leq g_1, \; U \wedge (r_{\phi_1(1)+1} + 2r_{\phi_1(2)+3}) \leq g_1,$$
$$U \wedge \left(r_{\phi_1(1)+1} + r_{\phi_1(2)+3} + 2r_{\phi_1(3)+7}\right) \leq g_1$$

and so on. Then $U \wedge \left(\sum_{n=1}^{k} r_{\phi_1(n)+2^n-1}\right) \leq g_1$ holds for every positive integer k; hence $U \wedge \left(\sum_{n=1}^{\infty} r_{\phi_1(n)+2^n-1}\right) \leq g_1$. In a similar way, one proves that $U \wedge \left(\sum_{n=1}^{\infty} r_{\phi_N(n)+2^n-1}\right) \leq g_N$ holds, for every positive integer N. Now, we set $\omega(N) := \phi_N(N) + 2^N - 1$. For every natural number k, we have

$$U \wedge \left(\sum_{n=N}^{N+k} r_{\omega(n)}\right) = U \wedge \left(\sum_{n=N}^{N+k} r_{\phi_n(n)+2^n-1}\right) \leq U \wedge \left(\sum_{n=N}^{N+k} r_{\phi_N(n)+2^n-1}\right)$$
$$\leq U \wedge \left(\sum_{n=1}^{\infty} r_{\phi_N(n)+2^n-1}\right) \leq g_N.$$

From the arbitrariness of k, we obtain the assertion. □

The next result expresses the fact that, as soon as $\{(r_n^{(k)})_n : k \in \mathbb{N}\}$ is an equibounded countable family of (O)-sequences, there exists a single (O)-sequence, which can replace them all. More precisely, we have the following.

Lemma 2.8. *Let $\{(r_n^{(k)})_n : k \in \mathbb{N}\}$ be an equibounded countable family of (O)-sequences. Then there exists an (O)-sequence $(b_j)_j$ with the following property: For every $j, k \in \mathbb{N}$, an integer $n = n(j,k) > 0$ exists, such that $r_n^{(k)} \leq b_j$.*

Proof. Define $a_{k,n} = r_n^{(k)}$ for each k, n. Clearly, $(a_{k,n})_{k,n}$ is a D-sequence, hence there exists a sequence $(\phi_j)_j$ in Φ, such that $j \mapsto b_j := \bigvee_{n=1}^{\infty} a_{n,\phi_j(n)}$ defines an (O)-sequence. As above, we can assume that $\phi_j(n) < \phi_j(n+1)$ and $\phi_j(n) < \phi_{j+1}(n)$ for all j, n. Arbitrarily fix j and k, and choose $n = n(j,k) = \phi_j(k)$. We have then $r_n^{(k)} = a_{k,\phi_j(k)} \leq \bigvee_{k=1}^{\infty} a_{k,\phi_j(k)} = b_j$, which is the assertion. □

From now on, we denote by $\mathcal{F}$ an algebra and by $\mathcal{A}$ a σ-algebra of subsets of a nonempty arbitrary set X.

We now introduce the concept of s-boundedness in the context of (O)-convergence.

Definition 2.9. A finitely additive measure $m : \mathcal{F} \to R$ is said to be *s-bounded* if there exists an (O)-sequence $(b_j)_j$ such that, for every disjoint sequence $(H_k)_k$ in $\mathcal{F}$ and every index $j \in \mathbb{N}$, one can find an integer k_0 satisfying

$$|m(H_k)| \leq b_j \text{ for each } k \in \mathbb{N}, k \geq k_0. \tag{1}$$

We say that the measures $m_n : \mathcal{F} \to R$, $n \in \mathbb{N}$, are *uniformly s-bounded,* if the integer k_0 in (1) can be chosen independently of n.

We now introduce the notion of σ-additivity in the context of (O)-convergence. This concept is, in general, stronger than the classical one of order σ-additivity.

Definition 2.10. Let $m : \mathcal{F} \to R$ be a finitely additive measure. We say that m is *σ-additive* if there exists an (O)-sequence $(b_j)_j$ such that, for every sequence $(H_k)_k$ in $\mathcal{F}$, decreasing to the empty set, and for every positive integer j, there exists a natural number k_0 satisfying $|m(B)| \le b_j$ for every $B \in \mathcal{F}, B \subset H_{k_0}$. A similar definition concerns *uniform σ-additivity* for a family $\{m_i : \mathcal{F} \to R\}_i$ of measures.

We observe that, in case of an equibounded sequence $(m_n)_n$ of σ-additive measures, it is possible to find a unique (O)-sequence $(b_j)_j$ which is related to the σ-additivity of all the m_n. This is a consequence of Lemma 2.8. The same also holds for other properties,such as continuity, to be introduced later.

From the extension theorems found in [4], it is possible to deduce that any s-bounded positive order σ-additive measure is also σ-additive.

Some of these extension theorems will now be recalled, for completeness, and also for further reference.

