RESEARCH

Real Analysis Exchange
Vol. 33(1), 2007/2008, pp. 51–82

Erik Talvila,* Department of Mathematics and Statistics, University College of the Fraser Valley, Abbotsford, BC V2S 7M8, Canada.
email: Erik.Talvila@ucfv.ca

THE DISTRIBUTIONAL DENJOY INTEGRAL

Abstract

Let f be a distribution (generalized function) on the real line. If there is a continuous function F with real limits at infinity such that $F' = f$ (distributional derivative), then the distributional integral of f is defined as $\int_{-\infty}^{\infty} f = F(\infty) - F(-\infty)$. It is shown that this simple definition gives an integral that includes the Lebesgue and Henstock–Kurzweil integrals. The Alexiewicz norm leads to a Banach space of integrable distributions that is isometrically isomorphic to the space of continuous functions on the extended real line with uniform norm. The dual space is identified with the functions of bounded variation. Basic properties of integrals are established using elementary properties of distributions: integration by parts, Hölder inequality, change of variables, convergence theorems, Banach lattice structure, Hake theorem, Taylor theorem, second mean value theorem. Applications are made to the half plane Poisson integral and Laplace transform. The paper includes a short history of Denjoy's descriptive integral definitions. Distributional integrals in Euclidean spaces are discussed and a more general distributional integral that also integrates Radon measures is proposed.

1 Introduction.

We are fortunate to live in a richly diverse universe in which there are many integrals and many interesting ways of defining these integrals. Some of the major integrals are those of Riemann, Lebesgue, Denjoy and Henstock-Kurzweil. In this paper we will present a theory of integration based on the descriptive

Key Words: distributional Denjoy integral, continuous primitive integral, Henstock-Kurzweil integral, Schwartz distributions, Alexiewicz norm, Banach lattice
Mathematical Reviews subject classification: 26A39, 46E15, 46F05, 46G12
Received by the editors June 3, 2006
Communicated by: Stefan Schwabik
*Supported by the Natural Sciences and Engineering Research Council of Canada.

Denjoy method. The definition is simple and elegant. A distribution f is integrable if there is a continuous function F whose distributional derivative equals f. Then $\int_a^b f = F(b) - F(a)$. This is a very powerful integral that includes all of those mentioned above. To define it we only need the notion of distributional derivative and the Riemann integration of continuous functions. No measure theory is needed to define the integral and there are no partitions to construct. We will see that under the Alexiewicz norm (see Section 2), the space of integrable distributions forms a Banach space (and Banach lattice) that is isometrically isomorphic to the space of continuous functions on the extended real line with uniform norm. The dual space is identified with the space of functions of bounded variation. There are general versions of the Fundamental Theorem of Calculus, integration by parts and change of variables formulas, a Hölder inequality, convergence theorems, Taylor's theorem, Hake's theorem and the second mean value theorem. We give applications to the half plane Poisson integral and the Laplace transform. Absolute integration is also discussed. All of these results are easy to prove using only elementary results in distributions (generalized functions). Besides distributions, we will assume some familiarity with Riemann–Stieltjes integrals, functions of bounded variation, and basic notions of functional analysis, such as Cauchy sequences in the Banach space of continuous functions with uniform norm $\|\cdot\|_\infty$. Most of the results we use in distributions are summarized in Section 3. The reader should have a nodding acquaintance with Lebesgue measure and integration although it will be apparent that this approach to integration de-emphasizes measure and puts more emphasis on functional analytic aspects. Our setting will be integration on the real line with respect to Lebesgue measure λ. At the end of the paper we sketch out generalizations to integration in $\mathbb{R}^n$ and integration with respect to Radon measures.

2 Integrating Derivatives.

The Riemann and Lebesgue integrals are both absolute. This means that if function f is integrable, so is $|f|$. An outcome of this is that we get a weaker version of the Fundamental Theorem than we'd like. For example, the function $F(x) = x^2 \cos(x^{-2})$ for $x \neq 0$ and $F(0) = 0$ is differentiable at each point in $\mathbb{R}$ but F' is not continuous at 0 since $F'(x) \sim 2x^{-1} \sin(x^{-2})$ as $x \to 0$. And, $\int_0^1 F'$ does not exist in the Riemann or Lebesgue sense since $|F'|$ is not integrable in a neighborhood of 0. However, $\int_0^1 F'$ exists as a conditionally convergent improper Riemann integral and hence as a Henstock–Kurzweil integral. So, to be able to write $\int_0^1 F' = F(1) - F(0)$ we need to consider nonabsolute integrals. The problem of integrating derivatives was solved by Denjoy in the early part of the 20th century.

Arnaud Denjoy (pronounced rather like "dawn-djwah") was a French mathematician who was born in 1884 and lived for over 90 years. He produced three different solutions to the problem of integrating derivatives and is known for several other results in function theory, Fourier series, quasi-analytic functions and dynamical systems. See [14] for a photo.

Denjoy's solution was to use a descriptive definition of the integral. This defines the integral via its *primitive*. This is a continuous function whose derivative in some sense is equal to the integrand. For example, $F : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* (AC) if for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $\{(x_n, y_n)\}$ is a sequence of disjoint intervals with $\sum |x_n - y_n| < \delta$ we have $\sum |F(x_n) - F(y_n)| < \epsilon$. This definition readily generalizes to arbitrary measure spaces. We have the strict inclusions $C^1 \subsetneq AC \subsetneq C^0$. If F is AC, then it is continuous on $\mathbb{R}$ and is differentiable almost everywhere. If $F \in AC$, then $\int_a^b F' = F(b) - F(a)$. The descriptive definition of the Lebesgue integral is, then: $f : [a, b] \to \mathbb{R}$ is integrable if there is a function $F \in AC$, called the primitive, such that $F' = f$ almost everywhere. In this case, $\int_a^b f = F(b) - F(a)$. This is one half of the Fundamental Theorem of Calculus. The other half says that if $f \in L^1$, then $F(x) := \int_a^x f$ defines an AC function and $F' = f$ almost everywhere. The function $F(x) = x^2 \cos(x^{-2})$ at the beginning of this section is not AC.

The corresponding function space for Denjoy integrals is $ACG*$ (generalized absolute continuity in the restricted sense). The precise definition need not concern us here. If you are interested, see [12]. The important thing is that $C^1 \subsetneq AC \subsetneq ACG* \subsetneq C^0$ and we have a larger, more complicated space in which functions have derivatives almost everywhere. The Denjoy integral is then defined by saying that f is integrable if it has a primitive $F \in ACG*$ such that $F' = f$ almost everywhere. Then, $\int_a^b f = F(b) - F(a)$.

Since $AC \subsetneq ACG*$, the Denjoy integral properly contains the Lebesgue integral (with respect to Lebesgue measure on the real line). It turns out that if a continuous function is differentiable everywhere, then it is in $ACG*$. The same is true if the function has a derivative everywhere except in a countable set. Hence, we can integrate the function F' given at the beginning of this section. The Denjoy integral is equivalent to the Henstock-Kurzweil integral, which is defined using Riemann sums. It is also equivalent to the Perron integral, which is defined using major and minor functions [12].

The Denjoy integrable functions are made into a normed linear space via the Alexiewicz norm [1]. This is defined by $\|f\| = \sup_{a \le x \le b} |\int_a^x f|$. Unfortunately, this does not define a Banach space so we do not have analogues of the many wonderful results in L^p spaces. Real analysts delight in working with spaces such as $ACG*$ (see any volume of the journal *Real Analysis Exchange*). However, the attraction of such spaces has been less compelling for other mathematicians. One problem is that there is no canonical generalization to $\mathbb{R}^n$. A considerable amount of research was carried out in Denjoy integration until the end of the 1930's but these deficiencies caused this integral to be virtually abandoned by 1940. However, we get a much simpler and yet more powerful integral by using the distributional Denjoy integral. For this, we will need to briefly introduce some results in distributions.

3 Schwartz Distributions.

The theory of distributions, or generalized functions, extends the notion of function so that we no longer have pointwise values but all distributions have derivatives of all orders. Most of the final theory that emerged in the 1940's was due to Laurent Schwartz but of course he did not work in vacuum and names such as Dirac and Sobolev figure prominently. A good introduction is [11], while [25] is still an important work in the field.

Distributions are defined as continuous linear functionals on certain vector spaces. Define the space of *test functions* by $\mathcal{D} = C_c^\infty = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty \text{ and } \phi \text{ has compact support}\}$. The *support* of a function is the closure of the set on which it does not vanish. With the usual pointwise operations $\mathcal{D}$ is a vector space. An example of a test function is $\phi(x) = \exp(1/(|x|-1))$ for $|x| < 1$ and $\phi(x) = 0$, otherwise. The only analytic function in $\mathcal{D}$ is 0. We say a sequence $\{\phi_n\} \subset \mathcal{D}$ converges to $\phi \in \mathcal{D}$ if there is a compact set K such that all ϕ_n have support in K and for each integer $m \geq 0$, the sequence of derivatives $\phi_n^{(m)}$ converges to $\phi^{(m)}$ uniformly on K. The distributions are then defined as the dual space of $\mathcal{D}$, i.e., the continuous linear functionals on $\mathcal{D}$. For each $\phi \in \mathcal{D}$, the action of distribution T is denoted $\langle T, \phi \rangle \in \mathbb{R}$. Linear means that for all $a, b \in \mathbb{R}$ and all $\phi, \psi \in \mathcal{D}$ we have $\langle T, a\phi + b\psi \rangle = a\langle T, \phi \rangle + b\langle T, \psi \rangle$. Continuous means that if $\phi_n \to \phi$ in $\mathcal{D}$, then $\langle T, \phi_n \rangle \to \langle T, \phi \rangle$ in $\mathbb{R}$. The space of distributions is denoted $\mathcal{D}'$.

If f is a locally integrable function, then $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f\phi$ defines a distribution since $\phi \in \mathcal{D}$ has compact support, integrals are linear and dominated convergence or uniform convergence allows us to take limits under the integral. Hence, for $1 \leq p \leq \infty$ all the functions in L^p are distributions. An example of a distribution that is not given by a function is the Dirac distribution. It is defined by $\langle \delta, \phi \rangle = \phi(0)$.

If $f \in C^1$ and ϕ is a test function then integration by parts shows that $\int_{-\infty}^{\infty} f'\phi = -\int_{-\infty}^{\infty} f\phi'$. For all $T \in \mathcal{D}'$ we can mimic this behavior by defining the derivative via $\langle T', \phi \rangle = -\langle T, \phi' \rangle$. With this definition, all distributions have derivatives of all orders and each derivative is a distribution. For example, $\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$. In electrostatics, δ models a point charge and δ' models a dipole. If T is a function, we will write its distributional derivative as T' and its pointwise derivative as $T'(x)$ where $x \in \mathbb{R}$. From now on, all derivatives will be distributional derivatives unless stated otherwise.

If $f \in C^0$, then T_f is a distribution. We can recover its pointwise value at any point $x \in \mathbb{R}$ by evaluating the limit $\langle T, \phi_n \rangle$ for a sequence $\{\phi_n\} \subset \mathcal{D}$ such that for each n, $\phi_n \geq 0$, $\int_{-\infty}^{\infty} \phi_n = 1$, and the support of ϕ_n tends to $\{x\}$ as $n \to \infty$. Such a sequence is termed a *delta sequence*.

The distributional derivative subsumes pointwise and approximate derivatives and so is very general. An integration process that inverts it leads to a very general integral.

4 The Distributional Denjoy Integral.

Denote the extended real numbers by $\overline{\mathbb{R}} = [-\infty, \infty]$. We define $C^0(\overline{\mathbb{R}})$ to be the continuous functions such that $\lim_{\infty} F$ and $\lim_{-\infty} F$ both exist in $\mathbb{R}$. To be in $C^0(\overline{\mathbb{R}})$, F must have real limits at infinity. We can then define $F(\pm\infty) = \lim_{\pm\infty} F$. Thus, no definition of $F(x) = e^x$ at $\pm\infty$ can put F in $C^0(\overline{\mathbb{R}})$. Similarly with $G(x) = \sin(x)$. However, $H(x) = \arctan(x)$ is in $C^0(\overline{\mathbb{R}})$ if we define $H(\pm\infty) = \pm\pi/2$. Let

$$\mathcal{B}_C = \{F \in C^0(\overline{\mathbb{R}}) \mid F(-\infty) = 0\}.$$

Note that $\mathcal{B}_C$ is a Banach space with the uniform norm $\|F\|_\infty = \sup_{\mathbb{R}} |F| = \max_{\overline{\mathbb{R}}} |F|$. We now define the space of integrable distributions by

$$\mathcal{A}_C = \{f \in \mathcal{D}' \mid f = F' \text{ for some } F \in \mathcal{B}_C\}.$$

A distribution f is integrable if it is the distributional derivative of a function $F \in \mathcal{B}_C$; i.e., for all $\phi \in \mathcal{D}$ we have $\langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} F\phi'$. Since F and ϕ' are continuous and ϕ' has compact support, this exists as a Riemann integral. If $f \in \mathcal{A}_C$, then its integral is defined as $\int_{-\infty}^{\infty} f = F(\infty)$. An obvious alternative would have been to take $F \in C^0(\overline{\mathbb{R}})$ and then $\int_{-\infty}^{\infty} f = F(\infty) - F(-\infty)$. The function F is a primitive of f.

This definition seems to have been first proposed by P. Mikusiński and K. Ostaszewski [18]. (See also [21], [22] and [19].) Without reference to these papers, it was developed in detail in the plane by D. D. Ang, K. Schmidt, L. K. Vy [3] (and repeated in [4]). Several of our results come from this paper. All of these papers work with the integral in a compact Cartesian interval.

