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ON SOME THEOREMS OF RICHTER AND STEPHANI FOR SYMMETRICAL QUASICONTINUITY AND SYMMETRICAL CLIQUISHNESS

Abstract

In this article we prove that some results of Richter and Stephani concerning the cluster sets of quasicontinuous and cliquish real functions ([6]) are also true for the special quasicontinuities introduced by Piotrowski and Vallin in [5].

Let (X, T_X) and (Y, T_Y) be topological spaces. A function $g : X \to \mathbb{R}$ is called:

(1) quasicontinuous (resp. cliquish) at a point $x \in X$ if for every set $U \in T_X$ containing x and for each positive real ε there is a nonempty set $U' \in T_X$ contained in U such that $g(U') \subset (g(x)-\varepsilon, g(x)+\varepsilon)$ (resp. the diameter $\operatorname{diam}(g(U')) = \sup\{|g(t) - g(u)| : t, u \in U'\} < \varepsilon$) ([3], [4]);

A function $f : X \times Y \to \mathbb{R}$ is said to be:

(2) quasicontinuous at (x, y) with respect to the first coordinate (alternatively to the second coordinate) if for every set $U \times V \in T_X \times T_Y$ containing (x, y) and for each positive real ε there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $f(U' \times V') \subset (f(x, y) - \varepsilon, f(x, y) + \varepsilon)$ ([5]);

(3) cliquish at (x, y) with respect to the first coordinate (alternatively to the second coordinate) if for every set $U \times V \in T_X \times T_Y$ containing (x, y) and for each positive real ε there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $\operatorname{diam}(f(U' \times V')) < \varepsilon$ ([1]);

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(4) symmetrically quasicontinuous (resp. symmetrically cliquish) at (x, y) if it is quasicontinuous (alternatively cliquish) at (x, y) with respect to the first and with respect to second coordinate ([5], [1]).

Recall that a set $A \subset X$ is semi-open if $A \subset \mathrm{cl}(\mathrm{int}(A))$ and that $g : X \to \mathbb{R}$ is quasicontinuous at $x \in X$ if and only if for each positive real ε there is a semi-open set $A \ni x$ with $g(A) \subset (g(x) - \varepsilon, g(x) + \varepsilon)$ ([4]).

Denote by $SO(X)$ (resp. by $SO(X, Y)$) the family of all semi-open sets in X (resp. in $X \times Y$). Moreover, if $A \subset X \times Y$, then for $x \in X$ (resp. $y \in Y$) let $A_x = \{v \in Y : (x, v) \in A\}$ (resp. $A^y = \{u \in X : (u, y) \in A\}$) be the vertical (resp. horizontal) section of A. Let

$$SO_1(X, Y) = \{A \subset X \times Y : \text{if } (x, y) \in A, \text{ then } \; y \in \mathrm{cl}((\mathrm{int}(A))_x)\}$$

and

$$SO_2(X, Y) = \{A \subset X \times Y : \text{if } (x, y) \in A, \text{ then } x \in \mathrm{cl}((\mathrm{int}(A))^y)\}.$$

By standard arguments we obtain the following assertions.

Remark 1. *A function $f : X \times Y \to \mathbb{R}$ is quasicontinuous with respect to the first coordinate at a point (u, v) if and only if for each positive real ε there is a set $A \in SO_1(X, Y)$ containing (u, v) and such that $f(A) \subset (f(u, v) - \varepsilon, f(u, v) + \varepsilon)$.*

Remark 2. *A function $f : X \times Y \to \mathbb{R}$ is quasicontinuous with respect to the second coordinate at a point (u, v) if and only if for each positive real ε there is a set $A \in SO_2(X, Y)$ containing (u, v) and such that $f(A) \subset (f(u, v) - \varepsilon, f(u, v) + \varepsilon)$.*

Remark 3. *A function $f : X \times Y \to \mathbb{R}$ is symmetrically quasicontinuous at a point (u, v) if and only if for each positive real ε there is a set $A \in SO_1(X, Y) \cap SO_2(X, Y)$ containing (u, v) and such that $f(A) \subset (f(u, v) - \varepsilon, f(u, v) + \varepsilon)$.*

