RESEARCH

Real Analysis Exchange
Vol. 33(1), 2007/2008, pp. 41–50

Anna Loranty, Faculty of Mathematics, University of Łódź, Banach 22, 90-238 Łódź, Poland. email: loranta@math.uni.lodz.pl

THE INTERSECTION CONDITIONS FOR $\langle s \rangle$-DENSITY SYSTEMS OF PATHS

Abstract

We investigate the intersection conditions for an $\langle s \rangle$-density system of paths. We show, for example, that for every unbounded and nondecreasing sequence of positive numbers $\langle s \rangle$ such that $\liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0$, there exists a system of paths connected with $\langle s \rangle$-density points which does not satisfy the intersection conditions. Moreover, we show that a function $f : \mathbb{R} \to \mathbb{R}$ is $\langle s \rangle$-approximately continuous if and only if f is continuous with respect to some $\langle s \rangle$-density system of paths.

1 Introduction.

The notion of a density point was introduced by Lebesgue at the beginning of the 20th century. Together with his fundamental theorem that almost all points of a Lebesgue measurable set are density points of that set, they turned out to play an important role in the theory of real functions.

The density topology connected with the notion of a density point was discovered by Haupt and Pauc [5] and was subsequently studied in detail by Goffman, Neugebauer, Nishiura, Waterman and Tall. It turned out that a function is approximately continuous if and only if it is continuous with respect to the density topology.

Some generalizations of the notion of a Lebesgue density point on the real line were introduced by Taylor [8]. Wilczyński presented the concept of a density point with respect to category [10]. Further, Filipczak and Hejduk [3], generalized the notion of a density point using unbounded and nondecreasing sequences of positive numbers. Such an approach to density points gave rise to $\langle s \rangle$-density topologies and $\langle s \rangle$-approximately continuous functions. Some

Key Words: density topology, system of paths, intersection conditions, approximately continuous function
Mathematical Reviews subject classification: 26A21, 54A10
Received by the editors June 1, 2006
Communicated by: B. S. Thomson

properties of the latter were studied in the papers [3], [2], [4], [6], [7]. Although an $\langle s\rangle$-density topology is often essentially bigger than the density topology, it turns out that most of its properties are analogous to those of the density topology.

In this paper, we show that, although an $\langle s\rangle$-density topology and the density topology are so similar, the systems of paths generated by these topologies have different properties. We concentrate on the intersection conditions for relevant systems of paths.

2 Notation and Definitions.

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, m (m^*) standard Lebesgue measure (Lebesgue outer measure) on the real line $\mathbb{R}$ and $\mathcal{L}$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. By $\mathcal{T}_d$ we denote the density topology in $\mathbb{R}$ and by S the family of all unbounded and nondecreasing sequences of positive numbers. The symbol $\langle s\rangle$ denotes a sequence $\{s_n\}_{n\in\mathbb{N}} \in S$. Moreover, a set $H \in \mathcal{L}$ is a measurable hull of $A \subset \mathbb{R}$, if $A \subset H$ and every Lebesgue measurable set $Y \subset H \setminus A$ has Lebesgue measure zero.

We introduce the following notation: $S_0 = \{\langle s\rangle \in S : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}$ and $S_1 = \{\langle s\rangle \in S : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0\}$. Of course, we have that $S = S_0 \cup S_1$.

This paper concerns two systems of paths generated by density points and $\langle s\rangle$-density points. We start with some definitions.

Definition 1 ([1])**.** Let $x \in \mathbb{R}$. A *path leading* to x is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and x is a point of accumulation of E_x.

Definition 2 ([1])**.** A *system of paths* is a collection $E = \{E_x : x \in \mathbb{R}\}$ such that each E_x is a path leading to x.

Definition 3 ([3])**.** We say that $x \in \mathbb{R}$ is *a density point of the set* $A \in \mathcal{L}$ *with respect to the sequence* $\{s_n\}_{n\in\mathbb{N}} \in S$, also called an $\langle s\rangle$*-density point*, if

$$\lim_{n\to\infty} \frac{m(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} = 1.$$

For any $\langle s\rangle \in S$ and $A \in \mathcal{L}$, let

$$\Phi_{\langle s\rangle}(A) = \{x \in \mathbb{R} : x \text{ is an } \langle s\rangle\text{-density point of A}\}.$$

It is known that $\Phi_{\langle s\rangle}$ is a lower density operator, the family $\mathcal{T}_{\langle s\rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle s\rangle}(A)\}$ is a topology in $\mathbb{R}$ and this topology is equal to the density topology if and only if $\langle s\rangle \in S_1$ [3].