Notice that if $f \in \mathcal{A}_C$, then f has many primitives in $C^0(\overline{\mathbb{R}})$, all differing by a constant, but f has exactly one primitive in $\mathcal{B}_C$. If $F_1, F_2 \in \mathcal{B}_C$ and $F_1' = f, F_2' = f$, then linearity of the derivative shows $(F_1 - F_2)' = 0$. It is known that the only solutions of this differential equation are constants [11, §2.4]. The condition at $-\infty$ now shows $F_1 = F_2$. Hence, the integral is unique.

We can define $\int_a^b f = \int_{-\infty}^b f - \int_{-\infty}^a = F(b) - F(a)$ for all $a, b \in \overline{\mathbb{R}}$, where F is a primitive of f. The integral is then additive: $\int_a^b f + \int_b^c f = \int_a^c f$. Also, for open interval $I \subset \mathbb{R}$, define $\mathcal{D}(I) = \{\phi : I \to \mathbb{R} \mid \phi \in C^\infty(I)$ and ϕ has compact support in $I\}$. We then have the distributions on I, $\mathcal{D}'(I)$, being the continuous linear functionals on $\mathcal{D}(I)$. If $f \in \mathcal{D}'(I)$, then f is integrable on $\overline{I}$ if there is $F \in C^0(\overline{I})$ such that $F' = f$. Then $\int_a^b f = F(b) - F(a)$. It is easy to see that these two definitions of $\int_a^b f$ are equivalent. For, if $f \in \mathcal{D}'$, then $f \in \mathcal{D}'(I)$ since $\mathcal{D}(I) \subset \mathcal{D}$. If $F \in C^0(\overline{\mathbb{R}})$ with $F' = f$ on $\mathcal{D}$ then we also have $F \in C^0(\overline{I})$ so both definitions give $\int_a^b f = F(b) - F(a)$. If $f \in \mathcal{D}'(I)$, then in general we cannot extend f to $\mathcal{D}'$. For example, f could be a function with a non-integrable singularity at an endpoint of I. However, if f is integrable on $\overline{I}$, then we have $F \in C^0(\overline{I})$ such that $F' = f$ on $\mathcal{D}(I)$. Write $I = (a, b)$. Define $G = 0$ on $[-\infty, a]$, $G = F - F(a)$ on I, $G = F(b) - F(a)$ on $[b, \infty]$. Then $G \in \mathcal{B}_C$ and $G' = f$ on $\mathcal{D}(I)$. And, $G(b) - G(a) = F(b) - F(a)$.

Since the derivative is linear, the operations $\langle af + g, \phi\rangle = a\langle f, \phi\rangle + \langle g, \phi\rangle$ ($a \in \mathbb{R}$; $f, g \in \mathcal{A}_C$; $\phi \in \mathcal{D}$) make $\mathcal{A}_C$ into a vector space and $\int_{-\infty}^{\infty} af + g = aF(\infty) + G(\infty)$. We will use the convention that when f, g, f_1, etc. are in $\mathcal{A}_C$, then we will denote their corresponding primitives in $\mathcal{B}_C$ by upper case letters F, G, F_1, etc.

Here are some examples that show the extent of applicability of our definition.

Example 1. 1. If f is Riemann integrable on $[a, b]$, then the Riemann integral $F(x) = \int_a^x f$ is a Lipshitz continuous function and $F'(x) = f(x)$ at all points of continuity of f. By Lebesgue's characterization of the Riemann integral, f is continuous almost everywhere. Hence, $\langle F', \phi\rangle = \int_{-\infty}^{\infty} F'\phi = \int_{-\infty}^{\infty} f\phi$ since changing F' on a set of measure zero doesn't affect the value of this last integral. Therefore, if $F'(x) = f(x)$ almost everywhere, then $F' = f$ on $\mathcal{D}$. The distributional integral then contains the Riemann integral.

2. If $f \in L^1$, then $F(x) = \int_{-\infty}^x f$ defines $F \in AC \cap C^0(\overline{\mathbb{R}}) \subsetneq \mathcal{B}_C$. Since $F' = f$ almost everywhere, the distributional integral then contains the Lebesgue integral. Note that to define L^1 primitives on the real line we have to include the condition that $F \in C^0(\overline{\mathbb{R}})$ with $F \in AC$.

3. If f is Denjoy integrable, then its primitive is an $ACG*$ function and by the same reasoning as above, the distributional integral contains the Denjoy integral. This integral includes the improper Riemann and Cauchy-Lebesgue extensions of the Riemann and Lebesgue integrals, respectively. The function F' given at the beginning of Section 2 has an improper Riemann integral. Only

the origin is a point of *nonabsolute summability*, i.e., over no open interval containing the origin is $|F'|$ integrable. However, the Denjoy integral can integrate functions whose set of points of nonabsolute summability has positive measure, provided it is nowhere dense on the real line. For such functions it is impossible to define an integral by limits of integrals over subintervals as is done with the improper Riemann and Cauchy-Lebesgue processes. Denjoy used a transfinite induction process, which he called *totalization*, to define an integral in terms of limits of Lebesgue integrals. This was his second solution to the problem of integrating derivatives. This integral turned out to be equivalent to the integral defined using $ACG*$ functions. See [6] for references to this history.

Denjoy's third solution to the problem of integrating derivatives was to define an integration process that integrated the approximate derivative of ACG functions. Here ACG is yet another complicated function class of continuous functions that have some differentiability properties. In this case, $ACG* \subsetneq ACG \subsetneq C^0$. See [7] or [12] for details. The *wide* or *generalized* Denjoy integral of f is $\int_a^x f = F(x) - F(a)$ where $F \in ACG$ and $D_{ap}F = f$ almost everywhere. Using integration by parts for the wide Denjoy integral [7, p. 33] we can show that if $D_{ap}F = f$ almost everywhere, then $F' = f$ on $\mathcal{D}$. Hence, the distributional integral contains the wide Denjoy integral.

4. Let F be a continuous function such that $F'(x)$ does not exist for any $x \in \mathbb{R}$. Then $F' \in \mathcal{A}_C$ and $\int_a^b F' = F(b) - F(a)$ for all $a, b \in \mathbb{R}$. This example shows the following difference between Denjoy and distributional integrals. If $\int_a^b f$ exists as a Denjoy integral, then there is a subinterval $I \subset [a, b]$ such that $|f|$ is integrable over I, i.e., $f \in L^1(I)$. The corresponding result is false for the distributional integral since F would have to be AC on I and thus differentiable almost everywhere in I but F is differentiable nowhere.

5. Let F be a continuous, increasing, singular function on $[0, 1]$, such as the Cantor–Lebesgue function. Then $F'(x) = 0$ for almost all $x \in [0, 1]$. Since F is of bounded variation (see Section 5), its derivative is integrable in the Lebesgue sense and $\int_0^x F'(t)\, dt = 0$ for all $x \in [0, 1]$. But, $F' \in \mathcal{A}_C$ and the distributional integral is $\int_0^x F' = F(x) - F(0)$ for all $x \in [0, 1]$.

6. The distributional Denjoy integral is included in the Riemann-Stieltjes integral since for any function F we have $\int_a^b dF = F(b) - F(a)$. A valuable feature of the distributional integral is that it confines itself to the Banach space $\mathcal{A}_C$ so we can work directly with the integrand F' rather than have to deal with the differential dF or its attendant finitely additive measure.

We now consider the Banach space structure of $\mathcal{A}_C$. For $f \in \mathcal{A}_C$, define the Alexiewicz norm by $\|f\| = \|F\|_\infty = \sup_{\mathbb{R}} |F| = \max_{\overline{\mathbb{R}}} |F|$.

Theorem 2. *With the Alexiewicz norm, $\mathcal{A}_C$ is a Banach space.*

Proof. The fact that $\mathcal{A}_C$ is a vector space follows from the linearity of the derivative, so we will start by proving that $\|\cdot\|$ is a norm. Let $f, g \in \mathcal{A}_C$.

(i) First, $\|0\| = \|0\|_\infty = 0$. And, if $\|f\| = 0$, then $\|F\|_\infty = 0$ so $F(x) = 0$ for all $x \in \overline{\mathbb{R}}$. But then $F' = 0$.

(ii) Let $a \in \mathbb{R}$. Then $(aF)' = aF'$. Note that this means $\langle (aF)', \phi\rangle = -\int_{-\infty}^{\infty}(aF(x))\phi'(x)\,dx = -a\int_{-\infty}^{\infty} F(x)\phi'(x)\,dx = a\langle F', \phi\rangle$ for all $\phi \in \mathcal{D}$. We then have $\|af\| = \|aF\|_\infty = |a|\|F\|_\infty$.

(iii) Since $(F+G)' = F' + G'$ we get $\|f+g\| = \|F+G\|_\infty \le \|F\|_\infty + \|G\|_\infty = \|f\| + \|g\|$.

And, $\mathcal{A}_C$ is a normed linear space. To show it is complete, suppose $\{f_n\}$ is a Cauchy sequence in $\|\cdot\|$. Since we have $\|F_n - F_m\|_\infty = \|f_n - f_m\|$ it follows that $\{F_n\}$ is Cauchy in $\|\cdot\|_\infty$. There is $F \in \mathcal{B}_C$ such that $\|F - F_n\|_\infty \to 0$. But then $\|F' - f_n\| = \|F - F_n\|_\infty \to 0$ so $f_n \to F'$ in $\|\cdot\|$. Since $F \in \mathcal{B}_C$ we have $F' \in \mathcal{A}_C$. □

Three equivalent norms are considered in Theorem 29.

The definition of the integral shows that $\mathcal{A}_C$ and $\mathcal{B}_C$ are isometrically isomorphic [3]. They are isomorphic because a bijection is given by $f \leftrightarrow F$ where $f \in \mathcal{A}_C$ and F is its primitive in $\mathcal{B}_C$. This mapping is a linear isometry since for all $F, G \in \mathcal{B}_C$ and all $a \in \mathbb{R}$, $(aF + G)' = aF' + G'$ and $\|f\| = \|F\|_\infty$. This also shows $\mathcal{A}_C$ is separable and that L^1 and the spaces of Denjoy and wide Denjoy integrable functions are dense in $\mathcal{A}_C$.

Theorem 3. *$\mathcal{A}_C$ is separable and L^1 and the spaces of Denjoy and wide Denjoy integrable functions are dense in $\mathcal{A}_C$.*

Proof. Functions, Φ, for which there is a polynomial p and an interval $[a, b] \subset \mathbb{R}$ with $p(a) = 0$ such that $\Phi = 0$ on $(-\infty, a]$, $\Phi = p$ on $[a, b]$, and $\Phi = p(b)$ on $[b, \infty)$, are dense in $\mathcal{B}_C$ with $\|\cdot\|_\infty$. But such functions are absolutely continuous, so L^1 is dense in $\mathcal{A}_C$. It follows that the spaces of Denjoy and wide Denjoy integrable functions are dense in $\mathcal{A}_C$. Polynomials on $[a, b]$ with rational coefficients form a countable dense set in $C^0([a, b])$ so $\mathcal{A}_C$ is separable. □

Note that $C^0(K)$ is separable exactly when K is compact [8, Exercise V.7 17]. Our two-point compactification of the real line makes $\overline{\mathbb{R}}$ into a compact Hausdorff space. A topological base is the set of all intervals (a, b), $[-\infty, b)$, $(a, \infty]$ and $[-\infty, \infty]$, for all $-\infty \le a < b \le \infty$. That is, we declare all such intervals open in $\overline{\mathbb{R}}$.

One half of the Fundamental Theorem is built into the definition. The other half follows easily.

Theorem 4 (Fundamental Theorem of Calculus)**.** (a) *Let* $f \in \mathcal{A}_C$ *and let* $\Phi(x) = \int_{-\infty}^{x} f$*. Then* $\Phi \in \mathcal{B}_C$ *and* $\Phi' = f$*.*
(b) *Let* $F \in C^0(\overline{\mathbb{R}})$*. Then* $\int_{-\infty}^{x} F' = F(x) - F(-\infty)$ *for all* $x \in \overline{\mathbb{R}}$*.*

Proof. (a) By the uniqueness of the integral, $\Phi = F \in \mathcal{B}_C$. Then $\Phi' = F' = f$. (b) There is a constant $c \in \mathbb{R}$ such that $F + c \in \mathcal{B}_C$. But then $F' = (F+c)' \in \mathcal{A}_C$ and the result follows from the definition of the integral. □

At this stage it is hoped that the reader appreciates what we have accomplished. With minimal effort we have proven a very general version of the Fundamental Theorem and have proven that the space of integrable distributions is a Banach space. To prove the corresponding results for the Lebesgue integral requires considerably more machinery. For example, part (b) of Theorem 4 (Lebesgue differentiation theorem) uses the Vitali covering theorem. And, one usually requires convergence theorems to prove that L^1 is complete.

5 Integration by Parts, Hölder's Inequality.

If $g : \overline{\mathbb{R}} \to \mathbb{R}$, its *variation* is $Vg = \sup \sum_n |g(x_n) - g(y_n)|$ where the supremum is taken over every sequence $\{(x_n, y_n)\}$ of disjoint intervals in $\overline{\mathbb{R}}$. The set of functions with bounded variation is denoted $\mathcal{BV}$. It is known that functions of bounded variation are bounded and have left and right limits at each point (from the right at $-\infty$ and from the left at ∞.) Thus, if $g : \mathbb{R} \to \mathbb{R}$ is of bounded variation on $\mathbb{R}$, then the limits $\lim_{-\infty} g$ and $\lim_{\infty} g$ exist and we will use these to extend the domain of g to $\overline{\mathbb{R}}$. If $g \in \mathcal{BV}$ we can change g on a countable set so that it is right continuous on $[-\infty, \infty)$ and left continuous at ∞, i.e., $\lim_{x \to a^+} g(x) = g(a)$ for all $a \in [-\infty, \infty)$ and $\lim_{x \to \infty} g(x) = g(\infty)$. We will say such functions are of *normalized bounded variation* ($\mathcal{NBV}$). (This is slightly different from the usual definition but more convenient for our purposes. See [8, p. 241].) The space $\mathcal{BV}$ is a Banach space with norm $\|g\|_{\mathcal{BV}} = |g(-\infty)| + Vg$.