Definition 1. (cf. [6]). Let $f : X \times Y \to \mathbb{R}$ be a function and let $(u, v) \in X \times Y$ be a point. Then the set

$${}_{SO_1}C(f; (u, v)) = \{\gamma \in \mathbb{R} : \text{ for every } \varepsilon > 0 \text{ there is a set } S \in SO_1(X, Y) \text{ with}$$

$$(u, v) \in \mathrm{cl}(S), S \cup \{(u, v)\} \in SO_1(X, Y) \text{ and } |f(s, t) - \gamma| < \varepsilon \text{ for all } (s, t) \in S\}$$

is called the SO_1-cluster set of the function f at the point (u, v).

Replacing in Definition 1 the set "$SO_1(X, Y)$" by "$SO_2(X, Y)$" (or by "$SO_1(X, Y) \cap SO_2(X, Y)$") we define the SO_2-cluster set ${}_{SO_2}C(f; (u, v))$ (or the SOS-cluster set ${}_{SOS}C(f; (u, v))$) of the function f at the point (u, v).

Moreover, we use the standard symbol $C(g;u)$ for the cluster set of a function $g : X \to \mathbb{R}$ at a point u. Observe that (see [6])

$$C(g;u) = \bigcap_{U\in\mathcal{B}(u)} \mathrm{cl}(f(U)) = \{\gamma \in \mathbb{R} : \text{ for each } \varepsilon > 0 \text{ there is a set } A \subset X \text{ with } u \in \mathrm{cl}(A) \text{ and } |g(t) - \gamma| < \varepsilon \text{ for all } t \in A\},$$

where $\mathcal{B}(u)$ is an arbitrary basis of the neighborhood system $\mathcal{U}(u)$ of u.

In [6, Proposition 1] the authors observed that a function $g : X \to \mathbb{R}$ is quasicontinuous at a point $u \in X$ if and only if $g(u) \in {}_{SO}C(g;u)$. Standard arguments yield the following analogue.

Theorem 1. *A function $f : X \times Y \to \mathbb{R}$ is quasicontinuous with respect to the first coordinate at a point $(u,v) \in X \times Y$ if and only if $f(u,v) \in {}_{SO_1}C(f;(u,v))$.*

Theorem 2. *A function $f : X \times Y \to \mathbb{R}$ is quasicontinuous with respect to the second coordinate at a point $(u,v) \in X \times Y$ if and only if $f(u,v) \in {}_{SO_2}C(f;(u,v))$.*

Theorem 3. *A function $f : X \times Y \to \mathbb{R}$ is symmetrically quasicontinuous at a point $(u,v) \in X \times Y$ if and only if $f(u,v) \in {}_{SOS}C(f;(u,v))$.*

If $f : X \times Y \to \mathbb{R}$ is a function and $(u,v) \in X \times Y$ is a point, then the functions $f_u(t) = f(u,t)$, $t \in Y$, and $f^v(z) = f(z,v)$, $z \in X$, are called the vertical and horizontal sections of f. In [6, Proposition 2] the authors prove that if a function $g : X \to \mathbb{R}$ is quasicontinuous, then ${}_{SO}C(g;x) = C(g;x)$ for all points $x \in X$. An analogue of this is the following.

Theorem 4. *If a function $f : X \times Y \to \mathbb{R}$ is quasicontinuous with respect to the first coordinate, then for each point $(u,v) \in X \times Y$ the equality ${}_{SO_1}C(f;(u,v)) = C(f_u;v)$ is true.*

Proof. Fix a point $(u,v) \in X \times Y$ and observe that the inclusion

$${}_{SO_1}C(f;(u,v)) \subset C(f_u;v)$$

is true. For the proof of the opposite inclusion fix a real $\gamma \in C(f_u;v)$ and a real $\varepsilon > 0$. There is a set $A \subset Y$ with

$$v \in \mathrm{cl}(A) \text{ and } |f(u,y) - \gamma| < \frac{\varepsilon}{2} \text{ for } y \in A.$$