We first introduce some notation: given an algebra $\mathcal{F}$ of subsets of any nonempty set X, we denote by $\sigma(\mathcal{F})$ the σ-algebra generated by $\mathcal{F}$.

A stronger notion of convergence is also needed.

Definition 2.11. Let T be any nonempty set, and $(f_n : T \to R)_n$ be any sequence of functions. We say that the sequence $(f_n)_n$ is *(RO)-convergent* to a limit function f if there exists an (O)-sequence $(p_j)_j$ in R such that, for every positive integer j and every element $t \in T$, a natural number n_0 can be found, $n_0 = n_0(j, t)$, for which $|f_n(t) - f(t)| \le p_j \;\; \forall n \ge n_0$.

This definition will be mainly used for sequences of R-valued *measures.*

The theorems we list deal mainly with the so-called *Stone Isomorphism* technique. The well-known Stone Isomorphism Theorem asserts that any Boolean algebra $\mathcal{F}$ is algebraically isomorphic with the algebra Σ of clopen sets in a suitable compact, totally disconnected, Hausdorff space S. Denoting by $\psi : \mathcal{F} \to \Sigma$ such an isomorphism, any finitely additive measure $m : \mathcal{F} \to R$ can be associated with the measure $m \circ \psi^{-1} : \Sigma \to R$. Since Σ turns out to be perfect, any finitely additive measure on Σ is also order σ-additive.

Thus a suitable extension procedure, inspired by Carathéodory's construction, yields the following theorem [4].

Theorem 2.12. *Let $m : \mathcal{F} \to R_0^+$ be any finitely additive, s-bounded measure. There exists a σ-additive measure $\widetilde{m} : \sigma(\Sigma) \to R_0^+$ such that $\widetilde{m}_{|\Sigma} = m \circ \psi^{-1}$. Moreover, there exists a suitable (O)-sequence $(b_j)_j$ in R, such that, for every element $A \in \sigma(\Sigma)$ and each positive integer j, an element $F \in \mathcal{F}$ can be found, satisfying $\widetilde{m}(A \Delta \psi(F)) \leq b_j$.*

Combining this theorem with a convergence theorem of the type of Vitali-Hahn-Saks [3, 4], the following result can be deduced.

Theorem 2.13. *Let $(m_n : \mathcal{A} \to R)_n$ be an equibounded sequence of s-bounded finitely additive measures, defined on a σ-algebra $\mathcal{A}$. Assuming that the sequence $(m_n(A))_n$ is (RO)-convergent to a limit $m(A)$, $\forall A \in \mathcal{A}$, then: (i) the sequence $(m_n)_n$ is uniformly s-bounded, and therefore m is s-bounded too; and, (ii) the sequence $(\widetilde{m_n}(B))_n$ is (RO)-convergent to $\widetilde{m}(B)$, $\forall B \in \sigma(\Sigma)$, where as usual Σ denotes the Stone algebra isomorphic with $\mathcal{A}$, and $\widetilde{m_n}$, $\widetilde{m}$ are the Stone extensions to the σ-algebra $\sigma(\Sigma)$ of m_n and m respectively.*

3 Continuous and Atomic Measures.

We now introduce the concept of continuity for l-group-valued measures [1, 7].

Definition 3.1. We say that a finitely additive measure $m : \mathcal{F} \to R_0^+$ is *continuous* if $\inf_{P\in\Pi} [\sup_{D\in P} m(D)] = 0$, where P is any finite partition of X, and the infimum is taken with respect to the totality Π of such partitions. The finitely additive measures $m_n : \mathcal{F} \to R_0^+$, $n \in \mathbb{N}$, are said to be *uniformly continuous* if $\inf_{P\in\Pi} [\sup_{D\in P} (\sup_n m_n(D))] = 0$.

The following result is a characterization of continuity for finitely additive positive measures.

Proposition 3.2. *A finitely additive measure $m : \mathcal{F} \to R_0^+$ is continuous if and only if there exists an (O)-sequence $(b_j)_j$ in R such that for all $j \in \mathbb{N}$ there exists a finite partition Q_j of X into sets $D_1, \ldots, D_{h_j}$, for which*

$$m(D_i) \leq b_j \quad \forall\, i = 1, \ldots, h_j. \tag{2}$$

Analogously, the finitely additive measures $m_n : \mathcal{F} \to R_0^+$, $n \in \mathbb{N}$, are uniformly continuous if, and only if, there exists an (O)-sequence $(b_j)_j$ in R such that for all $j \in \mathbb{N}$, there exists a finite partition Q_j of X into sets $D_1, \ldots, D_{h_j}$, satisfying $\sup_n m_n(D_i) \leq b_j \quad \forall\, i = 1, \ldots, h_j$.