The *essential variation* is defined as $essvar\, g = \sup \int_{-\infty}^{\infty} g\phi'$ where now the supremum is taken over all $\phi \in C_c^1$ with $\|\phi\|_\infty \leq 1$. Denote the functions of essential variation by $\mathcal{EBV}$. Changing a function at even one point can affect its variation but changing a function on a set of measure zero will not affect its essential variation. And, $\mathcal{BV} \subsetneq \mathcal{EBV}$ but changing a function in $\mathcal{EBV}$ on a certain set of measure zero will put it into $\mathcal{BV}$. The space $\mathcal{EBV}$ is a Banach space with norm $\|g\|_{\mathcal{EBV}} = \|g\|_\infty + essvar\, g$. If $g \in \mathcal{NBV}$, then its variation and essential variation are identical.

As with the Denjoy integral, functions of bounded variation play an important role in the distributional integral. They form the dual space, tell us about integration by parts and Hölder's inequality. Theorem 8 below shows that results that hold for functions of bounded variation also hold for functions of essential bounded variation.

If $F \in C^0(\overline{\mathbb{R}})$ and $g \in \mathcal{BV}$, then it is known that the Riemann–Stieltjes integral $\int_{-\infty}^{\infty} F\,dg$ exists. It can be defined using a partition of $\overline{\mathbb{R}}$. The integral exists, with value $\int_{-\infty}^{\infty} F\,dg \in \mathbb{R}$, if for all $\epsilon > 0$ there is $\delta > 0$ such that if $-\infty = x_0 < x_1 < \ldots < x_N = \infty$, $\max_{2\leq n\leq N-1}(x_n - x_{n-1}) < \delta$, $x_1 < -1/\delta$, and $x_{N-1} > 1/\delta$, then for all $z_n \in [x_{n-1}, x_n]$, we have

$$\Big|\sum_{n=1}^{N} F(z_n)[g(x_n) - g(x_{n-1})] - \int_{-\infty}^{\infty} F\,dg\Big| < \epsilon.$$

To integrate over $[a, b] \subset \overline{\mathbb{R}}$ we use partitions of $[a, b]$. The integral can also be defined by taking limits of Riemann–Stieltjes integrals over finite subintervals; specifically

$$\int_{-\infty}^{\infty} F\,dg = \lim_{\substack{B\to\infty\\ A\to-\infty}} \int_A^B F\,dg + F(\infty)\big[g(\infty) - \lim_{x\to\infty} g(x)\big] + F(-\infty)\big[\lim_{x\to-\infty} g(x) - g(-\infty)\big].$$

See [15, p. 187] and [27] for details.

Proposition 5. *Let $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$ and let $H(x) = F(x)g(x) - \int_{-\infty}^{x} F(t)\,dg(t)$. Then $H \in \mathcal{B}_C$.*

Proof. Since g is of bounded variation, it is bounded. Write $|g| \leq M$ for some $M \in \mathbb{R}$. Let $x \in \mathbb{R}$ and $y \geq x$. Because $\int_x^y dg = g(y) - g(x)$, we can write $H(x) - H(y) = [F(x) - F(y)]\,g(x) + \int_x^y [F(t) - F(y)]\,dg(t)$. Now,

$$\begin{aligned}|H(x) - H(y)| \leq& |F(x) - F(y)|M + \max_{x\leq t\leq y} |F(t) - F(y)|Vg \\ \to& 0 \text{ as } y \to x \text{ since } F \text{ is uniformly continuous.}\end{aligned} \tag{1}$$

Similarly if $y \leq x$. Hence, $H \in C^0(\mathbb{R})$. To prove $H \in \mathcal{B}_C$, let $x \in \mathbb{R}$. Then $|H(x)| \leq |F(x)|M + \|F\chi_{(-\infty,x]}\|_\infty Vg \to 0$ as $x \to -\infty$. From (1), the sequence $\{H(n)\}$ is Cauchy and so has a limit as $n \to \infty$. Hence, $H \in \mathcal{B}_C$. □

We now get the integration by parts formula.

Definition 6 (Integration by parts)**.** Let $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$ and let $fg = H'$ where $H(x) = F(x)g(x) - \int_{-\infty}^{x} F\,dg$. Then $fg \in \mathcal{A}_C$ and $\int_{-\infty}^{\infty} fg = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F\,dg$.

Notice that in $H(x) = F(x)g(x) - \int_{-\infty}^{x} F\,dg$ we really mean $g(x)$ and not the left or right limit of g at x, including the cases when $x = \pm\infty$. Although g has a limit at infinity, it might also have a jump discontinuity at infinity. Changing g on a countable set will in general change the value of both $F(x)g(x)$ and $\int_{-\infty}^{x} F\,dg$ but will not affect $H(x)$. To see this, it suffices to prove that if $g \in \mathcal{BV}$ and $g = 0$, except perhaps on a countable set, then $H = 0$. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Since $\lim_{-\infty} F = 0$, we can take $A < x$ such that $g(A) = 0$ and $|\int_{-\infty}^{A} F\,dg| \le \|F\chi_{(-\infty,A]}\|_\infty Vg < \epsilon/3$. Since F is continuous at x, we can take $A < B < x$ such that $g(B) = 0$ and

$$\begin{aligned}\Big|F(x)g(x) - \int_B^x F\,dg\Big| =& \Big|F(x)[g(x) - g(B)] - \int_B^x F\,dg\Big| \\ =& \Big|\int_B^x [F(x) - F(t)]dg(t)\Big| \\ \le& \max_{B \le t \le x} |F(x) - F(t)|\, Vg \le \epsilon/3.\end{aligned}$$

And, since F is uniformly continuous, there are $A = a_0 < a_1 < \cdots < a_N = B$ such that $g(a_n) = 0$ for all $0 \le n \le N$ and $\max_{a_{n-1} \le t \le a_n} |F(a_n) - F(t)| < \epsilon/[3(1 + Vg)]$. Then

$$\begin{aligned}\Big|\int_A^B F\,dg\Big| =& \Big|\sum_{n=1}^{N} \int_{a_{n-1}}^{a_n} F\,dg\Big| = \Big|\sum_{n=1}^{N} \int_{a_{n-1}}^{a_n} [F(a_n) - F(t)]\,dg(t)\Big| \\ \le& \sum_{n=1}^{N} \max_{a_{n-1} \le t \le a_n} |F(a_n) - F(t)|\, V(g\chi_{[a_{n-1},a_n]}) \le \frac{\epsilon\, Vg}{3(1 + Vg)}.\end{aligned}$$

Combining these results shows that $H(x) = 0$.

A general distribution $T \in \mathcal{D}'$ can be multiplied by a smooth function $h \in C^\infty$ using $\langle hT, \phi\rangle = \langle T, h\phi\rangle$. This works because $h\phi \in \mathcal{D}$ for all $\phi \in \mathcal{D}$. We can multiply $f \in \mathcal{A}_C$ by any function $g \in \mathcal{BV}$. Let $fg = H'$. Then

$$\begin{aligned}\langle fg, \phi\rangle =& \langle H', \phi\rangle = -\langle H, \phi'\rangle \\ =& -\int_{-\infty}^{\infty} \big[F(x)g(x) - \int_{-\infty}^{x} F(t)\,dg(t)\big]\phi'(x)\,dx.\end{aligned}$$

Since ϕ is of compact support, Fubini's theorem tells us we can interchange orders of integration to write $\langle fg, \phi\rangle = \langle (Fg)', \phi\rangle - \int_{-\infty}^{\infty} F(t)\phi(t)\,dg(t)$. This agrees with the usual definition when $g \in C^\infty$ since then for $\phi \in \mathcal{D}$ we have $g\phi \in \mathcal{BV}$ and $\langle fg, \phi\rangle = \langle f, g\phi\rangle = \langle (Fg)', \phi\rangle - \int_{-\infty}^{\infty} F(t)\phi(t)\,dg(t)$, upon integrating by parts.

The integration by parts formula agrees with the usual one when f has a Lebesgue, Henstock-Kurzweil or wide Denjoy integral. Note that we have defined $fg = H'$ but we have no way of proving this. However, we can use the norm to show this is the correct definition. Suppose $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$ with $|g| \leq M$. By Theorem 3, there is a sequence $\{f_n\} \subset L^1$ such that $\|f_n - f\| \to 0$ as $n \to \infty$. Define $H_n(x) := \int_{-\infty}^{x} f_n g = F_n(x)g(x) - \int_{-\infty}^{x} F_n\, dg$ by the usual integration by parts formula. As in (1), $|H_n(x) - H(x)| \leq \|F_n - F\|_\infty (M + Vg) \to 0$ as $n \to \infty$. It follows that $\|H_n - H\|_\infty \to 0$, which justifies our definition $fg = H'$.

Note that for $(a, b) \subset \mathbb{R}$ we have $\int_a^b fg = F(b)g(b) - F(a)g(a) - \int_a^b F\, dg$, where $F' = f$ and $F \in C^0([a, b])$. A consequence is that if $f \in \mathcal{A}_C$ then f is integrable on every subinterval of the real line. For compact interval $[a, b]$,

$$\int_a^b f = \int_{-\infty}^{\infty} f\chi_{[a,b]} = F(\infty)\chi_{[a,b]}(\infty) - \int_{-\infty}^{\infty} F\, d\chi_{[a,b]} = F(b) - F(a).$$

We also have $\int_I f = F(b) - F(a)$ when $I = [a, b]$, $[a, b)$, $(a, b]$ or (a, b). This can be seen by letting $g = \chi_I$ and integrating by parts.

The integration by parts formula shows that the distributional integral is compatible with Schwartz's definition of integral [25]. If $f \in \mathcal{D}'$ such that $f(1)$ is defined, then $\int_{-\infty}^{\infty} f := f(1)$. Since the function $1 \in \mathcal{BV}$, integration by parts gives, $f(1) = \int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f1 = F(\infty)1 - \int_{-\infty}^{\infty} f\, d1 = F(\infty)$. For another type of distributional integral, see the final paragraph of Section 11.

As a corollary to Proposition 5 we have a version of the Hölder inequality.

Theorem 7 (Hölder inequality)**.** *Let $f \in \mathcal{A}_C$. If $g \in \mathcal{NBV}$, then* $\left|\int_{-\infty}^{\infty} fg\right| \leq |\int_{-\infty}^{\infty} f| \inf_{\mathbb{R}} |g| + 2\|f\| Vg$. *If $g \in \mathcal{BV}$, then* $\left|\int_{-\infty}^{\infty} fg\right| \leq 2\|f\| \|g\|_{\mathcal{BV}}$.

The first inequality was proved in [28, Lemma 24] for the Henstock–Kurzweil integral and the same proof works here. The second inequality is similar. The factor of '2' is replaced by '1' if we use the equivalent norm on $\mathcal{A}_C$, $\|f\|' := \sup_I |\int_I f|$ where the supremum is taken over all intervals $I \subset \mathbb{R}$.

We now get a new interpretation of the action of $f \in \mathcal{A}_C$ as a distribution. Let $\phi \in \mathcal{D}$. Since $\phi \in \mathcal{BV} \subset C^1$, we have

$$\begin{aligned}\langle f, \phi\rangle =& \langle F', \phi\rangle = -\langle F, \phi'\rangle = -\int_{-\infty}^{\infty} F\phi' \\ =& -\int_{-\infty}^{\infty} F\, d\phi = \int_{-\infty}^{\infty} f\phi - F(\infty)\phi(\infty) = \int_{-\infty}^{\infty} f\phi.\end{aligned}$$

Hence, the action of f on test function ϕ is interpreted as the integral of the product $f\phi$, as in the case when f is a locally integrable function.

The Hölder inequality shows that f is a continuous linear functional on $\mathcal{BV}$. Suppose $\{g_n\} \subset \mathcal{BV}$ and $\|g_n\|_{\mathcal{BV}} \to 0$ as $n \to \infty$. Then f is continuous:

$$|\langle f, g_n\rangle| = \left|\int_{-\infty}^{\infty} f g_n\right| \leq 2\|f\|\|g_n\|_{\mathcal{BV}} \to 0.$$

And, for $a \in \mathbb{R}$; $g_1, g_2 \in \mathcal{BV}$;

$$\begin{aligned}\langle f, ag_1 + g_2\rangle =& F(\infty)\,[ag_1 + g_2](\infty) - \int_{-\infty}^{\infty} F\,d(ag_1 + g_2)\\ =& aF(\infty)g_1(\infty) + F(\infty)g_2(\infty) - a\int_{-\infty}^{\infty} F\,dg_1 - \int_{-\infty}^{\infty} F\,dg_2\\ =& a\langle f, g_1\rangle + \langle f, g_2\rangle.\end{aligned}$$

So, we know that the dual of $\mathcal{BV}$ contains $\mathcal{A}_C$, i.e., $\mathcal{A}_C \subset \mathcal{BV}^*$. In fact, $\mathcal{BV}^*$ is much larger than $\mathcal{A}_C$ since it contains measures not in $\mathcal{A}_C$ such as the Dirac measure. However, we do know that $\mathcal{A}_C^* = \mathcal{BV}$. If $\{f_n\} \subset \mathcal{A}_C$ and $\|f_n\| \to 0$, then for $g \in \mathcal{BV}$ it follows that $\left|\int_{-\infty}^{\infty} f_n g\right| \leq 2\|f_n\|\|g\|_{\mathcal{BV}} \to 0$ so $g \in \mathcal{A}_C^*$, since we also have linearity $\langle af_1 + f_2, g\rangle = a\langle f_1, g\rangle + \langle f_2, g\rangle$. The Riesz Representation Theorem says that if $[a, b]$ is a compact interval, then $C^0([a, b])^* = \mathcal{BV}$. Since our two-point compactification of the real line makes $\mathcal{B}_C$ homeomorphic to the continuous functions on $[a, b]$ vanishing at a, it also true that $\mathcal{A}_C^* = \mathcal{BV}$. Hence, the functions of bounded variation are the multipliers for the distributional integral ($g \in \mathcal{BV}$ implies $fg \in \mathcal{A}_C$ for all $f \in \mathcal{A}_C$) and $\mathcal{BV}$ also forms the dual space (the set of continuous linear functionals on $\mathcal{A}_C$).