Since f is quasicontinuous with respect to the first coordinate at all points (u,y), $y \in A$, for all points $y \in A$ there are sets $U(y) \in SO_1(X,Y)$ containing (u,y) and such that

$$f(U(y)) \subset \left(f(u,y) - \frac{\varepsilon}{2}, f(u,y) + \frac{\varepsilon}{2}\right) \subset (\gamma - \varepsilon, \gamma + \varepsilon).$$

Consequently, the set $E = \bigcup_{y\in A} \mathrm{int}(U(y))$ satisfies

$$E,\ E\cup\{(u,v)\} \in SO_1(X,Y),\ (u,v)\in \mathrm{cl}(E) \ \text{ and } \ f(E)\subset(\gamma-\varepsilon,\gamma+\varepsilon).$$

So, $\gamma \in {}_{SO_1}C(f;(u,v))$ and the proof is completed. □

Using similar methods as for the previous theorem one can prove the next two theorems.

Theorem 5. *If a function $f : X \times Y \to \mathbb{R}$ is quasicontinuous with respect to the second coordinate, then for each point $(u,v) \in X \times Y$ the equality ${}_{SO_2}C(f;(u,v)) = C(f^v;u)$ is true.*

Theorem 6. *If a function $f : X \times Y \to \mathbb{R}$ is symmetrically quasicontinuous then for each point $(u,v) \in X \times Y$ the equality ${}_{SOS}C(f;(u,v)) = C(f_u;v) \cap C(f^v;u)$ is true.*

In [6, Proposition 3] the authors show that if a function $g : X \to \mathbb{R}$ and a point $u \in X$ are such that ${}_{SO}C(g;u) \neq \emptyset$, then g is cliquish at u, and conversely, if g is cliquish at u and locally bounded at u, then ${}_{SO}C(g;u) \neq \emptyset$. We obtain the following analogue.

Theorem 7. *Let $f : X \times Y \to \mathbb{R}$ be a function and let $(u,v) \in X \times Y$ be a point. If ${}_{SO_1}C(f;(u,v)) \neq \emptyset$, then f is cliquish with respect to the first coordinate at (u,v). Conversely, if f is locally bounded at (u,v) and f is cliquish with respect to the first coordinate, then ${}_{SO_1}C(f;(u,v)) \neq \emptyset$.*

PROOF. We apply a modification of the reasoning from [6]. Observe that if ${}_{SO_1}C(f;(u,v)) \neq \emptyset$, then f is cliquish with respect to the first coordinate at (u,v). Assume that f is locally bounded and cliquish with respect to the first coordinate at (u,v). There are a positive real M and a base $\mathcal{B}((u,v))$ for the neighborhood system $\mathcal{U}((u,v))$ such that for $B \in \mathcal{B}((u,v))$ the images $f(B) \subset [-M,M]$. For each set $B \in \mathcal{B}((u,v))$ and each real $\varepsilon > 0$ put

$$H(f;u;B;\varepsilon) = \{\gamma\in\mathbb{R} : \text{ there exists a nonempty open set } G\subset B \\ \text{with } u\in Pr_X(G) \text{ and } f(G)\subset(\gamma-\varepsilon,\gamma+\varepsilon)\},$$

where $Pr_X(G)$ denotes the projection of G onto X. Since f is cliquish with respect to the first coordinate at (u,v), the sets $H(f;u;B;\varepsilon)$ are nonempty. They are also bounded. Observe that

$${}_{SO_1}C(f;(u,v)) = \bigcap_{B\in\mathcal{B}((u,v)),\varepsilon>0} \mathrm{cl}(H(f;u;B;\varepsilon)).$$

Thus, if the set ${}_{SO_1}C(f;(u,v))$ is empty, then there is a finite intersection

$$\bigcap_{i=1}^{n} \mathrm{cl}(H(f;u;B_i;\varepsilon_i)) = \emptyset.$$

But

$$\bigcap_{i=1}^{n} \mathrm{cl}(H(f;u;B_i;\varepsilon_i)) \supset \bigcap_{i=1}^{n} H(f;u;B_i;\varepsilon_i) \supset H\left(f;u;\bigcap_{i=1}^{n} B_i; \min_{i\le n}\varepsilon_i\right) \neq \emptyset,$$

and this contradiction implies that ${}_{SO_1}C(f;(u,v)) \neq \emptyset$. □

In the same spirit as the preceding theorem one can prove also the next two theorems.