Proof. We give the proof only for the first part of the proposition. Let m be a positive continuous finitely additive measure. Since R is super Dedekind complete, then there exists a sequence $(Q_j)_j$ of finite partitions of X, such that $\inf_j \left[\sup_{D\in Q_j} m(D)\right] = 0$. Without loss of generality, we can assume that the sequence $(Q_j)_j$ is increasing with respect to the refinement order; i.e., we assume that, for every positive integer j, each element in Q_j is the union of some elements from Q_{j+1}.

For every index $j \in \mathbb{N}$, define $b_j := \sup_{D\in Q_j} m(D)$. Thus, $(b_j)_j$ turns out to be an (O)-sequence, and the sequence $(Q_j)_j$ is the required one.

Conversely, let $(Q_j)_j$ be a sequence of finite partitions of X, satisfying (2). We get $\inf_j \left[\sup_{D\in Q_j} m(D)\right] = 0$ and, a fortiori, $\inf_{P\in\Pi} \left[\sup_{D\in P} m(D)\right] = 0$, that is, continuity of m. □

Definition 3.3. A finitely additive measure $m : \mathcal{A} \to R_0^+$ is said to be *atomic* if 0 is the unique finitely additive and continuous measure $\nu : \mathcal{A} \to R_0^+$, satisfying $\nu \le m$.

We note that the given definition of atomic measure agrees with the classical one of atomic $\mathbb{R}_0^+$-valued measure, by virtue of the Sobczyk-Hammer decomposition theorem for scalar measures [7].

In the sequel, a suitable notion of absolute continuity is needed. We introduce it now, only for σ-additive measures, and will then deduce a useful result about continuity of measures.

Definition 3.4. Let $\nu : \mathcal{A} \to R_0^+$ be any σ-additive measure, defined on a σ-algebra $\mathcal{A}$. Given any other σ-additive measure $m : \mathcal{A} \to R_0^+$, we say m is *absolutely continuous* with respect to ν if $\nu(A) = 0 \Rightarrow m(A) = 0$.

Here is the theorem concerning continuity.

Theorem 3.5. *Let m, $\nu : \mathcal{A} \to R_0^+$, and assume m and ν are σ-additive, m is absolutely continuous with respect to ν and ν is continuous. Then m is continuous.*

PROOF. Set $U := m(X) + \nu(X)$. Fix an (O)-sequence $(b_j)_j$ in R, agreeing with the σ-additivity of m, with σ-additivity and with continuity of ν. Thanks to Lemma 2.7, there exists a subsequence $(b_{j_k})_k$ such that $(\rho_N)_N$ is an (O)-sequence, where $\rho_N := U \wedge \sum_{k\ge N} b_{j_k}$ for all N. Set $b'_N := b_{j_N}$, and define $\alpha_N := \sup\{m(A) : \nu(A) \le b'_N\}, \forall N \in \mathbb{N}$. We shall prove that $(\alpha_N)_N$ is an (O)-sequence. Clearly, $(\alpha_N)_N$ is decreasing, so all that must be shown is $\alpha := \inf_N \alpha_N$ coincides with 0.

If this is not the case, then there exists an integer N such that $\alpha \not\le b_N$; hence $\alpha_k \not\le b_N$ $\forall k$. This means that, for every natural number k, an element $A_k \in \mathcal{A}$ can be found, such that $\nu(A_k) \le b'_k$, but $m(A_k) \not\le b_N$. For all positive integers s, define $A_s^* := \bigcup_{k\ge s} A_k$. We get $\nu(A_s^*) \le U \wedge \sum_{k\ge s} b'_k \le \rho_s$, but

$$m(A_s^*) \not\le b_N, \tag{3}$$

for all s. The sequence $(A_s^*)_s$ is decreasing. Denoting its limit by A, we get $\nu(A) = 0$ by σ-additivity, and then $m(A) = 0$. Hence, the sequence $(A_s^* \setminus A)_s$ is decreasing to $\emptyset$, and $m(A_s^* \setminus A) = m(A_s^*)$ for all s. Thus, by σ-additivity, a positive integer τ_N can be found, satisfying $m(A_{\tau_N}^*) \le b_N$. But this is contrary to (3). We must conclude that $\alpha = 0$.

Now, we can easily prove the continuity of m. Indeed, for each $N \in \mathbb{N}$, a partition P of X exists, such that $\nu(D) \le b'_N$ for all $D \in P$. Then $m(D) \le \alpha_N$ for all $D \in P$, and the assertion is proved. □

We now prove that, for continuous equibounded finitely additive measures, uniform s-boundedness is a sufficient condition for uniform continuity. To this aim, we begin with the following definition.

Definition 3.6. Let $m : \mathcal{F} \to R_0^+$ be any set function, and fix $u \in R_0^+$. Given any set $A \in \mathcal{F}$, we say that A is *u-decomposable* (with respect to m), if there exists a finite partition of A into sets $D_1, \dots, D_k$ of $\mathcal{F}$, such that $m(D_i) \le u$ for all $i = 1, \dots, k$. Thus we get that a positive finitely additive measure m is continuous if, and only if, there exists an (O)-sequence $(r_j)_j$ in R such that, for each $j \in \mathbb{N}$, the set X is r_j-decomposable (see also Proposition 3.2).