Although it is prohibited to discuss measure and distribution $f \in \mathcal{A}_C$ in the same breath, measure-theoretic arguments apply to $g \in \mathcal{BV}$. Using a density argument, we see that changing g on a set of measure 0 does not affect the value of $\int_{-\infty}^{\infty} fg$.

Theorem 8. *Let $f \in \mathcal{A}_C$ and let $g \in \mathcal{EBV}$. Let $\{\phi_n\} \subset \mathcal{D}$ with $\|f - \phi_n\| \to 0$. Define $\int_{-\infty}^{\infty} fg = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n g$. Let $\tilde{g}$ be the unique function in $\mathcal{NBV}$ such that essvar $g = V\tilde{g}$. Then $\int_{-\infty}^{\infty} fg = \int_{-\infty}^{\infty} f\tilde{g}$.*

Proof. Note that such a sequence $\{\phi_n\}$ exists since $\mathcal{D}$ is dense in $\mathcal{A}_C$. For each $n \in \mathbb{N}$, the integral $\int_{-\infty}^{\infty} \phi_n g$ exists as a Lebesgue integral since ϕ_n is smooth with compact support and $g \in L^1_{loc}$. We can then change g on a set of measure zero to get $\int_{-\infty}^{\infty} \phi_n g = \int_{-\infty}^{\infty} \phi_n \tilde{g} \to \int_{-\infty}^{\infty} f\tilde{g}$, using a convergence theorem for Henstock–Kurzweil integrals [27, Corollary 3.3]. The definition does not depend on the choice of $\{\phi_n\}$ since if $\{\psi_n\} \subset \mathcal{D}$ with $\|f - \psi_n\| \to 0$, then

$$\Big|\int_{-\infty}^{\infty}\phi_n g-\int_{-\infty}^{\infty}\psi_n g\Big|=\Big|\int_{-\infty}^{\infty}(\phi_n-\psi_n)\tilde{g}\Big|$$
$$\leq 2\|\phi_n-\psi_n\|\|\tilde{g}\|_{\mathcal{BV}}\to 0 \text{ as } n\to\infty.$$

Hence we are justified in writing $\int_{-\infty}^{\infty} fg=\int_{-\infty}^{\infty} f\tilde{g}$ for all $f\in\mathcal{A}_C$. □

Corollary 9. $\mathcal{A}_C^*=\mathcal{EBV}$.

The Hölder inequality also shows that if $f\in\mathcal{A}_C$, then f is a distribution of order one and hence is tempered. See [11] for the definitions.

6 Change of Variables.

In order to write a change of variables formula, we need to be able to compose a distribution in $\mathcal{A}_C$ with a function. For $(\alpha,\beta)\subset\mathbb{R}$, we can define $\mathcal{D}((\alpha,\beta))$ to be the test functions with compact support in (α,β) and then $\mathcal{D}'((\alpha,\beta))$ is the corresponding space of distributions. Suppose $(\alpha,\beta),(a,b)\subset\mathbb{R}$. If we have distribution $T\in\mathcal{D}'((\alpha,\beta))$, let $G:(a,b)\to(\alpha,\beta)$ be a C^∞ bijection such that $G'(x)\neq 0$ for any $x\in(a,b)$. Then $T\circ G\in\mathcal{D}'((a,b))$ is defined by $\langle T\circ G,\phi\rangle=\langle T,\frac{\phi\circ G^{-1}}{G'\circ G^{-1}}\rangle$ for all $\phi\in\mathcal{D}((a,b))$. This definition follows from the change of variables formula for smooth functions. See [11, §7.1]. For $f\in\mathcal{A}_C$ and G as above, this then leads to the formula $\int_\alpha^\beta f=\int_a^b(f\circ G)\,G'$ when G is increasing, with a sign change if G is decreasing. However, using the properties of $\mathcal{A}_C$, we can do much better than this. We will show below that the norm validates this formula when the only condition on G is that it be continuous. First we need to define the derivative of the composition of two continuous functions.

Definition 10 (Derivative of composition of continuous functions)**.** Let $F,G\in C^0(\overline{\mathbb{R}})$. Then $(F'\circ G)G':=(F\circ G)'$, i.e., $\langle(F'\circ G)G',\phi\rangle=\langle(F\circ G)',\phi\rangle=-\langle F\circ G,\phi'\rangle=-\int_{-\infty}^{\infty}(F\circ G)(t)\,\phi'(t)\,dt$ for all $\phi\in\mathcal{D}$.

The Alexiewicz norm shows this definition is compatible with the usual definition for smooth functions. Suppose $F,G\in C^0(\overline{\mathbb{R}})$. Let $\epsilon>0$. Take $\delta>0$ such that whenever $|x-y|<\delta$ we have $|F(x)-F(y)|<\epsilon/2$. This is possible since F is uniformly continuous on $\overline{\mathbb{R}}$. There are C^1 functions p and q such that $\|F-p\|_\infty<\epsilon/2$ and $\|G-q\|_\infty<\delta$. Note that $F\circ G\in C^0(\overline{\mathbb{R}})$ so $(F\circ G)'\in\mathcal{A}_C$. And, $(p\circ q)'(t)=(p'\circ q)(t)\,q'(t)$ for all $t\in\mathbb{R}$. We have

$$\begin{aligned}\|(F\circ G)'-(p'\circ q)q'\|&=\|F\circ G-p\circ q\|_\infty\\&\leq\|F\circ G-F\circ q\|_\infty+\|(F-p)\circ q\|_\infty\\&<\epsilon/2+\epsilon/2.\end{aligned}$$

With this definition we then have the following change of variables formula.

Theorem 11. *Suppose $f \in \mathcal{A}_C$ and $F' = f$ where $F \in C^0(\overline{\mathbb{R}})$. Let $-\infty \leq a < b \leq \infty$. If $G \in C^0([a,b])$, then*

$$\int_{G(a)}^{G(b)} f = \int_a^b (f \circ G)\, G' = (F \circ G)(b) - (F \circ G)(a).$$

If $G \in C^0((a,b))$ and $\lim_{t \to a^+} G(t) = -\infty$ and $\lim_{t \to b^-} G(t) = \infty$, then

$$\int_{-\infty}^{\infty} f = \int_a^b (f \circ G)\, G' = F(\infty) - F(-\infty).$$

The first statement follows from Definition 10 and the second from Theorem 25 below. This is a remarkable formula because it demands so little of f and G. For Lebesgue integrals, the usual formula requires $f \in L^1$ and $G \in AC$ and monotonic [16, §38.4]. Even invoking Stieltjes integrals leads to a change of variables formula requiring monotonicity or differentiability properties of G. See [8, Exercises III.13 4. and 5.]. Similarly for the Denjoy integral. See [15, §2.7, §7.9]. J. Foran [10] cites references to further theorems in Denjoy integration. See [5] and [24] for good change of variables theorems for Riemann integrals.

7 Convergence Theorems.

Two of the main reasons the Lebesgue integral so easily replaced the Riemann integral in the first part of the twentieth century were that the space L^1 is a Banach space and there are excellent convergence theorems. We have already shown that $\mathcal{A}_C$ is a Banach space. Now we will look at convergence theorems.

A sequence $\{f_n\} \in \mathcal{A}_C$ is said to converge *strongly* to $f \in \mathcal{A}_C$ if $\|f_n - f\| \to 0$. It converges *weakly* in $\mathcal{D}$ if $\langle f_n - f, \phi \rangle = \int_{-\infty}^{\infty} (f_n - f)\phi \to 0$ for each $\phi \in \mathcal{D}$. And, $\{f_n\}$ converges *weakly* in $\mathcal{BV}$ if $\int_{-\infty}^{\infty} (f_n - f)g \to 0$ for each $g \in \mathcal{BV}$.

Theorem 12. *Weak convergence in $\mathcal{BV}$ implies weak convergence in $\mathcal{D}$. Strong convergence implies weak convergence in $\mathcal{D}$ and $\mathcal{BV}$. Weak convergence in $\mathcal{D}$ does not imply weak convergence in $\mathcal{BV}$. Weak convergence in $\mathcal{BV}$ does not imply strong convergence.*

PROOF. Since $\mathcal{D} \subset \mathcal{BV}$, weak convergence in $\mathcal{BV}$ implies weak convergence in $\mathcal{D}$. Suppose $\|f_n - f\| \to 0$. Then $\|F_n - F\|_\infty \to 0$. Let $g \in \mathcal{BV}$. By the Hölder inequality,

$$\left| \langle f_n - f, g \rangle \right| = \left| \int_{-\infty}^{\infty} (f_n - f)\, g \right| \leq 2\|F_n - F\|_\infty \|g\|_{\mathcal{BV}} \to 0.$$

To see that weak convergence in $\mathcal{D}$ does not imply weak convergence in $\mathcal{BV}$, let $f_n = \chi_{(n,n+1)}$. For $\phi \in \mathcal{D}$ we have $\int_{-\infty}^{\infty} f_n\phi = \int_n^{n+1} \phi \to 0$ but if $g = 1$, then $\int_{-\infty}^{\infty} f_n g = 1 \not\to 0$. Let $f_n = \chi_{(n-1,n)} - \chi_{(n,n+1)}$. Then $\{f_n\} \subset \mathcal{A}_C$. For $g \in \mathcal{BV}$, we have $\int_{-\infty}^{\infty} f_n g \to 0$ by dominated convergence since $\|f_n\|_\infty = 1$, g is bounded and $f_n \to 0$ pointwise on $\mathbb{R}$. As $\|f_n\| = 1$, weak convergence in $\mathcal{BV}$ (and hence in $\mathcal{D}$) does not imply strong convergence. □

Suppose $\{f_n\} \subset \mathcal{A}_C$. Strong convergence $\|f_n - f\| \to 0$ implies $f \in \mathcal{A}_C$ since $\mathcal{A}_C$ is a Banach space. If $f_n \to f$ weakly in $\mathcal{BV}$ then by definition $f \in \mathcal{A}_C$. But, if $f_n \to f$ weakly in $\mathcal{D}$, then f need not be in $\mathcal{A}_C$.

Example 13. There is a sequence $\{f_n\} \subset \mathcal{A}_C$ that converges weakly in $\mathcal{D}$ to $f \in \mathcal{D}' \setminus \mathcal{A}_C$. Let $f_n = \chi_{[-n,n]}$. Then $f_n \in \mathcal{A}_C$ for each $n \in \mathbb{N}$. Let $\phi \in \mathcal{D}$. By dominated convergence (or Weierstrass M-test), $\langle f_n, \phi\rangle = \int_{\mathrm{supp}(\phi)} \chi_{[-n,n]}\phi \to \int_{-\infty}^{\infty} \phi = \langle 1, \phi\rangle$. Hence, f_n converges weakly in $\mathcal{D}$ to $1 \in \mathcal{D}' \setminus \mathcal{A}_C$.

Now suppose we are interested in conditions on f_n so that $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$.

Theorem 14. *Let $\{f_n\} \subset \mathcal{A}_C$ and $f \in \mathcal{A}_C$. If $\|f_n - f\| \to 0$, then $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$. The converse is false. If $f_n \to f$ weakly in $\mathcal{BV}$, then $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$. There is a sequence $\{f_n\} \subset \mathcal{A}_C$ and a distribution $f \in \mathcal{A}_C$ such that $f_n \to f$ weakly in $\mathcal{D}$ and $\int_{-\infty}^{\infty} f_n \not\to \int_{-\infty}^{\infty} f$. There is a sequence $\{f_n\} \subset \mathcal{A}_C$ that does not converge weakly in $\mathcal{D}$ but $\{\int_{-\infty}^{\infty} f_n\}$ converges in $\mathbb{R}$.*

Proof. Certainly we have $|\int_{-\infty}^{\infty} f_n - f| \le \|F_n - F\|_\infty = \|f_n - f\|$ so $\|f_n - f\| \to 0$ and the triangle inequality imply $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$. Let $f_n(t) = n^2 \sin(nt)$ for $|t| \le \pi$ and $f_n(t) = 0$ for $|t| > \pi$. Then for each $n \in \mathbb{N}$, $\int_{-\infty}^{\infty} f_n = 0$ but $\|f_n\| = n^2 \int_0^{\pi/n} \sin(nt)\,dt = 2n \to \infty$. Now suppose $f_n \to f$ weakly in $\mathcal{BV}$. Since $1 \in \mathcal{BV}$ we have $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$. And, define

$$F_n(t) = \begin{cases} 0, & t \le n \\ t-n, & n \le t \le n+1 \\ 1, & t \ge n+1. \end{cases}$$

Then $F_n \in \mathcal{B}_C$ and $f_n(t) := F_n'(t) = 1$ for $n \le t \le n+1$ and $f_n(t) = 0$, otherwise. For $\phi \in \mathcal{D}$, $\langle f_n, \phi\rangle = \int_n^{n+1} \phi \to 0$ since ϕ has compact support. But, $\int_{-\infty}^{\infty} f_n = F_n(\infty) = 1$. This phenomenon can also occur on compact intervals. Let $F_n(t) = t^n$ for $t \in [0,1]$. Then $\int_0^1 f_n = F_n(1) = 1$ and yet, for $\phi \in \mathcal{D}((0,1))$, $|\langle f_n, \phi\rangle| = |-\int_0^1 t^n \phi'(t)\,dt| \le \|\phi'\|_\infty \int_0^1 t^n\,dt = \frac{\|\phi'\|_\infty}{n+1} \to 0$.