Theorem 8. *Let $f : X \times Y \to \mathbb{R}$ be a function and let $(u,v) \in X \times Y$ be a point. If ${}_{SO_2}C(f;(u,v)) \neq \emptyset$, then f is cliquish with respect to the second coordinate at (u,v). Conversely, if f is locally bounded at (u,v) and f is cliquish with respect to the second coordinate, then ${}_{SO_2}C(f;(u,v)) \neq \emptyset$.*

Theorem 9. *Let $f : X \times Y \to \mathbb{R}$ be a function and let $(u,v) \in X \times Y$ be a point. If ${}_{SOS}C(f;(u,v)) \neq \emptyset$, then f is symmetrically cliquish at (u,v). Conversely, if f is locally bounded at (u,v) and f is symmetrically cliquish, then ${}_{SOS}C(f;(u,v)) \neq \emptyset$.*

Let $f : X \times Y \to \mathbb{R}$ be a function and let $C(f)$ be the set of all continuity points of f. A function $h_f : X \times Y \to \mathbb{R}$ is said to be an admissible modification of a function f if $h_f(x) = f(x)$ for all $x \in C(f)$ and $C(f) \subset C(h_f)$. In [6, Theorem 3] the authors prove that if $g : X \to \mathbb{R}$ is such that ${}_{SO}C(g;u) \neq \emptyset$ for all $u \in X$, then each function $h : X \to \mathbb{R}$, with $h(u) \in {}_{SO}C(g;u)$ for $u \in X$, is a quasicontinuous admissible modification of g such that ${}_{SO}C(h;u) = {}_{SO}C(g;u)$ for all $u \in X$. As an analogue of that result we have the following.

Theorem 10. *Let $f : X \times Y \to \mathbb{R}$ be a function such that ${}_{SO_1}C(f;(u,v)) \neq \emptyset$ for each point $(u,v) \in X \times Y$. Then each function $h_f : X \times Y \to \mathbb{R}$ with $h_f(u,v) \in {}_{SO_1}C(f;(u,v))$ is an admissible modification of f quasicontinuous with respect to the first coordinate such that ${}_{SO_1}C(h_f;(u,v)) = {}_{SO_1}C(f;(u,v))$ for all $(u,v) \in X \times Y$.*

Proof. Since

$${}_{SO_1}C(f;(u,v)) \subset {}_{SO}C(f;(u,v)) \text{ for all } (u,v) \in X \times Y,$$

[6, Theorem 3] shows that h_f is an admissible modification of f. For the proof of the remaining part of our theorem we repeat also the reasoning from the proof of [6, Theorem 3]. For the proof of the coincidence of ${}_{SO_1}C(h_f;(u,v))$ and ${}_{SO_1}C(f;(u,v))$ fix a point (u,v) and $\gamma \in {}_{SO_1}C(h_f;(u,v))$. Let $\varepsilon > 0$ be a real and let U be an open neighborhood of (u,v). There is a point $t \in Y$ such that $(u,t) \in U$ and $|h_f(u,t) - \gamma| < \frac{\varepsilon}{2}$. Since $h_f(u,t) \in {}_{SO_1}C(f;(u,t))$ and $(u,t) \in U$, there is an open set $G \subset U$ such that

$$u \in Pr_X(G) \text{ and } |f(w,z) - h_f(u,t)| < \frac{\varepsilon}{2} \text{ for } (w,z) \in G.$$

So, $f(G) \subset (\gamma-\varepsilon, \gamma+\varepsilon)$ and consequently, ${}_{SO_1}C(h_f;(u,v)) \subset {}_{SO_1}C(f;(u,v))$.