Let $(m_n : \mathcal{F} \to R_0^+)_n$ be a sequence of finitely additive equibounded measures. For every $A \in \mathcal{F}$, set $M(A) := \sup_{n\in\mathbb{N}} m_n(A)$. Given $A \in \mathcal{F}$, we say that A is *uniformly u-decomposable* if it is u-decomposable with respect to M.

Thus we get that the m_n's are uniformly continuous if, and only if, there exists an (O)-sequence $(r_j)_j$ in R such that, for each $j \in \mathbb{N}$, the set X is uniformly r_j-decomposable.

We also need the following lemma.

Lemma 3.7. *Let $(m_n : \mathcal{F} \to R_0^+)_n$ be a sequence of continuous, equibounded and uniformly s-bounded measures, and let $(r_j)_j$ be an (O)-sequence, according with the uniform s-boundedness and continuity of each measure m_n. Assume that, for some $j \in \mathbb{N}$, there exists $A \in \mathcal{F}$, with $M(A) \not\le 2r_j$. Then there exists $H \subset A$, $H \in \mathcal{F}$, uniformly r_j-decomposable and such that $M(H) \not\le 2r_j$.*

Proof. Let $(r_j)_j$ be as in the hypotheses. We note that such a sequence does exist, by virtue of Lemma 2.8 and equiboundedness. Without loss of generality, suppose $j = 1$, and set $r = r_1$. By contradiction, suppose that there exists $A \in \mathcal{F}$, with $M(A) \not\le 2r$, such that there are no uniformly r-decomposable subsets $H \in \mathcal{F}$, with $M(H) \not\le 2r$. Without loss of generality, suppose that $m_1(A) \not\le 2r$. By the continuity of m_1, there exists a partition $\mathcal{D}_A$ of A into sets $D_1, \dots, D_k$, such that $m_1(D_i) \le r$ for all $i = 1, \dots, k$. By virtue of the assumed contradiction, we get $M(D_{k_1}) \not\le 2r$ for some index $k_1 \le k$, and thus, in correspondence with k_1, there exists an integer $n_1 > 1$ such that

$m_{n_1}(D_{k_1}) \not\leq 2r$. Set $A_1 := D_{k_1}$ and $B_1 := A \setminus A_1$. By difference, we get $m_1(B_1) \not\leq r$. By the continuity of $m_1, m_2, \dots m_{n_1}$, there exists a partition $\mathcal{D}$ of A_1 into sets $D'_1, D'_2, \dots, D'_s$, such that $m_1(D'_i) \vee m_2(D'_i) \vee \dots \vee m_{n_1}(D'_i) \leq r$ for each $i = 1, \dots, s$. Again by contradiction, it follows: $M(D'_{k_2}) \not\leq 2r$ for a suitable index $k_2 \leq s$, and hence there exists an integer $n_2 > n_1$ such that $m_{n_2}(D'_{k_2}) \not\leq 2r$. Put $A_2 := D'_{k_2}$, and $B_2 := A_1 \setminus A_2$. By difference, we get $m_{n_1}(B_2) \not\leq r$. By virtue of continuity of $m_1, m_2, \dots, m_{n_2}$, select a partition of A_2 and a set $A_3 \subset A_2$, $A_3 \in \mathcal{F}$, such that $M(A_3) \not\leq 2r$ and $m_{n_2}(A_3) \leq r$. Setting $B_3 := A_2 \setminus A_3$, by difference we have $m_{n_2}(B_3) \not\leq r$. Proceeding in this way, we find an increasing sequence of natural numbers $(n_h)_h$ and a decreasing sequence of sets $(A_h)_h$ in $\mathcal{F}$, such that $m_{n_h}(B_{h+1}) \not\leq r$ for all $h \in \mathbb{N}$, where $(B_h := A_h \setminus A_{h+1})_h$ is a disjoint sequence in $\mathcal{F}$. This contradicts uniform s-boundedness, and the lemma is proved. □

We finally turn to the announced useful result.

Theorem 3.8. *Under the same notations as in Lemma 3.7, if the m_n's are positive, continuous, equibounded and uniformly s-bounded finitely additive R-valued measures, then they are uniformly continuous.*