Definition 4 ([1])**.** A system of paths $E = \{E_x : x \in \mathbb{R}\}$ is said to be of $(1,1)$-*density type*, if x is a density point of the set E_x for each $x \in \mathbb{R}$.

Definition 5. A system of paths $E = \{E_x : x \in \mathbb{R}\}$ is said to be of $\langle s \rangle$-*density type* if x is an $\langle s \rangle$-density point of the set E_x for each $x \in \mathbb{R}$.

The following properties of $(1,1)$-density systems of paths have been established [1].

type	bilateral	non-porous	I.C	I.I.C	E.I.C	E.I.C.[w]	one-sided E.I.C.[w]
(1,1)-density	yes	yes	yes	yes	yes	yes	yes

In the next part of this paper, we will examine whether an $\langle s \rangle$-density system of paths fulfills these conditions. We start with the following definitions.

Definition 6 ([9])**.** Let $A \subset \mathbb{R}$ be any set. The *right hand porosity (left hand porosity, bilateral porosity)* of the set A at the point x is defined as $p^+(A,x) = \limsup_{r \to 0^+} \dfrac{l(A,x,x+r)}{r}$ $(p^-(A,x) = \limsup_{r \to 0^+} \dfrac{l(A,x,x-r)}{r}$, $p(A,x) = \limsup_{r \to 0} \dfrac{l(A,x,x+r)}{|r|})$, where $l(A,x,x+r)$ $(l(A,x,x-r))$ denotes the length of the largest open subinterval of $(x,x+r)$ $((x-r,x))$ containing no point of A.

Definition 7 ([1])**.** Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths.

1. E is said to be *bilateral*, if x is a bilateral point of accumulation of E_x for each $x \in \mathbb{R}$.
2. E is said to be *nonporous*, if for each $x \in \mathbb{R}$ the set E_x has bilateral porosity 0 at x.

Definition 8 ([1])**.** Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. E is said to satisfy the condition listed below, if there is associated with E a positive function δ on $\mathbb{R}$ so that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$, the sets E_x and E_y intersect in the stated fashion:

1. *intersection condition (I.C.)*: $E_x \cap E_y \cap [x,y] \neq \emptyset$,
2. *internal intersection condition (I.I.C.)*: $E_x \cap E_y \cap (x,y) \neq \emptyset$,
3. *external intersection condition (E.I.C.)*: $E_x \cap E_y \cap (y, 2y-x) \neq \emptyset$ and $E_x \cap E_y \cap (2x-y, x) \neq \emptyset$,
4. *external intersection condition with parameter* $[w]$ (where $w > 0$) (E.I.C. $[w]$): $E_x \cap E_y \cap (y, (w+1)y - wx) \neq \emptyset$ and $E_x \cap E_y \cap ((w+1)x - wy, x) \neq \emptyset$,
5. *one-sided external intersection condition with parameter* $[w]$ (where $w > 0$): $E_x \cap E_y \cap (y, (w+1)y - wx) \neq \emptyset$ or $E_x \cap E_y \cap ((w+1)x - wy, x) \neq \emptyset$.

Obviously, any system of paths having the I.I.C. property has the I.C. property. Moreover, if a system of paths does not satisfy the one-sided external intersection condition with parameter $[w]$, then it does not satisfy E.I.C.$[w]$ nor E.I.C..

From the definition of an $\langle s\rangle$-density point it is clear that an $\langle s\rangle$-density system of paths is bilateral for each $\langle s\rangle \in S$.

3 The $\langle s\rangle$-Approximately Continuous Functions.

Now consider functions continuous with respect to a system of paths.

Definition 9 ([1])**.** Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a point x_0 with respect to system of paths E* (E-continuous), if $f(x_0) = \lim_{\substack{x\to x_0,\\ x\in E_{x_0}}} f(x)$.