Finally, let $f_n(t) = a_n$ for $1 \le t \le 2$, $f_n(t) = -a_n$ for $-2 \le t \le -1$, and $f_n(t) = 0$, otherwise. Here, $\{a_n\}$ is an arbitrary sequence of real numbers. Then, $\int_{-\infty}^{\infty} f_n = 0$ for each $n \in \mathbb{N}$ but, unless $\lim_{n\to\infty} a_n = 0$, $\{f_n\}$ is not weakly convergent in $\mathcal{D}$ since we can always take a test function that has support in $[0, 3]$ that is identically 1 on $[1, 2]$. $\square$

Theorem 14 indicates that to have $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$ we should look for some condition between weak convergence in $\mathcal{BV}$, which is sufficient but not necessary, and weak convergence in $\mathcal{D}$, which is neither necessary nor sufficient. Note that for $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$ we will really want $F_n(x) \to F(x)$ for each $x \in \overline{\mathbb{R}}$. Indeed, a corollary to Theorem 14 is that strong convergence or weak convergence in $\mathcal{BV}$ of $f_n \to f$ both imply $\int_{-\infty}^{x} f_n \to \int_{-\infty}^{x} f$ for all $x \in \overline{\mathbb{R}}$. If we do not have convergence on subintervals, then each f_n could be an arbitrary distribution in $\mathcal{A}_C$ with integral 0 and we would then not expect there to be any sensible condition on f_n that ensures $\int_{-\infty}^{\infty} f_n \to 0$.

Note that strong convergence $\|f_n - f\| \to 0$ is the same as uniform convergence of $F_n \to F$ on $\overline{\mathbb{R}}$. If each function $F_n \in \mathcal{B}_C$, then uniform convergence of $F_n \to F$ guarantees F is continuous on $\overline{\mathbb{R}}$. Since each $F_n(-\infty) = 0$, we also have $F(-\infty) = 0$ so $F \in \mathcal{B}_C$ and $\int_{-\infty}^{x} f_n \to \int_{-\infty}^{x} F'$ for each $x \in \overline{\mathbb{R}}$. But, uniform convergence is not necessary for the limit of a sequence of continuous functions to be continuous. The necessary and sufficient condition is *quasi-uniform* convergence. See [13] or [8, IV.6.10].

Definition 15 (Quasi-uniform convergence)**.** Let $\{F_n\} \subset C^0(\overline{\mathbb{R}})$ and suppose $F : \overline{\mathbb{R}} \to \mathbb{R}$. If $F_n(x) \to F(x)$ at each point $x \in \overline{\mathbb{R}}$, then $F_n \to F$ quasi-uniformly at $x \in \mathbb{R}$ if for each $\epsilon > 0$ and each $N \in \mathbb{N}$ there is $\delta > 0$ and $n \ge N$ such that whenever $|x - y| < \delta$ we have $|F_n(y) - F(y)| < \epsilon$. For quasi-uniform convergence at $x = \infty$, replace the condition involving δ with $y > 1/\delta$, with a similar condition for $x = -\infty$.

Theorem 16. *Let $\{f_n\} \subset \mathcal{A}_C$ and $F : \overline{\mathbb{R}} \to \mathbb{R}$. If $F_n \to F$ quasi-uniformly on $\overline{\mathbb{R}}$, then $F \in \mathcal{B}_C$ and $\int_{-\infty}^{x} f_n \to \int_{-\infty}^{x} F'$ for each $x \in \overline{\mathbb{R}}$.*

The following three results give sufficient conditions for $\int_{-\infty}^{x} f_n$ to converge to $\int_{-\infty}^{x} f$. Each involves weak convergence of $f_n \to f$ in $\mathcal{D}$.

Theorem 17 ([3], Theorem 8)**.** *Let $\{f_n\} \subset \mathcal{A}_C$ and $F \in C^0(\overline{\mathbb{R}})$. Suppose $\{F_n\}$ is uniformly bounded on each compact interval in $\mathbb{R}$ and $F_n \to F$ on $\overline{\mathbb{R}}$. Then $f_n \to F'$ weakly in $\mathcal{D}$ and $\int_{-\infty}^{x} f_n \to \int_{-\infty}^{x} F'$ for each $x \in \mathbb{R}$.*

PROOF. Since $F(-\infty) = \lim_{n\to\infty} F_n(-\infty) = \lim_{n\to\infty} 0 = 0$ we have $F \in \mathcal{B}_C$. Let $\phi \in \mathcal{D}$ with support in the compact interval $I \subset \mathbb{R}$. Then $|\langle F_n, \phi\rangle| = |\int_I F_n \phi| \le \|F_n \phi \chi_I\|_\infty \lambda(I)$. By dominated convergence (or the Weierstrass

M-test), $\int_{-\infty}^{\infty} F_n\phi \to \int_{-\infty}^{\infty} F\phi$; i.e., $F_n \to F$ weakly in $\mathcal{D}$. And, since $\phi' \in \mathcal{D}$, $\langle f_n, \phi\rangle = -\langle F_n, \phi'\rangle \to -\langle F, \phi'\rangle = \langle F', \phi\rangle$. Therefore, $f_n \to F'$ weakly in $\mathcal{D}$. And, $\int_{-\infty}^{x} f_n = F_n(x) \to F(x) = \int_{-\infty}^{x} F'$ for each $x \in \overline{\mathbb{R}}$. □

Corollary 18 ([3], Theorem 9)**.** *Let $\{f_n\} \subset \mathcal{A}_C$ and $F \in C^0(\overline{\mathbb{R}})$. Suppose $\{F_n\}$ is uniformly bounded on each compact interval in $\mathbb{R}$ and $F_n \to F$ on $\overline{\mathbb{R}}$. Suppose $f_n \to f$ weakly in $\mathcal{D}$ for some $f \in \mathcal{D}'$. Then $f = F' \in \mathcal{A}_C$ and $\int_{-\infty}^{x} f_n \to \int_{-\infty}^{x} f$ for each $x \in \overline{\mathbb{R}}$.*

Proof. As in the theorem, $F_n \to F$ weakly in $\mathcal{D}$. Therefore, for $\phi \in \mathcal{D}$, $\langle f_n, \phi\rangle = -\langle F_n, \phi'\rangle \to -\langle F, \phi'\rangle$. By the uniqueness of limits in $\mathcal{D}$, $f = F' \in \mathcal{A}_C$. □

A sequence of functions $\{F_n\} \subset \mathcal{B}_C$ is *equicontinuous* at $x \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 1$, if $y \in \mathbb{R}$ such that $|x - y| < \delta$, then $|F_n(x) - F_n(y)| < \epsilon$. We can define equicontinuity at ∞ by replacing the condition involving δ with $y > 1/\delta$. Similarly at $-\infty$. The point is that one δ works for all $n \in \mathbb{N}$. If $\{F_n\}$ is equicontinuous at each point of $\overline{\mathbb{R}}$ we say this sequence is equicontinuous on $\overline{\mathbb{R}}$.

Corollary 19 ([3], Corollary 3)**.** *Let $\{f_n\} \subset \mathcal{A}_C$ such that $f_n \to f$ weakly in $\mathcal{D}$ for some $f \in \mathcal{D}'$. Suppose $\{F_n\}$ is equicontinuous on $\overline{\mathbb{R}}$. Then $f \in \mathcal{A}_C$ and $\|f_n - f\| \to 0$.*

The proof depends on the Arzelà–Ascoli theorem. See [3].

Example 20. Let $\{a_n\}$ be a sequence of positive real numbers that increases to infinity. Define f_n as the step function

$$f_n(t) = \begin{cases} 0, & t \leq n-1 \\ a_n, & n-1 < t \leq n \\ -a_n, & n < t \leq n+1 \\ 0, & t > n+1. \end{cases}$$

Then $f_n \in \mathcal{A}_C$ for each $n \in \mathbb{N}$ and F_n is the piecewise linear function

$$F_n(x) = \begin{cases} 0, & x \leq n-1 \\ a_n(x-n+1), & n-1 \leq x \leq n \\ a_n(n+1-x), & n \leq x \leq n+1 \\ 0, & x \geq n+1. \end{cases}$$

It follows that $\|F_n\|_\infty = a_n$. Note that $F_n \to 0$ on $\overline{\mathbb{R}}$ and that the convergence is quasi-uniform but not uniform. To see that it is not uniform, notice that

$F_n(n) = a_n \to \infty$. By Theorem 16, $\int_{-\infty}^{\infty} f_n \to 0$. Note that $\{F_n\}$ is uniformly bounded on compact intervals: $\|F_n\chi_{[a,b]}\|_\infty \leq a_m$ where m is the largest integer such that $a-1 \leq m \leq b+1$. Hence, f_n converges weakly to 0 in $\mathcal{D}$. Theorem 17 and Corollary 18 allow us to conclude that $\int_{-\infty}^{\infty} f_n \to 0$. Also, $\{F_n\}$ is equicontinuous on $\mathbb{R}$ but not at ∞, since if $\delta > 0$, then for integer $n > 1/\delta$ we have $F_n(n) = a_n$ and this can be made arbitrarily large by taking n large enough. Hence, Corollary 19 is not applicable.

Although $f_n \to 0$ weakly in $\mathcal{D}$, $\{f_n\}$ does not converge weakly in $\mathcal{BV}$. Define $g = \sum_n b_n \chi_{[2n-1,2n]}$ where $\{b_n\}$ is a sequence of positive real numbers. Then $Vg = 2\sum_n b_n$. We have $\langle f_{2n}, g\rangle = \int_{2n-1}^{2n+1} f_{2n} g = a_{2n} b_n$. If $a_n = n^3$ and $b_n = 1/n^2$, then $g \in \mathcal{BV}$ but $\langle f_{2n}, g\rangle = 8n \to \infty$.

Each function f_n is Riemann integrable and $f_n \to 0$ pointwise on $\mathbb{R}$ but the sequence of integrals $\int_{-\infty}^{\infty} f_n$ does not converge uniformly so the usual convergence theorems for Riemann integration do not apply.

Convergence theorems for Lebesgue integration also do not apply, even though each function $f_n \in L^1$. There is no L^1 function that dominates $|f_n|$ for all $n \in \mathbb{N}$ so the dominated convergence theorem is not applicable. The Vitali convergence theorem [8] gives necessary and sufficient conditions for taking limits under Lebesgue integrals but is also not applicable here since $\int_{-\infty}^{\infty} |f_n| = 2a_n \to \infty$, even though $\int_{-\infty}^{\infty} f_n = 0$ for each $n \in \mathbb{N}$.

Example 21. Let $\{a_n\}$ be a sequence of positive real numbers such that a_n/n increases to infinity. Define f_n as the step function

$$f_n(t) = \begin{cases} 0, & t \leq 0 \\ a_n, & 0 < t \leq 1/n \\ -a_n, & 1/n < t \leq 2/n \\ 0, & t > 2/n. \end{cases}$$

Then $f_n \in \mathcal{A}_C$ for each $n \in \mathbb{N}$ and F_n is the piecewise linear function

$$F_n(x) = \begin{cases} 0, & x \leq 0 \\ a_n x, & 0 \leq x \leq 1/n \\ a_n(\frac{2}{n} - x), & 1/n \leq x \leq 2/n \\ 0, & x \geq 2/n. \end{cases}$$

It follows that $\|F_n\|_\infty = a_n/n$. Note that $F_n \to 0$ on $\overline{\mathbb{R}}$ and that the convergence is quasi-uniform but not uniform, since $F_n(1/n) = a_n/n \to \infty$. By Theorem 16, $\int_{-\infty}^{\infty} f_n \to 0$. Note that $\{F_n\}$ is not uniformly bounded on $[0,1]$. Theorem 17 and Corollary 18 are not applicable. Also, F_n is not equicontinuous on $[0,1]$ so Corollary 19 is not applicable. As with Example 20, convergence theorems for Riemann and Lebesgue integration are not useful here.

With Lebesgue integration, the dominated convergence theorem is particularly useful because it is often easy to find an integrable function that dominates each function in a sequence of functions. There is a notion of ordering in $\mathcal{A}_C$ that permits monotone and dominated convergence theorems. If f and g are in $\mathcal{A}_C$ then $f \geq g$ if $\langle f, \phi\rangle \geq \langle g, \phi\rangle$ for all $\phi \in \mathcal{D}$ such that $\phi \geq 0$. Then $f \geq g$ if and only if $f - g \geq 0$. It is known that if $f \in \mathcal{D}'$ and $f \geq 0$, then f is a Radon measure; i.e., a Borel measure that is inner and outer regular, and is finite on compact sets. See [3] for convergence theorems based on this ordering. A different ordering, more compatible with the Alexiewicz norm, is described in Section 9 below.

Instead of dominated convergence we have the following convergence theorem. We will see in the next section that it is quite useful.

Theorem 22. *Let $f \in \mathcal{A}_C$. Suppose $\{g_n\} \subset \mathcal{BV}$ such that there is $M \in \mathbb{R}$ so that for all $n \in \mathbb{N}$, $Vg_n \leq M$. If $g_n \to g$ on $\overline{\mathbb{R}}$ for a function $g \in \mathcal{BV}$ then $\lim_{n\to\infty} \int_{-\infty}^{\infty} fg_n = \int_{-\infty}^{\infty} fg$.*

The theorem is based on Helly's theorem for Riemann-Stieltjes integrals. See [27] for a proof. This paper also contains convergence theorems for products $f_n g_n$ when f_n is Henstock–Kurzweil integrable. The proofs carry over to $\mathcal{A}_C$ with no change.