Proof. By virtue of equiboundedness and thanks to Lemma 2.8, there exists an (O)-sequence $(r_j)_j$, agreeing both with the uniform s-boundedness and with the continuity of each measure m_n. To prove the theorem, we shall show that, for every $j \in \mathbb{N}$, X is uniformly $2r_j$-decomposable. Fix any positive integer j, and set $r = r_j$. Suppose, for purposes of contradiction, that X is not uniformly $2r$-decomposable. (From now on in this proof, the term "decomposable" will always mean "uniformly decomposable.") Then $M(X) \not\leq 2r$, and thus, by Lemma 3.7, there exists a set $A_1 \in \mathcal{F}$, with $M(A_1) \not\leq 2r$, but r-decomposable. By the assumed contradiction, $X \setminus A_1$ is not $2r$-decomposable, and hence we have $M(X \setminus A_1) \not\leq 2r$. Again by Lemma 3.7, there exists an r-decomposable set $A_2 \subset X \setminus A_1$, with $A_2 \in \mathcal{F}$, such that $M(A_2) \not\leq 2r$. Since A_1 and A_2 are r-decomposable and X is not, then $X \setminus (A_1 \cup A_2)$ is not $2r$-decomposable, and thus, again by Lemma 3.7, there exists an r-decomposable set A_3 in $\mathcal{F}$, disjoint both from A_1 and from A_2, such that $M(A_3) \not\leq 2r$. Thus we get the existence of a sequence $(A_k)_k$ of pairwise disjoint r-decomposable sets of $\mathcal{F}$, such that $M(A_k) \not\leq 2r, \forall\, k \in \mathbb{N}$. This contradicts uniform s-boundedness of the measures m_n, and thus X is $2r$-decomposable. So the assertion follows. □

4 Sobczyk-Hammer Decompositions.

In this section, we deduce existence and convergence theorems for Sobczyk-Hammer decompositions, first for σ-additive measures, and then for finitely additive ones.

For σ-additive measures m, we shall obtain a decomposition of *sectional* type; i.e., we shall find a suitable set $H \in \mathcal{A}$, such that the restrictions of m to H and to H^c are continuous and atomic respectively.

We begin with a lemma.

Lemma 4.1. *Suppose $m : \mathcal{A} \to R$ is a positive, σ-additive measure. If m is not atomic, then there exists at least a set $F \in \mathcal{A}$ with $m(F) \neq 0$, such that $m_{|F}$ is continuous.*

Proof. Since m is not atomic, there exists a continuous non-trivial measure $\mu : \mathcal{A} \to R_0^+$, such that $\mu \leq m$. It is readily seen that μ is σ-additive. Let now $\mathcal{H}$ be the family of all sets $H \in \mathcal{A}$ such that $\mu(H) = 0$, and define

$$\alpha = \sup\{m(H) : H \in \mathcal{H}\}. \tag{4}$$

By super Dedekind completeness, there exists a sequence $(K_n)_n$ in $\mathcal{H}$, such that $\alpha = \sup_n\ m(K_n)$. Without loss of generality, we can assume that the sequence $(K_n)_n$ is non-decreasing. Then, $m(K) = \alpha$ and $\mu(K) = 0$, where $K = \bigcup_{n=1}^{\infty} K_n$.

Put $F = K^c$, and let us show that F is the required set. First of all, it is easy to check that the measure $m_{|F}$ is absolutely continuous with respect to μ. Indeed, if there exists a set $G \in \mathcal{A}$ such that $\mu(G) = 0$ and $m_{|F}(G) > 0$, then $m[(G \cap F) \cup K] = m(G \cap F) + m(K) > \alpha$. But $\mu[(G \cap F) \cup K] = \mu(G \cap F) + \mu(K) = 0$, and therefore α could not be the supremum in (4), a contradiction. Thus, by virtue of Theorem 3.5, $m_{|F}$ is continuous. Finally we observe that $m(F) \neq 0$, otherwise $\mu \equiv 0$, which is impossible. $\square$

A useful consequence is the following.

Proposition 4.2. *Let m_1 and m_2 be two atomic (continuous) σ-additive positive R-valued measures, defined on $\mathcal{A}$. Then $m_1 + m_2$ is atomic (continuous).*

Proof. The assertion concerning continuous measures is easy, so we deal only with the atomic case. Let us assume that m_1 and m_2 are atomic, and $m_1 + m_2$ is not. Then there exists a set F, according with the previous Lemma. Thus $m_{1|F}$ is continuous, and hence null. Similarly, $m_{2|F}$ is null; hence we obtain $(m_1 + m_2)_{|F} = 0$, a contradiction. $\square$

We now turn to the following Sobczyk-Hammer-type theorem.

Theorem 4.3. *If $m : \mathcal{A} \to R$ is a σ-additive positive measure, then there exists a set $E \in \mathcal{A}$, such that $m_{|E}$ is continuous and $m_{|E^c}$ is atomic.*

PROOF. If m is atomic, it is enough to take $E = \emptyset$. Otherwise, by virtue of Lemma 4.1, there exists $F \in \mathcal{A}$ such that $m(F) \neq 0$ and $m_{|F}$ is continuous. Denote by $\mathcal{K}$ the family of such sets, and write $\alpha := \sup\{m(F) : F \in \mathcal{K}\}$. By super Dedekind completeness of R, there exists an increasing sequence $(F_n)_n$ in $\mathcal{K}$, such that $\sup_n m(F_n) = \alpha$. Put $E = \bigcup_{n=1}^{\infty} F_n$. We prove that E is the requested set. First of all, note that

$$m_{|E}(A) = \sup_n m_{|F_n}(A) \tag{5}$$

for all $A \in \mathcal{A}$. Moreover, it is easy to check that the $m_{|F_n}$'s, $n \in \mathbb{N}$, are uniformly s-bounded. By Theorem 3.8, these measures are uniformly continuous, and hence, by (5), $m_{|E}$ is σ-additive and continuous.