Definition 10 ([6])**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $\langle s\rangle \in S$. We say that f is *$\langle s\rangle$-approximately continuous at a point* $x_0 \in \mathbb{R}$, if there exists a set $A_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi_{\langle s\rangle}(A_{x_0})$ and $f(x_0) = \lim_{\substack{x\to x_0,\\ x\in A_{x_0}}} f(x)$.

Definition 11 ([6])**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $\langle s\rangle \in S$. We say that f is *$\langle s\rangle$-approximately continuous*, if f is $\langle s\rangle$-approximately continuous at every point $x \in \mathbb{R}$.

The family of all $\langle s\rangle$-approximately continuous functions is equal to the family of functions continuous with respect to the topology $\mathcal{T}_{\langle s\rangle}$ [6]. Moreover, it is clear that the following theorem holds.

Theorem 12. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\langle s\rangle \in S$. Then f is $\langle s\rangle$-approximately continuous if and only if f is continuous with respect to some $\langle s\rangle$-density system of paths.*

We will need two more theorems

Theorem 13 ([7])**.** *Let $\langle s\rangle \in S$. If $f : \mathbb{R} \to \mathbb{R}$ is $\langle s\rangle$-approximately continuous, then f is in the first class of Baire.*

Theorem 14 ([1])**.** *Let f be in the first class of Baire and E-continuous for a choice of paths $E = \{E_x : x \in \mathbb{R}\}$. If the system E is bilateral, then f has the Darboux property.*

From Theorems 12, 13 and 14, we conclude the following.

Corollary 15. *Let $\langle s\rangle \in S$. If $f : \mathbb{R} \to \mathbb{R}$ is $\langle s\rangle$-approximately continuous, then f has the Darboux property.*

There is another proof of Corollary 15, which is independent of the properties of systems of paths [4, Cor. 9].

4 The Intersection Conditions for $\langle s\rangle$-Density Systems of Paths.

Let us return to the properties of an $\langle s\rangle$-density system of paths. We will consider only sequences $\langle s\rangle$ belonging to S_0 because otherwise the notions of an $\langle s\rangle$-density point and a density point are equivalent [3]. Consequently, for the sequences belonging to S_1, the properties of an $\langle s\rangle$-density system of paths and (1,1)-density system of paths coincide. Consider a sequence $\langle s\rangle$ belonging to S_0. There exists a subsequence $\{s_{k_n}\}_{n\in\mathbb{N}}$ of the sequence $\langle s\rangle$ such that $\lim\limits_{n\to\infty}\frac{s_{k_n}}{s_{k_{n+1}}}=0$. Putting, for $x\in\mathbb{R}$ and $M\in\mathbb{N}$,

$$\begin{aligned}E_x^M=&\bigcup_{n=M}^{\infty}((x+\frac{1}{s_{k_{n+1}}},x+\frac{1}{s_{k_n+1}})\cup(x+\frac{1}{\sqrt{s_{k_n}s_{k_n+1}}},x+\frac{1}{s_{k_n}}))\\&\cup\bigcup_{n=M}^{\infty}((x-\frac{1}{s_{k_n}},x-\frac{1}{\sqrt{s_{k_n}s_{k_n+1}}})\cup(x-\frac{1}{s_{k_n+1}},x-\frac{1}{s_{k_{n+1}}}))\cup\{x\},\end{aligned}$$

we have that the family $E^M=\{E_x^M:x\in\mathbb{R}\}$ is an $\langle s\rangle$-density system of paths, because, for each $x\in\mathbb{R}$, the point x is an $\langle s\rangle$-density point of the set E_x^M [3].