8 The Poisson Integral and Laplace Transform.

A common use of integrals is the integration of functions from a certain class against a fixed kernel. We will look at two typical cases, the Poisson integral and Laplace transform.

The upper half plane Poisson integral is given by the convolution $u(x, y) = K(x - \cdot, y) * f = \int_{-\infty}^{\infty} f(t)K(x - t, y)\, dt$, where the Poisson kernel is $K(x, y) = y/[\pi(x^2 + y^2)]$. It is known that if $f \in L^p$ $(1 \leq p \leq \infty)$, then u is harmonic in the upper half plane. This is also true in $\mathcal{A}_C$. Fix $x \in \mathbb{R}$ and $y > 0$. Let $f \in \mathcal{A}_C$. The kernel $t \mapsto K(x - t, y)$ is of bounded variation on $\overline{\mathbb{R}}$. Therefore, the product $f(\cdot)K(x - \cdot, y)$ is in $\mathcal{A}_C$ and u exists on the upper half plane. To show that we can differentiate under the integral sign, let h be a nonzero real number and consider

$$t \mapsto \frac{K(x + h - t, y) - K(x - t, y)}{h} = \frac{-y(2x - 2t + h)}{\pi\,[(x + h - t)^2 + y^2]\,[(x - t)^2 + y^2]}.$$

This function is of bounded variation on $\overline{\mathbb{R}}$, uniformly for $h \neq 0$. Hence, using Theorem 22, we can differentiate under the integral sign to get $u_1(x, y) = -\frac{2y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)(x-t)\,dt}{[(x-t)^2+y^2]^2}$. Similarly, $u_2(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)[(x-t)^2-y^2]dt}{[(x-t)^2+y^2]^2}$. And, using these two new kernels and Theorem 22, we see that $\Delta u(x,y)=\int_{-\infty}^{\infty} f(t)\Delta K(x-t,y)\,dt=0$ and u is harmonic in the upper half plane.

Using our change of variables Theorem 11 with $G(t)=x-t$, $a=-\infty$ and $b=\infty$, we can show that $u(x,y)=f(x-\cdot)*K(\cdot,y)=\int_{-\infty}^{\infty} f(x-t)K(t,y)\,dt$. It is also possible to show that boundary conditions are taken on in the Alexiewicz norm, i.e., $\|u(\cdot,y)-f(\cdot)\|\to 0$ as $y\to 0^+$.

Let $\mathbb{R}^+=(0,\infty)$ and $f\in\mathcal{A}_C(\mathbb{R}^+)$. We will say that the variation of a complex-valued function is the sum of the variations of the real and imaginary parts. Let $x,y\in\mathbb{R}$ and write $z=x+iy$. The function $t\mapsto e^{-zt}$ is of bounded variation on $[0,\infty]$ if $x>0$ or if $z=0$. Hence, the Laplace transform of f is $\hat{f}(z)=\int_0^\infty f(t)\,e^{-zt}\,dt$ and exists for $x>0$ or $z=0$. We can now prove some basic properties of the Laplace transform. First we will prove $\widehat{f}$ is differentiable. Fix $x>0$ and take $h\in\mathbb{C}$ such that $0<|h|<x/2$. For fixed $z=x+iy$ with $x>0$ write $g_h(t)=[\exp(-(z+h)t)-\exp(-zt)]/h$. Then

$$|g_h'(t)|=e^{-xt}\left|\frac{(z+h)e^{-ht}-z}{h}\right|=e^{-xt}\left|\left(e^{-ht}-1\right)\frac{z}{h}+e^{-ht}\right|.$$

By Cauchy's theorem,

$$e^{-ht}=1+\frac{h}{2\pi i}\int_C\frac{e^{-st}\,ds}{s(s-h)}$$

where C is the circle with center 0 and radius $x/2$ in the complex plane. Then $|g_h'(t)|\leq(2|z|/[x-2|h|]+1)e^{-xt/2}$ and $Vg_h\leq(2|z|/[x-2|h|]+1)(2/x)$ so that g_h is of bounded variation on $[0,\infty]$, uniformly as $h\to 0$. By Theorem 22, $d\widehat{f}(z)/dz=-\int_0^\infty f(t)\,te^{-zt}\,dt$. Similarly, we can differentiate under the integral sign to get $d^n\widehat{f}(z)/dz^n=(-1)^n\int_0^\infty f(t)\,t^ne^{-zt}\,dt$ for all $n\in\mathbb{N}$.

One difference between Laplace transforms in $\mathcal{A}_C(\mathbb{R}^+)$ and Laplace transforms of distributions is that we get a different growth condition as $z\to\infty$. Write $z=x+iy$ with $x>0, y\in\mathbb{R}$. Let $\delta>0$. Integrate by parts to get $\widehat{f}(z)=z\int_0^\delta F(t)e^{-zt}\,dt+z\int_\delta^\infty F(t)e^{-zt}\,dt$. Then $|\widehat{f}(z)|\leq(|z|/x)\max_{[0,\delta]}|F|+(|z|/x)\|F\|_\infty e^{-x\delta}$. Given $\epsilon>0$, take δ small enough so that $\max_{[0,\delta]}|F|<\epsilon$. Let $0\leq\alpha<\pi/2$. We then have $\widehat{f}(z)=o(1)$ as $z\to\infty$ in the cone $|\arg(z)|\leq\alpha$. We can show this estimate is sharp by showing it is sharp as $z=x$ goes to infinity on the positive real axis. Suppose $A\colon(0,\infty)\to(0,1)$ with $\lim_\infty A=0$. First show A has a suitably smooth majorant. Define $B(s)=\sup_{0<t\leq s} eA(1/t)$. Then $B(s)\geq eA(1/s)$ for all $s>0$, B is increasing and $\lim_{s\to 0^+} B(s)=0$. Now define

$$F(s)=\begin{cases}\left[B\left(\frac{1}{n}\right)-B\left(\frac{1}{n+1}\right)\right](n+1)(n+2)s & \\ \quad -(n+1)B\left(\frac{1}{n}\right)+(n+2)B\left(\frac{1}{n+1}\right), & \frac{1}{n+2}\leq s\leq\frac{1}{n+1} \text{ for some } n\in\mathbb{N}\\ 0, & s=0\\ B(1), & s\geq 1/2.\end{cases}$$

Then $F\in\mathcal{B}_C(\mathbb{R}^+)$, $F(s)\geq eA(1/s)$ for all $s\in(0,1/2]$. Since F is increasing and piecewise linear, $F\in AC(\mathbb{R}^+)\cap C^0([0,\infty])$. Let $f=F'$ and let $s\in(0,1/2]$. Then $\widehat{f}(x)\geq\int_0^s f(t)e^{-xt}\,dt\geq F(s)e^{-xs}$. Now suppose $x\geq 2$. Let $s=1/x$. Then $\widehat{f}(x)\geq F(1/x)e^{-1}\geq A(x)$. Hence, the estimate $\widehat{f}(z)=o(1)$ ($z\to\infty$, $|\arg(z)|\leq\alpha$) is sharp, not only in $\mathcal{A}_C$ but in L^1 as well. Note that for the Dirac distribution, $\widehat{\delta}(z)=\exp(0)=1$ so the estimate does not hold for measures or distributions that are the second derivative of a continuous function. For distributions in general, the Laplace transform can have polynomial growth. See [32, p. 236, 237].

Since the kernel decays exponentially, we can define a Laplace transform under weaker conditions. Define the locally integrable distributions on $[0,\infty)$ by $\mathcal{A}_C(loc)=\{f\in\mathcal{D}'(\mathbb{R}^+)\mid f=F' \text{ for some } F\in C^0([0,\infty))\}$. In this case, $f=F'$ means that for all $\phi\in\mathcal{D}(\mathbb{R}^+)$ we have $\langle f,\phi\rangle=-\langle F,\phi'\rangle$. For $f\in\mathcal{A}_C(loc)$ there is a continuous function F such that $\int_0^x f=F(x)-F(0)$ for all $x\in[0,\infty)$. Note that $\lim_\infty F$ need not exist. Let $r\in\mathbb{R}$. Define $F_r(x)=\int_0^x f(t)e^{-rt}\,dt=F(x)e^{-rx}-F(0)+r\int_0^x F(t)e^{-rt}\,dt$. Note that $F_r(0)=0$ and $F_r\in C^0([0,\infty))$. Now we can define the weighted space $\mathcal{A}_C[e^{r\cdot}]=\{f\in\mathcal{A}_C(loc)\mid f=F' \text{ for some } F\in C^0([0,\infty)) \text{ such that } \lim_\infty F_r \text{ exists in } \mathbb{R}\}$. For example, if F is a continuous function such that $F(x)e^{rx}/x^2$ is bounded as $x\to\infty$, then $F\in\mathcal{A}_C[e^{r\cdot}]$. We then have $\int_0^\infty f(t)e^{-rt}\,dt=\lim_{x\to\infty}F_r(x)$. The limit is independent of which primitive $F\in C^0([0,\infty))$ is used. If $f\in\mathcal{A}_C[e^{r\cdot}]$, then $\hat{f}(z)$ exists for all $z\in\mathbb{C}$ such that $\mathcal{R}e(z)>r$ or $\mathcal{R}e(z)\geq r,\mathcal{I}m(z)=0$. If f is in one of these exponentially weighted spaces there are similar differentiation and growth results as to when $f\in\mathcal{A}_C(\mathbb{R}^+)$. Using an analogous technique, we can define weighted integrals $\int_{-\infty}^\infty fg$ for functions g that are of locally bounded variation.

9 Banach Lattice.

In $\mathcal{B}_C$ there is the pointwise order: for $F,G\in\mathcal{B}_C$, $F\leq G$ if and only if $F(x)\leq G(x)$ for all $x\in\overline{\mathbb{R}}$. It is easy to see that this relation is *reflexive* ($F\leq F$), *antisymmetric* ($F\leq G$ and $G\leq F$ imply $F=G$), and *transitive* ($F\leq G$ and $G\leq H$ imply $F\leq H$). This puts a *partial order* on $\mathcal{B}_C$.

As $\mathcal{A}_C$ is isomorphic to $\mathcal{B}_C$, it inherits this partial order. For $f, g \in \mathcal{A}_C$, we define $f \leq g$ if and only if $F \leq G$. For example, let $f(t) = \sin(t)/t$ for $t > 0$ and $f(t) = 0$ for $t < 0$. Then $f \in \mathcal{A}_C$. We have $F(x) = \int_0^x f$ for $x \geq 0$ and $F(x) = 0$ for $x \leq 0$. This is the *sine integral*, $\mathrm{Si}(x)$, and it is easy to show $F(x) \geq 0$ for all $x \in \mathbb{R}$. Hence, $f \geq 0$ in $\mathcal{A}_C$. This ordering on $\mathcal{A}_C$ is then not compatible with the usual pointwise ordering that we can use in L^1, i.e., $f \geq g$ if and only if $f(t) \geq g(t)$ for almost all $t \in \mathbb{R}$. The function defined by $\max(f(t), 0)$ is not in $\mathcal{A}_C$. Nor is our ordering compatible with the usual one for distributions: if $T \in \mathcal{D}'$, then $T \geq 0$ if and only if T is a Radon measure. The function $f(t) = \sin(t)/t$ is not positive in the distributional sense. It is not even the difference of two positive, Lebesgue integrable functions so it is not a signed measure. In $\mathcal{A}_C$, the relation $f \geq 0$ means that for each $x \in \mathbb{R}$, the integral over $(-\infty, x]$ is not negative, i.e., to the left of x there is more positive stuff than negative stuff. It is a not a *linear ordering*. For example, $f(t) = -2t\exp(-t^2)$ and $g(t) = -2(t-1)\exp(-(t-1)^2)$ are not comparable.

Now, $\mathcal{B}_C$ is closed under the operations $(F \vee G)(x) = \sup(F(x), G(x)) = \max(F(x), G(x))$ and $(F \wedge G)(x) = \inf(F(x), G(x)) = \min(F(x), G(x))$. It is then a *lattice*. And, $\mathcal{B}_C$ is also a *Banach lattice*. This means that the order is compatible with the vector space operations and norm. For all $F, G \in \mathcal{B}_C$,

(i) $F \leq G$ implies $F + H \leq G + H$ for all $H \in \mathcal{B}_C$
(ii) if $F \leq G$, then $aF \leq aG$ for all real numbers $a \geq 0$
(iii) $|F| \leq |G|$ implies $\|F\|_\infty \leq \|G\|_\infty$.

A good introduction to lattices can be found in [2].

As usual, in $\mathcal{B}_C$ we define $F^+ = F \vee 0$, $F^- = F \wedge 0$ and $|F| = F \vee (-F)$. The *Jordan decomposition* is $F = F^+ - F^-$. It is also true that $|F| = F^+ + F^-$. In $\mathcal{A}_C$, $f^+ = (F^+)'$, $f^- = (F^-)'$ and $|f| = |F|'$. These definitions make sense since $F \in \mathcal{B}_C$ so F^+, F^- and $|F|$ are all in $\mathcal{B}_C$ and then their derivatives are in $\mathcal{A}_C$. For the function $f(t) = \sin(t)/t$ when $t > 0$ and $f(t) = 0$, otherwise, we have $f^+ = |f| = f$ and $f^- = 0$.

Theorem 23. *$\mathcal{A}_C$ is a Banach lattice.*

PROOF. First we need to show that $\mathcal{A}_C$ is closed under the operations $f \vee g$ and $f \wedge g$. For $f, g \in \mathcal{A}_C$, we have $f \vee g = \sup(f, g)$. This is h such that $h \geq f$, $h \geq g$, and if $h_1 \geq f$, $h_1 \geq g$, then $h_1 \geq h$. This last statement is equivalent to $H \geq F$, $H \geq G$, and if $H_1 \geq F$, $H_1 \geq G$, then $H_1 \geq H$. But then $H = \max(F, G)$ and $h = H'$ so $f \vee g = (F \vee G)' \in \mathcal{A}_C$. Similarly, $f \wedge g = (F \wedge G)' \in \mathcal{A}_C$.