Finally we prove that $m_{|E^c}$ is atomic. Otherwise, by Theorem 4.1, there exists a set $H \in \mathcal{A}$, such that $m(H) \neq 0$, $H \cap E = \emptyset$ and $m_{|H}$ is continuous. Then $H \cup E \in \mathcal{K}$ and $m(H \cup E) > \alpha$, a contradiction. □

We now prove some convergence theorems for Sobczyk-Hammer-type decompositions.

First, consider the case of positive, σ-additive, (RO)-convergent measures. Recall that a sequence $(m_k)_k$ of finitely additive measures, defined on an algebra $\mathcal{F}$ with values in R, is *(RO)-convergent* to a measure m, if there exists an (O)-sequence $(r_n)_n$ such that, for all $A \in \mathcal{F}$ and $n \in \mathbb{N}$, an integer k_0 can be found, such that $|m_k(A) - m(A)| \leq r_n$ for all $k \geq k_0$.

In order to give convergence theorems for decompositions, we first prove the following.

Proposition 4.4. *Let $(m_n)_n$ be a sequence of σ-additive positive R-valued measures, defined on a σ-algebra $\mathcal{A}$. If the m_n's are equibounded, then there exists $H \in \mathcal{A}$ such that $m_{n|H}$ is continuous and $m_{n|H^c}$ is atomic for every $n \in \mathbb{N}$.*

PROOF. For any $N \in \mathbb{N}$, denote by μ_N the measure $\mu_N := \sum_{i=1}^{N} m_i$, and let (A_N, A_N^c) be a Sobczyk-Hammer sectional decomposition of μ_N (See Theorem 4.3.). Set now, for positive integers N, L, with $N \leq L$: $B_{N,L} := \bigcup_{p=0}^{L-N} A_{N+p}$, $B_N := \bigcup_{j=N}^{\infty} B_{N,j} = \bigcup_{j=N}^{\infty} A_j$. We have

$$m_{N|B_{N,L}} \leq \sum_{p=0}^{L-N} (\mu_{N+p})_{|A_{N+p}}$$

and hence $m_{N|B_{N,L}}$ is continuous. Letting $L \to \infty$, we deduce, by uniform s-boundedness, that the measure $m_{N|B_N}$ is continuous (see Theorem 3.8).

Clearly, $\mu_{N|B_N^c} \leq \mu_{N|A_N^c}$ is atomic, for every N. Set now

$$H := \bigcap_{N=1}^{\infty} B_N.$$

We shall show that H is the requested set. Indeed, for every $N \in \mathbb{N}$, we have $m_{N|H} \leq m_{N|B_N}$; hence $m_{N|H}$ is continuous.

We finally prove that $m_{N|H^c}$ is atomic, for all N. To this aim, fix any integer $N \in \mathbb{N}$, and let β be any positive measure, $\beta \leq m_{N|H^c}$: we deduce easily that $\beta_{|B^c_{N+p}} \leq m_{N|B^c_{N+p}} \leq (\mu_{N+p})_{|B^c_{N+p}}$ for all $p \in \mathbb{N}$, and hence $\beta_{|B^c_{N+p}}$ is null, by atomicity of $(\mu_{N+p})_{|B^c_{N+p}}$. But $\beta_{|H^c} = (RO) \lim_{p\to\infty} \beta_{|B^c_{N+p}}$; hence β is null. This concludes the proof, by arbitrariness of N. □

We now prove a first convergence theorem for Sobczyk-Hammer-type decompositions.

Theorem 4.5. *Let $(m_n)_n$ be a sequence of σ-additive positive R-valued measures, defined on a σ-algebra $\mathcal{A}$. Suppose the m_n are equibounded and (RO)-convergent to a measure m. Then m is σ-additive, and the sequences $(m_n^1)_n$ and $(m_n^2)_n$ are (RO)-convergent to the measures m^1 and m^2, where (m_n^1, m_n^2), (m^1, m^2) are the sectional Sobczyk-Hammer decompositions of m_n and m respectively, $n \in \mathbb{N}$.*

Proof. By virtue of the Vitali-Hahn-Saks theorem [4], the measures m_n are uniformly s-bounded, and hence the limit measure m is σ-additive. By applying Proposition 4.4 to the sequence $(m_n)_n$, we get the existence of a set $H \in \mathcal{A}$, which yields a sectional decomposition of the measures m_n, $n \in \mathbb{N}$, and of the limit measure m. The assertion follows immediately from (RO)-convergence. □

We now turn to the finitely additive case.