Theorem 16. *Let $\langle s\rangle\in S_0$. There exists an $\langle s\rangle$-density system of paths $E=\{E_x:x\in\mathbb{R}\}$ such that $p^+(E_x,x)=p^-(E_x,x)=1$ for each $x\in\mathbb{R}$.*

Proof. Let $\langle s\rangle\in S_0$ and consider an $\langle s\rangle$-density system of paths $E^1=\{E_x^1:x\in\mathbb{R}\}$. For each $x\in\mathbb{R}$ we have that $p^+(E_x^1,x)=p^-(E_x^1,x)=1$. Indeed, if we consider $r_n=\left(\sqrt{s_{k_n}s_{k_n+1}}\right)^{-1}$, we obtain that

$$1\geq\frac{l(,E_x^1,x,x+r_n)}{r_n}\geq\frac{\left(\sqrt{s_{k_n}s_{k_n+1}}\right)^{-1}-(s_{k_n+1})^{-1}}{\left(\sqrt{s_{k_n}s_{k_n+1}}\right)^{-1}}=1-\sqrt{\frac{s_{k_n}}{s_{k_n+1}}}.$$

Since $\lim\limits_{n\to\infty}(1-\sqrt{\frac{s_{k_n}}{s_{k_n+1}}})=1$, we have $\lim\limits_{n\to\infty}\frac{l(E_x^1,x,x+r_n)}{r_n}=1$ and finally $p^+(E_x^1,x)=\limsup\limits_{r\to0^+}\frac{l(E_x^1,x,x+r)}{r}=1$. In the same manner, we can see that $p^-(E_x^1,x)=1$. □

To prove the next theorem, we need the following lemma.

Lemma 17. *Let $\langle s\rangle\in S_0$ and $\delta:\mathbb{R}\to(0,\infty)$ be any function. Let $\{s_{k_n}\}$ be a subsequence of the sequence $\langle s\rangle$ and $A_n=\{x\in\mathbb{R}:\delta(x)>\frac{1}{s_{k_n}}\}$ for each $n\in\mathbb{N}$. For any $N_0\in\mathbb{N}$, there exists a positive integer $n_0\geq N_0$ and a point $x_0\in A_{n_0}$ such that for any set $C\in\mathcal{L}$, if there exists $r>1$ such that*

$$m\left(C\cap\left[x_0,x_0+(\sqrt{s_{k_n}s_{k_n+1}})^{-1}\right]\right)\geq(r\sqrt{s_{k_n}s_{k_n+1}})^{-1}$$

for each $n \geq n_0$, then $C \cap A_n \neq \emptyset$ for each $n \geq n_0$.

PROOF. Let $N_0 \in \mathbb{N}$. The proof starts with the obvious observation that

$$\mathbb{R} = \bigcup_{n=N_0}^{\infty} \delta^{-1}\left(\left(\frac{1}{s_{k_n}}, \infty\right)\right) = \bigcup_{n=N_0}^{\infty} A_n.$$

From this, we deduce there exists $n_0 \geq N_0$ such that $m^*(A_{n_0}) > 0$. Denote by B_0 a measurable hull of the set A_{n_0}. Since $m(B_0) > 0$, there exists a point $x_0 \in A_{n_0}$ that is a density point of B_0. Let $C \in \mathcal{L}$ be any set with the following property: there exists $r > 1$ such that

$$m\left(C \cap \left[x_0, x_0 + (\sqrt{s_{k_n} s_{k_n+1}})^{-1}\right]\right) \geq (r\sqrt{s_{k_n} s_{k_n+1}})^{-1}$$

for each $n \geq n_0$. We claim that $m(C \cap B_0) > 0$. Indeed, from the fact that x_0 is a density point of B_0, it follows that there exists $\alpha > 0$ such that for all h, if $0 < h < \alpha$, then $m(B_0 \cap [x_0, x_0 + h])/h > (2r-1)/2r$. Moreover for any $n \geq n_0$ we have

$$\frac{m\left(C \cap \left[x_0, x_0 + \left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}\right]\right)}{\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}} \geq \frac{1}{r}.$$

Since $\lim\limits_{n\to\infty} \left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1} = 0$, there exists $n_1 \geq n_0$ such that

$$\left(\sqrt{s_{k_{n_1}} s_{k_{n_1}+1}}\right)^{-1} < \alpha.$$

Consequently, $m\left(B_0 \cap C \cap \left[x_0, x_0 + \left(\sqrt{s_{k_{n_1}} s_{k_{n_1}+1}}\right)^{-1}\right]\right) > 0$, and finally $m(B_0 \cap C) > 0$. Hence, $C \cap A_{n_0} \neq \emptyset$. Since $\{A_n\}_{n\in\mathbb{N}}$ is increasing, it follows that $C \cap A_n \neq \emptyset$ for each $n \geq n_0$, completing the proof. □