If $f, g \in \mathcal{A}_C$ and $f \leq g$, then $F \leq G$. Let $h \in \mathcal{A}_C$. Then, $F + H \leq G + H$. But then $(F + H)' = F' + H' = f + h \leq g + h$. If $a \in \mathbb{R}$ and $a \geq 0$ then $(aF)' = aF' = af$ so $af \leq ag$. And, if $|f| \leq |g|$, then $|F|' \leq |G|'$ so $|F| \leq |G|$, i.e., $F(x) \leq G(x)$ for all $x \in \overline{\mathbb{R}}$. Then $\|f\| = \|F\|_\infty \leq \|G\|_\infty = \|g\|$. And, $\mathcal{A}_C$

is a Banach lattice that is isomorphic to $\mathcal{B}_C$. $\square$

Linearity of the derivative was necessary to prove conditions (i) and (ii), whereas, for (iii) we needed the fact that $\mathcal{B}_C$ and $\mathcal{A}_C$ are isometric. It is a fact that every Banach lattice is isomorphic to the vector space of continuous functions on some compact Hausdorff space. See, for example, [8, pp. 395].

The following results follow immediately from the definitions.

Theorem 24. *Let $f, g \in \mathcal{A}_C$.* (a) *If $f \le g$, then $F(x) \le G(x)$ for all $x \in \overline{\mathbb{R}}$.* (b) *If $\int_{-\infty}^x f \le \int_{-\infty}^x g$ for all $x \in \mathbb{R}$, then $f \le g$.* (c) *$|f| \in \mathcal{A}_C$ and $|\int_{-\infty}^x f| \le \int_{-\infty}^x |f|$ for all $x \in \overline{\mathbb{R}}$.* (d) $\| |f| \| = \| |F'| \| = \| |F| \|_\infty = \|f\|$.

The order on $\mathcal{A}_C$ gives us absolute integration since if F is continuous, so is $|F|$ and then integrability of f implies integrability of $|f|$. Notice that the definition of order allows us to integrate both sides of $f \le g$ in $\mathcal{A}_C$ to get $F \le G$ in $\mathcal{B}_C$. The isomorphism allows us to differentiate both sides of $F \le G$ in $\mathcal{B}_C$ to get $F' \le G'$ in $\mathcal{A}_C$. However, there is no pointwise implication. For example, $F(x) \ge 0$ for all $x \in \mathbb{R}$ does not imply $F'(x) \ge 0$ for all $x \in \mathbb{R}$. Take $F(x) = \exp(-x^2)$. And, if f and g are functions in $\mathcal{A}_C$ and $f(t) \le g(t)$ for all $t \in \mathbb{R}$, we cannot conclude that $f \le g$ in $\mathcal{A}_C$. This was shown with the $f(t) = \sin(t)/t$ function above. Note also that the partial ordering mentioned at the end of Section 7 fails to be a vector lattice. If $f \in \mathcal{A}_C$ is a function and $\langle f, \phi \rangle \ge 0$ for all $\phi \in \mathcal{D}$ with $\phi \ge 0$, then $f \ge 0$ almost everywhere. Hence, $\sup(f, 0)$ need not be in $\mathcal{A}_C$. This is the case for any function that has a conditionally convergent integral, as with our $\sin(t)/t$ function. In the next section we consider the more usual type of absolute integrability.

10 Absolute Convergence.

Suppose $f \in \mathcal{A}_C$. Let $\|f\|_{\mathcal{ABS}} = \sup_{\substack{\phi \in \mathcal{D} \\ \|\phi\|_\infty \le 1}} \langle f, \phi \rangle$ and define $\mathcal{ABS} = \{f \in \mathcal{A}_C \mid \|f\|_{\mathcal{ABS}} < \infty\}$. We will show that $\mathcal{ABS}$ provides a sensible extension of the notion of absolute integrability. If $f \in \mathcal{A}_C$ and its primitive is $F \in \mathcal{BV} \cap \mathcal{B}_C$, then, by the Hölder inequality,

$$|\langle f, \phi \rangle| = |\langle F', \phi \rangle| = \left| \int_{-\infty}^{\infty} F' \phi \right| \le 2VF \, \|\phi\|_\infty.$$

So, $f \in \mathcal{ABS}$. If $f \in \mathcal{ABS}$, then

$$\sup_{\substack{\phi \in \mathcal{D} \\ \|\phi\|_\infty \le 1}} \langle f, \phi \rangle = \sup_{\substack{\phi \in \mathcal{D} \\ \|\phi\|_\infty \le 1}} \int_{-\infty}^{\infty} F \phi' < \infty.$$

Since $F \in \mathcal{B}_C$ we have $VF = \operatorname{ess\,var} F < \infty$. Thus, $f \in \mathcal{ABS}$ if and only if $VF < \infty$. See Section 5 for the definition of the essential variation.

From the definition of variation it follows that $\|f\|_{\mathcal{ABS}} = VF$. We know $\mathcal{BV}$ is a Banach space. Clearly $\mathcal{B}_C \cap \mathcal{BV}$ is a subspace. To show it is complete, suppose $\{F_n\} \subset \mathcal{B}_C \cap \mathcal{BV}$ is Cauchy in the $\mathcal{BV}$ norm. Then there is $F \in \mathcal{BV}$ such that $V(F_n - F) \to 0$. We need to show $F \in \mathcal{B}_C$. Let $x \in \overline{\mathbb{R}}$. We have

$$\begin{aligned}|F(x) - F(y)| &\leq |F(x) - F_n(x) - F(y) + F_n(y)| + |F_n(x) - F_n(y)| \\ &\leq V(F_n - F) + |F_n(x) - F_n(y)|.\end{aligned}$$

Given $\epsilon > 0$ we can take n large enough so that $V(F_n - F) < \epsilon/2$. Since $F_n \in \mathcal{B}_C$ we can now take y close enough to x so that $|F_n(x) - F_n(y)| < \epsilon/2$. Hence, $F \in \mathcal{B}_C$ and $\mathcal{B}_C \cap \mathcal{BV}$ is a Banach space. The integral provides a linear isometry between $\mathcal{ABS}$ and $\mathcal{B}_C \cap \mathcal{BV}$. Hence, $\|f\|_{\mathcal{ABS}}$ is a norm and $\mathcal{ABS}$ is a Banach space. We identify $\mathcal{ABS}$ as the subspace of $\mathcal{A}_C$ consisting of absolutely integrable distributions by analogue with the fact that primitives of Denjoy or wide Denjoy integrable functions need not be of bounded variation but primitives of L^1 functions are absolutely continuous and hence of bounded variation.

11 Odds and Ends.

We collect here various other results. The first is that there are no improper integrals.

Theorem 25 (Hake Theorem)**.** *Suppose $f \in \mathcal{D}'$ and $f = F'$ for some $F \in C^0(\mathbb{R})$. If $\lim_{\infty} F$ and $\lim_{-\infty} F$ exist in $\mathbb{R}$, then $f \in \mathcal{A}_C$ and $\int_{-\infty}^{\infty} f = \lim_{x\to\infty} \int_0^x f + \lim_{x\to-\infty} \int_x^0 f$.*

PROOF. Define $\overline{F}(x) = F(x)$ for $x \in \mathbb{R}$, $\overline{F}(\infty) = \lim_{\infty} F$, $\overline{F}(-\infty) = \lim_{-\infty} F$. Then $\overline{F} \in C^0(\overline{\mathbb{R}})$ and $\overline{F}\,' = f$. Hence, $f \in \mathcal{A}_C$ and

$$\begin{aligned}\int_{-\infty}^{\infty} f &= \overline{F}(\infty) - \overline{F}(-\infty) = \lim_{\infty} F - \lim_{-\infty} F \\ &= \lim_{x\to\infty} [F(x) - F(0)] + \lim_{x\to-\infty} [F(0) - F(x)]\,. \qquad \square\end{aligned}$$

There are similar versions on compact intervals and intervals such as $[0, \infty)$. The corresponding result is false for Lebesgue integrals. For example, $\lim_{x\to\infty} \int_0^x \sin(t^2)\,dt = \sqrt{\pi}/(2^{3/2})$, but the function $t \mapsto \sin(t^2)$ is not in L^1. The integral is called a *Cauchy–Lebesgue* integral and in this case is also an improper Riemann integral. The theorem is true for Henstock-Kurzweil integrals. Proving the Hake theorem for the Henstock-Kurzweil or Perron integral is more involved. See [12], Theorem 9.21 and Theorem 8.18.

Theorem 26 (Second mean value theorem)**.** *Let $f \in \mathcal{A}_C$ and let $g : \overline{\mathbb{R}} \to \mathbb{R}$ be monotonic. Then $\int_{-\infty}^{\infty} fg = g(-\infty)\int_{-\infty}^{\xi} f + g(\infty)\int_{\xi}^{\infty} f$ for some $\xi \in \overline{\mathbb{R}}$.*

PROOF. Integrate by parts and use the mean value theorem for Riemann–Stieltjes integrals [15, §7.10]:

$$\begin{aligned}\int_{-\infty}^{\infty} fg &= F(\infty)g(\infty) - \int_{-\infty}^{\infty} F\,dg \\ &= F(\infty)g(\infty) - F(\xi)\int_{-\infty}^{\infty} dg \\ &= F(\infty)g(\infty) - F(\xi)[g(\infty) - g(-\infty)] \\ &= g(-\infty)F(\xi) + g(\infty)[F(\infty) - F(\xi)].\end{aligned}$$ □

This proof is taken from [7], where a proof of the Bonnet form of the second mean value theorem can also be found.

Using the distributional integral, it is possible to formulate a version of Taylor's theorem with integral remainder. For an approximation by an nth degree polynomial it is only required that $f^{(n)}$ be continuous.

Theorem 27 (Taylor)**.** *Suppose $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ and let $n \geq 0$ be an integer. If $f^{(n)} \in C^0([a,b])$, then for all $x \in [a,b]$ we have $f(x) = P_n(x) + R_n(x)$ where*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

and

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$

For each $x \in [a,b]$ we have the estimate

$$\begin{aligned}|R_n(x)| &\leq \frac{(x-a)^n \|f^{(n+1)}\chi_{[a,x]}\|}{n!} \leq \frac{(x-a)^n\|f^{(n+1)}\|}{n!} \\ &= \frac{(x-a)^n}{n!} \max_{a\leq\xi\leq x} |f^{(n)}(\xi) - f^{(n)}(a)|.\end{aligned}$$

And,

$$\|R_n\| \leq \|R_n\|_1 \leq \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n+1)}\| = \frac{(b-a)^{n+1}}{(n+1)!} \max_{a\leq\xi\leq b} |f^{(n)}(\xi) - f^{(n)}(a)|.$$

The remainder exists since the function $t \mapsto (x-t)^n$ is monotonic for each x. Repeated integration by parts establishes the integral remainder formula. Estimates of the remainder follow upon applying the second mean value theorem. See [29] for various other estimates of the remainder. Usual versions of Taylor's theorem require $f^{(n+1)}$ to be integrable. For the Lebesgue integral this means taking $f^{(n)}$ to be absolutely continuous. Here we only need $f^{(n)}$ continuous.

Theorem 28 (Homogeneity of Alexiewicz norm)**.** *Let $f \in \mathcal{A}_C$. For $t \in \mathbb{R}$, define the translation τ_t by $\langle \tau_t f, \phi \rangle = \langle f, \tau_{-t}\phi \rangle$ where $\tau_t\phi(x) = \phi(x-t)$ for $\phi \in \mathcal{D}$. The Alexiewicz norm is translation invariant: If $f \in \mathcal{A}_C$, then $\tau_t f \in \mathcal{A}_C$ and $\|\tau_t f\| = \|f\|$. Translation is continuous: $\|f - \tau_t f\| \to 0$ as $t \to 0$.*

PROOF. If $f \in \mathcal{A}_C$, then a change of variables shows

$$\begin{aligned}\langle \tau_t f, \phi \rangle = \langle f, \tau_{-t}\phi \rangle &= \int_{-\infty}^{\infty} f(s)\phi(s+t)\, ds = \int_{-\infty}^{\infty} f(s-t)\phi(s)\, ds \\ &= \int_{-\infty}^{\infty} F'(s-t)\phi(s)\, ds = \int_{-\infty}^{\infty} (\tau_t F)'(s)\phi(s)\, ds\end{aligned}$$

and $\tau_t F \in \mathcal{B}_C$ is the primitive of $\tau_t f$. Hence, $\tau_t f \in \mathcal{A}_C$. It is clear that $\|F\|_\infty = \|\tau_t F\|_\infty$ for all $t \in \mathbb{R}$. Hence, $\|f\| = \|\tau_t f\|$.

As well,

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{x} [f(s) - \tau_t f(s)] ds \right| &= \sup_{x\in\mathbb{R}} |F(x) - F(x-t)| \\ &\to 0 \text{ as } t \to 0 \text{ since } F \text{ is uniformly continuous.} \ \square\end{aligned}$$

See [30] for some other continuity properties of the Alexiewicz norm.