Theorem 4.6. *Let $m : \mathcal{F} \to R$ be any positive, finitely additive, s-bounded measure, defined on an algebra $\mathcal{F}$. There exists a decomposition $m = m^1 + m^2$ of m into two positive, finitely additive measures, such that m^1 is continuous and m^2 is atomic.*

Proof. We make use of the Stone isomorphism; namely we consider the Stone space S associated with $\mathcal{F}$, and the algebraic isomorphism ψ from $\mathcal{F}$ to the algebra Σ of all clopen subsets of S. From Theorem 2.12, the measure $m^0 := m \circ \psi^{-1}$ can be extended to a σ-additive measure $\widetilde{m}$ on $\sigma(\Sigma)$. Using Theorem 4.4, decompose $\widetilde{m}$ into the sum $\widetilde{m}^1 + \widetilde{m}^2$, in the sense of Sobczyk-Hammer, where $\widetilde{m}^1$ is continuous and $\widetilde{m}^2$ is atomic. Now, restrict $\widetilde{m}^1$ and $\widetilde{m}^2$ to the algebra Σ, thus obtaining two measures, denoted by m^1_S and m^2_S respectively.

We shall see that m^1_S and m^2_S are continuous and atomic, respectively. From this, it will follow immediately that the measures, $m^1 := m^1_S \circ \psi$ and $m^2 := m^2_S \circ \psi$, give the requested decomposition of m. Let us prove that m^1_S is continuous. By the continuity of $\widetilde{m}^1$, there exists an (O)-sequence $(a_j)_j$ such that, for all $j \in \mathbb{N}$, a finite partition $\{D_1, \dots, D_{h_j}\}$ in $\sigma(\Sigma)$ can be found, satisfying $\widetilde{m}^1(D_k) \leq a_j,\ \forall\, k = 1, \dots, h_j$. Now, by virtue of Theorem 2.12, there exists an (O)-sequence $(b_j)_j$ such that, for all $D \in \sigma(\Sigma)$ and $j \in \mathbb{N}$, it is possible to find $E \in \Sigma$ such that $\widetilde{m}^1(E\Delta D) \leq b_j$.

Moreover, thanks to Lemma 2.7, and denoting by U any majorant for all elements $m_n(X)$, $n \in \mathbb{N}$, there exists a subsequence $(b_{j_l})_l$ such that $N \mapsto U \wedge \sum_{l=N}^{\infty} b_{j_l}$ is still an (O)-sequence. Let us call $(\rho_N)_N$ such a sequence.

The (O)-sequence $(a_N + \rho_N)_N$ can be used to prove continuity of m^1_S.

To this aim, fix any $N \in \mathbb{N}$. We can find a partition $\{D_1, \dots, D_{h_N}\}$ in $\sigma(\Sigma)$, satisfying $\widetilde{m}^1(D_k) \leq a_N$ for all k. For each index k, from 1 to h_N, choose an element $E_k \in \Sigma$ such that $\widetilde{m}^1(E_k \Delta D_k) \leq b_{j_{(N+k)}}$. We have $\widetilde{m}^1(E_k) \leq a_N + \rho_N$, for all k.

Let $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, …, $F_{h_N} = E_{h_N} \setminus (E_1 \cup \ldots \cup E_{h_N - 1})$. Finally, set $F_{h_N+1} := X \setminus \bigcup_{k=1}^{h_N} F_k = X \setminus \bigcup_{k=1}^{h_N} E_k$.

Of course, all the sets F_k, $k \in \mathbb{N}$, belong to Σ, and

$$m^1_S(F_k) = \widetilde{m}^1(F_k) \leq b_N + \rho_N \ \forall\ k = 1, \dots, h_N.$$

As to F_{h_N+1}, we see that

$$\begin{aligned} m^1_S(F_{h_N+1}) =& \widetilde{m}^1(X) - \sum_{k=1}^{h_N} \widetilde{m}^1(F_k) = \widetilde{m}^1 \left(\bigcup_{k=1}^{h_N} D_k \setminus \bigcup_{k=1}^{h_N} E_k \right) \\ \leq& \widetilde{m}^1 \left(\bigcup_{k=1}^{h_N} (D_k \Delta E_k) \right) \leq U \wedge \sum_{k=1}^{h_N} b_{j_{(N+k)}} \leq \rho_N. \end{aligned}$$

Hence, the partition $\{F_1, \dots, F_{h_N}, F_{h_N+1}\}$ satisfies the condition $m^1_S(F_k) \leq \rho_N + a_N$ for all $k = 1, \dots, h_N + 1$, and thus m^1_S is continuous.