Theorem 18. *Let $\langle s\rangle \in S_0$. There exists an $\langle s\rangle$-density system of paths $E = \{E_x : x \in \mathbb{R}\}$ that does not satisfy the intersection condition (I.C.).*

PROOF. Since $\liminf\limits_{n\to\infty} \frac{s_n}{s_{n+1}} = 0$, there exists a subsequence $\{s_{k_n}\}_{n\in\mathbb{N}}$ of the sequence $\langle s\rangle$ such that $\lim\limits_{n\to\infty} \frac{s_{k_n}}{s_{k_n+1}} = 0$. Hence, there exists $N_0 \in \mathbb{N}$ such that $\frac{s_{k_n}}{s_{k_n+1}} < \frac{1}{16}$ for each $n \geq N_0$. Moreover, $s_{k_n} \neq s_{k_n+1}$ and $s_{k_n+1} > 4\sqrt{s_{k_n} s_{k_n+1}}$ for each $n \geq N_0$. Consider the $\langle s\rangle$-density system of paths E^{N_0} and suppose there exists a function $\delta : \mathbb{R} \to (0, \infty)$ such that the following condition is fulfilled:

For all x, y, if $0 < y - x < \min\{\delta(x), \delta(y)\}$, then $E_x \cap E_y \cap [x, y] \neq \emptyset$. (1)

By Lemma 17, there exists a positive number $n_0 \geq N_0$ and a point $x_0 \in A_{n_0} = \{x \in \mathbb{R} : \delta(x) > \frac{1}{s_{k_{n_0}}}\}$ such that for any set C, for which there exists $r > 1$ such that $m\left(C \cap \left[x_0, x_0 + (\sqrt{s_{k_n} s_{k_n+1}})^{-1}\right]\right) \geq (r\sqrt{s_{k_n} s_{k_n+1}})^{-1}$ for each $n \geq n_0$, we have $C \cap A_{n_0} \neq \emptyset$. In particular, if we consider the set

$$C = \bigcup_{n=n_0}^{\infty} \left(x_0 + \frac{3}{4}\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}, x_0 + \left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}\right),$$

we obtain that $C \cap A_{n_0} \neq \emptyset$. Consequently, there exists a point $y \in A_{n_0}$ and a positive number $n_2 \geq n_0$ such that

$$y \in \left(x_0 + \frac{3}{4}\left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}, x_0 + \left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}\right).$$

By the definition of the set $E_x^{N_0}$, we have $E_{x_0}^{N_0} \cap [x_0, y] \subset \left[x_0, x_0 + (s_{k_{n_2}+1})^{-1}\right)$ and $E_y^{N_0} \cap [x_0, y] \subset \left(y - (s_{k_{n_2}+1})^{-1}, y\right]$. Since $s_{k_{n_2}+1} > 4\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}$ and $y - x_0 > \frac{3}{4}\left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}$, we conclude that $E_{x_0}^{N_0} \cap E_y^{N_0} \cap [x_0, y] = \emptyset$. Simultaneously, $\min\{\delta(x_0), \delta(y)\} \geq (s_{k_{n_2}})^{-1} > \left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1} > y - x_0$. This contradicts (1), completing the proof. $\square$

Theorem 19. *Let $w > 0$ and $\langle s\rangle \in S_0$. There exists an $\langle s\rangle$-density system of paths $E = \{E_x : x \in \mathbb{R}\}$ that does not satisfy the one-sided external intersection condition with parameter $[w]$.*