Now, let $\gamma \in {}_{SO_1}C(f;(u,v))$. For proving $\gamma \in {}_{SO_1}C(h_f;(u,v))$ fix an open set $V \ni (u,v)$ and a real $\eta > 0$. There is an open set $H \subset V$ such that $u \in Pr_X(H)$and $f(H) \subset (\gamma-\eta, \gamma+\eta)$. The inclusion $f(H) \subset [\gamma-\eta, \gamma+\eta]$ implies that $h_f(H) \subset [\gamma-\eta, \gamma+\eta]$. Consequently, ${}_{SO_1}C(f;(u,v)) \subset {}_{SO_1}C(h_f;(u,v))$. This completes the proof of the equality ${}_{SO_1}C(f;(u,v)) = {}_{SO_1}C(h_f;(u,v))$. Since

$$h_f(u,v) \in {}_{SO_1}C(f;(u,v)) = {}_{SO_1}C(h_f;(u,v)),$$

at all points (u,v), by Theorem 1 we obtain that h_f is quasicontinuous with respect to the first coordinate. □

Now, we can use similar methods to show the following two theorems.

Theorem 11. *Let $f : X \times Y \to \mathbb{R}$ be a function such that ${}_{SO_2}C(f;(u,v)) \neq \emptyset$ for each point $(u,v) \in X \times Y$. Then each function $h_f : X \times Y \to \mathbb{R}$ with $h_f(u,v) \in {}_{SO_2}C(f;(u,v))$ is an admissible modification of f quasicontinuous with respect to the second coordinate such that ${}_{SO_2}C(h_f;(u,v)) = {}_{SO_2}C(f;(u,v))$ for all $(u,v) \in X \times Y$.*

Theorem 12. *Let $f : X \times Y \to \mathbb{R}$ be a function such that ${}_{SOS}C(f;(u,v)) \neq \emptyset$ for each point $(u,v) \in X \times Y$. Then each function $h_f : X \times Y \to \mathbb{R}$ with $h_f(u,v) \in {}_{SOS}C(f;(u,v))$ is an admissible modification of f symmetrically quasicontinuous such that ${}_{SOS}C(h_f;(u,v)) = {}_{SOS}C(f,(u,v))$ for all $(u,v) \in X \times Y$.*

In [6, Theorem 4] the authors show that for each admissible modification $h_g : X \to \mathbb{R}$ of a cliquish function $g : X \to \mathbb{R}$ on a Baire space X the equality ${}_{SO}C(g;u) = {}_{SO}C(h_g;u)$ is true at all points $u \in X$. An analogue of that theorem is the following.

Theorem 13. *Suppose that (Y, T_Y) is a Baire space and that a function $f : X \times Y \to \mathbb{R}$ is cliquish with respect to the first coordinate. Then for each admissible modification h_f of f the equality ${}_{SO_1}C(f;(u,v)) = {}_{SO_1}C(h_f;(u,v))$ is true at all points $(u,v) \in X \times Y$.*

Proof. We apply a modification of the proof of [6, Theorem 4]. Fix a point (u,v) and assume that ${}_{SO_1}C(h_f;(u,v)) \neq \emptyset$. Let $\gamma \in {}_{SO_1}C(h_f;(u,v))$, let $U \ni (u,v)$ be an open set and let ε be a positive real. There is an open set $G \subset U$ such that $u \in Pr_X(G)$and $h_f(G) \subset \left(\gamma - \frac{\varepsilon}{2}, \gamma + \frac{\varepsilon}{2}\right)$. Since f is cliquish with respect to the first coordinate, the section $(C(f))_u$ is dense in Y ([2]). Therefore we can find a point $(u,z) \in C(f) \cap G$. Consequently, there is an open set $H \subset G$ containing (u,z) such that

$$f(H) \subset \left(f(u,z) - \frac{\varepsilon}{2}, f(u,z) + \frac{\varepsilon}{2}\right).$$