We now prove the atomicity of m^2_S. Let $\nu : \Sigma \to R$ be a continuous positive finitely additive measure, such that $0 \leq \nu(A) \leq m^2_S(A)$, $\forall A \in \Sigma$. Then ν admits a Carathéodory-type extension $\widetilde{\nu}$ to the whole of $\sigma(\Sigma)$. The continuity of $\widetilde{\nu}$ follows immediately from the continuity of ν. Thus we get that $\widetilde{\nu}$ is a continuous finitely additive measure, such that $0 \leq \widetilde{\nu}(A) \leq \widetilde{m}^2(A)$, $\forall\, A \in \sigma(\Sigma)$. Thanks to the atomicity of $\widetilde{m}^2$, we get $\widetilde{\nu} \equiv 0$ on $\sigma(\Sigma)$, and thus, *a fortiori,* $\nu \equiv 0$ on Σ. □

Before stating our final convergence theorem, we introduce some definitions, in order to also consider measures taking values not necessarily positive.

Definition 4.7. Let $\mathcal{F}$ be any algebra of subsets of a nonempty set X. Assume that $m : \mathcal{F} \to R$ is any finitely additive bounded measure. We put:

$$\begin{aligned} m^+(A) &= \sup\{m(B) : B \in \mathcal{F},\, B \subset A\}, \\ m^-(A) &= -\inf\{m(B) : B \in \mathcal{F},\, B \subset A\}, \\ v(m)(A) &= \sup\{|m(B)| : B \in \mathcal{F},\, B \subset A\} \end{aligned}$$

for all $A \in \mathcal{F}$. The set functions m^+, m^-, $v(m)$ are called the *positive variation, negative variation* and *semivariation* of m respectively. It is easy to see that m^+ and m^- are positive finitely additive measures, $m^+ - m^- = m$, and $v(m) \le m^+ + m^- \le 2v(m)$.

For any bounded finitely additive measure $m : \mathcal{F} \to R$, we shall say that m is *continuous* (resp. *atomic*) if the measure $m^+ + m^-$ is.

From these definitions, it turns out immediately that any s-bounded finitely additive measure $m : \mathcal{F} \to R$ admits a Sobczyk-Hammer decomposition. It suffices to decompose m^+ and m^-, and then apply Proposition 4.2.

We now state our final theorem.

Theorem 4.8. *Let $(m_n)_n$ be any sequence of s-bounded, equibounded, finitely additive measures, defined on a σ-algebra $\mathcal{A}$ and taking values in R, and assume that $(RO)\lim_n\ m_n(A) = m(A)$ exists, for all $A \in \mathcal{A}$. Then, m is s-bounded, and the sequences $(m_n^1)_n$ and $(m_n^2)_n$ are (RO)-convergent to the measures m^1 and m^2, where (m_n^1, m_n^2), (m^1, m^2) are the continuous and the atomic parts which form the Sobczyk-Hammer decompositions of m_n, $n \in \mathbb{N}$ and m, respectively.*

Proof. Again, we make use of the Stone isomorphism technique. Denote by Σ the algebra of clopen sets, which is isomorphic to $\mathcal{A}$, and denote by $\psi : \mathcal{A} \to \Sigma$ such an isomorphism. Denote respectively by $\widetilde{m_n}$, $\widetilde{m_n^+}$, and so on, the countably additive extensions of m_n, m_n^+ and so on, to the σ-algebra $\sigma(\Sigma)$. (We observe that m_n, $n \in \mathbb{N}$, and m are s-bounded, and hence their Stone extensions, $\widetilde{m_n}$, $\widetilde{m}$ do exist.)

Thanks to Theorem 2.13, the sequence $(\widetilde{m_n})_n$ is (RO)-convergent to $\widetilde{m}$ in $\sigma(\Sigma)$. Now, apply Proposition 4.4 to the sequences $(\widetilde{m_n^+})_n$ and $(\widetilde{m_n^-})_n$; thus obtaining a set $H \in \sigma(\Sigma)$ such that:

(1) $\widetilde{m_n^+}|_H$ is continuous, $\widetilde{m_n^+}|_{H^c}$ is atomic (and the same for $\widetilde{m_n^-}$);
(2) $(RO)\lim_n \widetilde{m_n}|_H = \widetilde{m}|_H$, $\quad (RO)\lim_n \widetilde{m_n}|_{H^c} = \widetilde{m}|_{H^c}$.

If we denote by m^1_{nS}, m^2_{nS}, $m^1{}_S$, $m^2{}_S$ the restrictions to Σ of the measures $\widetilde{m_n}|_H, \widetilde{m_n}|_{H^c}, \widetilde{m}|_H, \widetilde{m}|_{H^c}$, respectively, then $m^1_n := m^1_{nS} \circ \psi, m^2_n := m^2_{nS} \circ \psi$ are the Sobczyk-Hammer decompositions of the measures m_n, and $m^1 := m^1{}_S \circ \psi$, $m^2 := m^2{}_S \circ \psi$ give the Sobczyk-Hammer decomposition of m. Clearly, (RO)-convergence of the measures $\widetilde{m_n}|_H$ and $\widetilde{m_n}|_{H^c}$ implies (RO)-convergence of m^1_n to m^1 and of m^2_n to m^2. □

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