PROOF. Since $\langle s\rangle \in S_0$, there exists a subsequence $\{s_{k_n}\}_{n\in\mathbb{N}}$ of $\langle s\rangle$ such that $\lim\limits_{n\to\infty} \frac{s_{k_n}}{s_{k_n+1}} = 0$. Hence, there exists $N_0 \in \mathbb{N}$ such that $\frac{s_{k_n}}{s_{k_n+1}} < \frac{1}{16(w+1)^2}$ for each $n \geq N_0$. It can easily be seen that $s_{k_n} \neq s_{k_n+1}$ and $s_{k_n+1} > 4(w+1)\sqrt{s_{k_n} s_{k_n+1}}$ for each $n \geq N_0$. Consider the $\langle s\rangle$-density system of paths E^{N_0} and suppose there exists a function $\delta : \mathbb{R} \to (0, \infty)$ such that the following condition is fulfilled:

$$\begin{aligned}&\text{For all } x, y \in \mathbb{R}, \text{ if } 0 < y - x < \min\{\delta(x), \delta(y)\}, \text{ then} \\ &(E_x \cap E_y \cap (y, (w+1)y - wx) \neq \emptyset) \vee E_x \cap E_y \cap ((w+1)x - wy, x) \neq \emptyset).\end{aligned} \tag{2}$$

By Lemma 17, there exists a positive number $n_0 \geq N_0$ and a point $x_0 \in A_{n_0}$ (where $A_n = \{x \in \mathbb{R} : \delta(x) > \frac{1}{s_{k_n}}\}$) such that for any set C for which there exists $r > 1$ such that $m\left(C \cap \left[x_0, x_0 + (\sqrt{s_{k_n} s_{k_n+1}})^{-1}\right]\right) \geq (r\sqrt{s_{k_n} s_{k_n+1}})^{-1}$ for each $n \geq n_0$, we have $C \cap A_{n_0} \neq \emptyset$. In particular, if we consider the set

$$C = \bigcup_{n=n_0}^{\infty} \left(x_0 + \frac{3}{4(w+1)}\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}, x_0 + \frac{1}{(w+1)}\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}\right),$$

we obtain, for $r = 4(w+1) > 1$, that

$$\begin{aligned}
&m\left(C \cap \left[x_0, x_0 + \left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}\right]\right) \\
\geq &m\left(\left[x_0 + \frac{3}{4(w+1)}\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}, x_0 + \frac{1}{(w+1)}\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1}\right]\right) \\
= &\frac{1}{4(w+1)}\left(\sqrt{s_{k_n} s_{k_n+1}}\right)^{-1},
\end{aligned}$$

for each $n \geq n_0$. Therefore, $A_{n_0} \cap C \neq \emptyset$ and there exists a point $y \in A_{n_0}$ and $n_2 \geq n_0$ such that

$$y \in \left(x_0 + \frac{3}{4(w+1)}\left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}, x_0 + \frac{1}{(w+1)}\left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}\right).$$

Hence, $(y, (w+1)y - wx_0) \subset \left(x_0 + \left(s_{k_{n_2}+1}\right)^{-1}, x_0 + \left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}\right)$. From this, and the definition of the set $E_{x_0}^{N_0}$, we conclude that

$$E_{x_0}^{N_0} \cap (y, (w+1)y - wx_0) = \emptyset.$$

Consequently, $E_{x_0}^{N_0} \cap E_y^{N_0} \cap (y, (w+1)y - wx_0) = \emptyset$. Moreover,

$$((w+1)x_0 - wy, x_0) \subset \left(y - \left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1}, y - \left(s_{k_{n_2}+1}\right)^{-1}\right).$$

It follows that

$$E_y^{N_0} \cap ((w+1)x_0 - wy, x_0) = \emptyset \text{ and } E_{x_0}^{N_0} \cap E_y^{N_0} \cap ((w+1)x_0 - wy, x_0) = \emptyset.$$

Simultaneously,

$$\min\{\delta(x_0), \delta(y)\} \geq \left(s_{k_{n_2}}\right)^{-1} > \frac{1}{(w+1)}\left(\sqrt{s_{k_{n_2}} s_{k_{n_2}+1}}\right)^{-1} > y - x_0.$$

This contradicts condition (2), and the proof is complete. □

In the table below we compare properties of the $\langle s \rangle$-density system of paths, obtained above, to the known properties of the $(1,1)$-density system of paths.

type	bilateral	non-porous	I.C	I.I.C	E.I.C	E.I.C.$[w]$	one-sided E.I.C.$[w]$
(1,1)-density	yes	yes	yes	yes	yes	yes	yes
$\langle s \rangle$-density	yes	no	no	no	no	no	no

In each case "no" means there exists an $\langle s \rangle$-density system of paths which does not satisfy the respective condition.

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