Observe that $f(H) \subset (\gamma - \varepsilon, \gamma + \varepsilon)$. This yields $\gamma \in {}_{SO_1}C(f;(u,v))$. So,

$${}_{SO_1}C(h_f;(u,v)) \subset {}_{SO_1}C(f;(u,v)).$$

Now let $\gamma \in {}_{SO_1}C(f;(u,v))$. Given $\eta > 0$ and an open neighborhood V of (u,v), we can apply the same steps as before. Since $C(f) \subset C(h_f)$ and $h_f|_{C(f)} = f|_{C(f)}$, there is an open set $D \subset V$ such that $u \in Pr_X(D)$ and $h_f(D) \subset (\gamma - \eta, \gamma + \eta)$. This means $\gamma \in {}_{SO_1}C(f;(u,v))$ and proves ${}_{SO_1}C(f;(u,v)) \subset {}_{SO_1}C(h_f;(u,v))$ and ${}_{SO_1}C(f;(u,v)) = {}_{SO_1}C(h_f;(u,v))$. □

By the same methods one can also show the next two theorems.

Theorem 14. *Suppose that (X,T_X) is a Baire space and that a function $f : X \times Y \to \mathbb{R}$ is cliquish with respect to the second coordinate. Then for each admissible modification h_f of f the equality ${}_{SO_2}C(f;(u,v)) = {}_{SO_2}C(h_f;(u,v))$ is true at all points $(u,v) \in X \times Y$.*

Theorem 15. *Suppose that (X,T_X) and (Y,T_Y) are Baire spaces and that a function $f : X \times Y \to \mathbb{R}$ is symmetrically cliquish. Then for each admissible modification h_f of f the equality ${}_{SOS}C(f;(u,v)) = {}_{SOS}C(h_f;(u,v))$ is true at all points $(u,v) \in X \times Y$.*

A function $g : X \to \mathbb{R}$ is called a semi-open step function (or an SO-step function) if there exists a partition $\mathcal{P} = \{P_i : i \in I\}$ of X into subsets $P_i \in SO(X)$ such that g is constant on the sets P_i. In [6] the authors observe that each semi-open step function is quasicontinuous and that each quasicontinuous function $h : X \to \mathbb{R}$ is the uniform limit of a sequence of semi-open step functions $h_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$.

Similarly, we say that a function $f : X \times Y \to \mathbb{R}$ is an SO_1-step function (alternatively SO_2-step function) [symmetrically semi-open step function] if there exists a partition $\mathcal{P} = \{P_i : i \in I\}$ of $X \times Y$ into subsets $P_i \in SO_1(X,Y)$ (alternatively $P_i \in SO_2(X,Y)$) $[P_i \in SO_1(X,Y) \cap SO_2(X,Y)]$ such that f is constant on the sets P_i. Evidently each SO_1-step function (alternatively

SO_2-step function) [symmetrically semi-open step function] is quasicontinuous with respect to the first coordinate (alternatively to the second coordinate) [symmetrically quasicontinuous].

The following problems are open.

Problem 1. Let $f : X \times Y \to \mathbb{R}$ be a quasicontinuous with respect to the first coordinate function. Does there exist a sequence of SO_1-step functions $f_n : X \times Y \to \mathbb{R}$, $n = 1, 2, \ldots$, which uniformly converges to f?

Problem 2. Let $f : X \times Y \to \mathbb{R}$ be a quasicontinuous with respect to the second coordinate function. Does there exist a sequence of SO_2-step functions $f_n : X \times Y \to \mathbb{R}$, $n = 1, 2, \ldots$, which uniformly converges to f?

Problem 3. Let $f : X \times Y \to \mathbb{R}$ be a symmetrically quasicontinuous function. Does there exist a sequence of symmetrically semi-open step functions $f_n : X \times Y \to \mathbb{R}$, $n = 1, 2, \ldots$, which uniformly converges to f?

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