A Banach space satisfying the conditions of Theorem 28 is called *homogeneous.*

Theorem 29 (Equivalent norms)**.** *The following norms on $\mathcal{A}_C$ are equivalent to $\|\cdot\|$. For $f \in \mathcal{A}_C$, define $\|f\|' = \sup_I |\int_I f|$ where the supremum is taken over all compact intervals $I \subset \mathbb{R}$; $\|f\|'' = \sup_g \int fg$, where the supremum is taken over all $g \in \mathcal{BV}$ such that $|g| \le 1$ and $Vg \le 1$; $\|f\|''' = \sup_g \int fg$, where the supremum is taken over all $g \in \mathcal{EBV}$ such that $\|g\|_\infty \le 1$ and essvar $g \le 1$.*

PROOF. We have $\|f\|' = \sup_{a<b} \left|\int_a^b f\right| = \sup_{a<b} |F(b) - F(a)| \le 2\|f\|$. And, $\|f\| \le \|f\|'$. Hence, $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. Let $g \in \mathcal{BV}$ with $|g| \le 1$ and $Vg \le 1$. By the Hölder inequality (Theorem 7),

$$\left| \int_{-\infty}^{\infty} fg \right| \le \|f\| \, [\inf |g| + 2Vg] \le 3\|f\|.$$

And,

$$\|f\|'' \geq \max\left(\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty} f\chi_{(-\infty,x]}, -\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty} f\chi_{(-\infty,x]}\right).$$

It follows that $\frac{1}{3}\|f\|'' \leq \|f\| \leq \|f\|''$. The proof for $\|\cdot\|'''$ is similar. □

The following definition allows us to integrate any distribution over a compact interval. The result is also a distribution. If $T \in \mathcal{D}'$ and $[a,b] \subset \mathbb{R}$, define

$$\begin{aligned}\left\langle \int_a^b T', \phi\right\rangle :=& \left\langle T', \int_a^b \tau_t\phi(\cdot)\,dt\right\rangle = -\langle T, \int_a^b \phi'(\cdot - t)\,dt\rangle \\ =& \langle T, \tau_b\phi\rangle - \langle T, \tau_a\phi\rangle = \langle \tau_{-b}T, \phi\rangle - \langle \tau_{-a}T, \phi\rangle.\end{aligned}$$

The translation τ_{-a} was defined in Theorem 28. In the case of $T' = f \in \mathcal{A}_C$ this gives $\left\langle \int_a^b f, \phi\right\rangle = \int_{-\infty}^{\infty} F(t)\phi(t-b)\,dt - \int_{-\infty}^{\infty} F(t)\phi(t-a)\,dt$, which is a convolution. Since F is continuous, we can recover the value $\int_a^b f \in \mathbb{R}$ by evaluating on a delta sequence $\{\phi_n\}$. See the end of Section 3. We then have $\langle \int_a^b f, \phi_n\rangle \to F(b) - F(a)$. This method of integration was developed by J. Mikusiński, J. A. Musielak and R. Sikorski in the 1950's and 1960's [17], [20], [26]. The advantage is that it can integrate every distribution over a compact interval. The disadvantage is that integrals over $(-\infty, \infty)$ must be treated as improper integrals since $\tau_{\pm\infty}\phi = 0$. As we saw in Theorem 25, there are no improper integrals in $\mathcal{A}_C$. And, of course $\mathcal{A}_C$ is a Banach space, whereas $\mathcal{D}'$ is not.

12 Further Threads.

In this final section we list several topics in passing and several ideas for further research.

12.1 What Happened to the Measure?

In Lebesgue and Henstock–Kurzweil integration the measure appears explicitly. With the distributional integral it is disguised in the formula $F' = f$, out of which $\langle f, \phi\rangle = -\langle F, \phi'\rangle$ for all $\phi \in \mathcal{D}$. The derivative is

$$\phi'(x) = \lim_{h\to 0}\frac{\phi(x+h) - \phi(x)}{h} = \lim_{h\to 0^+}\frac{\phi(I(x,h))}{\lambda(I(x,h))},$$

where $I(x,h)$ is the interval centered on x with radius h and we have replaced ϕ by the interval function $\phi((a,b)) = \phi(b) - \phi(a)$. Replacing Lebesgue measure

λ with some other measure μ gives the Radon-Nikodym derivative with respect to μ. To integrate f with respect to μ we need to use the Radon-Nikodym derivative when we define integration by parts for distributions. The test functions would have to have all their Radon-Nikodym derivatives continuous with respect to μ. The primitives would have to be continuous with respect to μ, rather than pointwise. For continuity at x this means that for all $\epsilon > 0$ there is $\delta > 0$ such that $\mu(I(x, |x-y|)) < \delta$ gives $|F(x) - F(y)| < \epsilon$, whereas replacing μ with λ gives the usual pointwise definition of continuity.

12.2 Integration in $\mathbb{R}^n$.

The Denjoy integral has not been easy to formulate in $\mathbb{R}^n$ due to the difficulty of defining $ACG*$ in $\mathbb{R}^n$. For the fearless, see Chapter 2 in [7]. There is, however, a distributional integral in $\mathbb{R}^n$. If $f \in \mathcal{D}'(\mathbb{R}^n)$, then f is integrable if there is a function $F \in C^0(\overline{\mathbb{R}}^n)$ such that $DF = f$. The differential operator is $D = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n}$. Now, $\langle f, \phi \rangle = \langle DF, \phi \rangle = (-1)^n \langle F, D\phi \rangle$ where ϕ is a C^∞ function with compact support in $\mathbb{R}^n$. For example, $\int_a^b \int_c^d F_{12} = F(b,d) - F(a,d) - F(b,c) + F(a,c)$ for each continuous function F. This is the form of the integral given in [18]. For details see [3], where there are applications to the wave equation and theorems of Fubini and Green. This definition extends the Lebesgue and Henstock–Kurzweil integrals. But, it is not invariant under rotations since the operator D is not invariant under rotations. For example, a rotation of $\pi/4$ for which $(x, y) \mapsto (\xi, \eta)$ transforms D into the wave operator $\partial^2/\partial\xi^2 - \partial^2/\partial\eta^2$. Hence, if f is integrable its rotation need not be integrable.

W. Pfeffer [23] has defined a nonabsolute integral that is invariant under rotations and other transformations but it is based on different principles. In some sense, his integral is designed to invert the divergence operator. A possible extension of Pfeffer's integral in the spirit of distributional integrals can be obtained with the following definitions. If $g \in L^1_{loc}(\mathbb{R}^n)$, then g is of *local bounded variation* if $\sup \int_U f \,\mathrm{div}\phi < \infty$ for each open ball $U \subset \mathbb{R}^n$, where the supremum is taken over all $\phi \in \mathcal{D}(U)$ with $\|\phi\|_\infty \leq 1$. A measurable set $E \subset \mathbb{R}^n$ has *locally finite perimeter* if χ_E is of local bounded variation. Sets with Lipshitz boundary have this property and thus polytopes do as well. Suppose $\Omega \subset \mathbb{R}^n$ is open and $E \subset \Omega$ has locally finite perimeter. Then $f \in \mathcal{D}'(\Omega)$ is integrable over E if there is a continuous function $F : \overline{E} \to \mathbb{R}^n$ such that $f = \mathrm{div}F$ in $\mathcal{D}'(\Omega)$. Then

$$\int_E f = \int_E \mathrm{div}F = \int_{\partial * E} F \cdot n \, d\mathcal{H}^{n-1}$$

where $\partial * E$ is the measure-theoretic boundary of E, n is the outward normal and $\mathcal{H}^{n-1}$ is Hausdorff measure. The final integral exists since F is continuous.

This definition of the integral is based on the Gauss-Green theorem, whose usual version requires F to be C^1. See [9] or [33].

Note that if F is a continuous function in $\mathbb{R}^2$ and $f = F_{21}$ in $\mathcal{D}'(\mathbb{R}^2)$, then $f = \mathrm{div}(F_2, 0)$. Since the boundary of a Cartesian interval in $\mathbb{R}^2$ is a union of four intervals in $\mathbb{R}$, the above integral can be used twice to obtain the formula $\int_a^b \int_c^d F_{12} = F(b,d) - F(a,d) - F(b,c) + F(a,c)$. Hence, the Gauss–Green integral includes the integral of Mikusiński and Ostaszewski [18]; Ang, Schmidt and Vy [3].

12.3 The Regulated Primitive Integral.

A function on the real line is *regulated* if it has a left and right limit at each point. It is known that the Riemann-Stieltjes integral $\int_{-\infty}^{\infty} F\,dg$ exists when one of F and g is regulated and the other is of bounded variation. We can then replace $\mathcal{B}_C$ with the space of regulated functions. Then we can integrate all distributions that are the distributional derivative of a regulated function. If $f = F'$, then there are four integrals $\int_{(a,b)} f = F(b-) - F(a+)$, $\int_{[a,b)} f = F(b-) - F(a-)$, $\int_{(a,b]} f = F(b+) - F(a+)$, $\int_{[a,b]} f = F(b+) - F(a-)$, which need not be same since the left and right limits of F are not necessarily equal. This will allow us to integrate signed Radon measures since if μ is a signed Radon measure, then $F(x) := \int_{-\infty}^{x} d\mu$ is a function of bounded variation and hence regulated. For example, the Dirac distribution is the derivative of the Heaviside step function, $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$, otherwise. And, $\int_{(0,1)} \delta = \int_{(0,1)} H' = H(1-) - H(0+) = 1 - 1 = 0$. Whereas, $\int_{[0,1)} \delta = H(1-) - H(0-) = 1 - 0 = 1$. The regulated primitive integral will be discussed in detail elsewhere [31].

It is not clear if we get a useful integral by replacing $\mathcal{B}_C$ with such Banach spaces as L^p $(1 \leq p \leq \infty)$ or $\mathcal{BV}$.

In light of the existence of other integrals that invert distributional derivatives, we propose the name *continuous primitive integral* for the integral described in this paper.

References

[1] A. Alexiewicz, *Linear functionals on Denjoy–integrable functions*, Colloquium Math., **1** (1948), 289–293.

[2] C. D. Aliprantis and W. Burkinshaw, *Principles of Real Analysis*, San Diego, Academic Press, 1998.

[3] D. D. Ang, K. Schmitt and L. K. Vy, *A multidimensional analogue of the Denjoy–Perron–Henstock–Kurzweil integral*, Bull. Belg. Math. Soc. Simon Stevin, **4** (1997), 355–371.

[4] D. D. Ang and L. K. Vy, *On the Denjoy-Perron-Henstock-Kurzweil integral*, Vietnam J. Math., **31** (2003), 381–389.

[5] R. Bagby, *The substitution theorem for Riemann integrals*, Real Anal. Exchange, **27** (2001-02), 309–314.

[6] P. S. Bullen, *Nonabsolute integrals in the twentieth century*, AMS special session on nonabsolute integration (P. Muldowney and E. Talvila, eds.), Toronto, 2000, `http://www.emis.de/proceedings/index.html`.

[7] V. G. Čelidze and A. G. Džvaršeĭšvili, *The Theory of the Denjoy Integral and Some Applications* (trans. P. S. Bullen), Singapore, World Scientific, 1989.

[8] N. Dunford and J. T. Schwartz, *Linear Operators, Vol. I*, New York, Interscience, 1957.

[9] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Boca Raton, CRC Press, 1992.

[10] J. Foran, *A chain rule for the approximate derivative and change of variables for the $\mathcal{D}$-integral*, Real Anal. Exchange, **8** (1982-83), 443–454.

[11] F. G. Friedlander and M. Joshi, *Introduction to the Theory of Distributions*, Cambridge, Cambridge University Press, 1999.

[12] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Providence, American Mathematical Society, 1994.

[13] R. A. Gordon, *When is a limit function continuous?*, Mathematics Magazine, **71** (1998), 306–308.

[14] *The MacTutor history of mathematics archive*, `http://www-history.mcs.st-and.ac.uk/history`.

[15] R. M. McLeod, *The Generalized Riemann Integral*, Washington, Mathematical Association of America, 1980.

[16] E. J. McShane, *Integration*, Princeton, Princeton University Press, 1944.

[17] J. Mikusiński and R. Sikorski, *The elementary theory of distributions, part I*, Rozprawy Mat., **12** (1957), 54 pp.

[18] P. Mikusińksi and K. Ostaszewski, *Embedding Henstock integrable functions into the space of Schwartz distributions*, Real Anal. Exchange, **14** (1988-89), 24–29.

[19] P. Mikusińksi and K. Ostaszewski, *The space of Henstock integrable functions II*, New integrals (P. S. Bullen, et al, eds.), Berlin, Springer-Verlag, 1990, 136–149.

[20] J. A. Musielak, *A note on integrals of distributions*, Prace Mat., **8** (1963/1964), 1–7.

[21] K. Ostaszewski, *Topology for the spaces of Denjoy integrable functions*, Real Anal. Exchange, **9** (1983-84), 79–85.

[22] K. Ostaszewski, *The space of Henstock integrable functions of two variables*, Internat. J. Math. Math. Sci., **11** (1988), 15–22.

[23] W. Pfeffer, *Derivation and Integration*, Cambridge, Cambridge University Press, 2001.

[24] D. N. Sarkhel and R. Výborný, *A change of variables theorem for the Riemann integral*, Real Anal. Exchange, **22** (1996-97), 390–395.

[25] L. Schwartz, *Thèorie des Distributions*, Paris, Hermann, 1966.

[26] R. Sikorski, *Integrals of distributions*, Studia Math., **20** (1961), 119–139.

[27] E. Talvila, *Limits and Henstock integrals of products*, Real Anal. Exchange, **25** (1999-2000), 907–918.

[28] E. Talvila, *Henstock-Kurzweil Fourier transforms*, Illinois J. Math., **46** (2002), 1207–1226.

[29] E. Talvila, *Estimates of the remainder in Taylor's theorem using the Henstock-Kurzweil integral*, Czechoslovak Math. J., **55(130)** (2005), 933–940.

[30] E. Talvila, *Continuity in the Alexiewicz norm*, Math. Bohem., **131** (2006), 189–196.

[31] E. Talvila, *The regulated primitive integral*, to appear.

[32] A. H. Zemanian, *Distribution Theory and Transform Analysis*, Dover, New York, 1987.

[33] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.