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ON THE APPROXIMATELY CONTINUOUS FORAN INTEGRAL: COMPLETING OUR CHART

Abstract

The paper is a follow-up to our previous work [P. Sworowski, *An answer to some questions of Ene*, Real Analysis Exchange, **30**(1) (2004/05), 183–192], where we have given a chart of relations between four approximately continuous Denjoy-Khintchine type integrals. Here we complete that chart with the approximate Foran integral.

1 Preliminaries.

Let us make use of *Preliminaries* of [10]. The only thing we would like to recall are the following definitions.

Definitions 1.1. We say that an $f\colon [a,b] \to \mathbb{R}$ is $\mathcal{F}_4$-integrable ($\mathcal{F}_2$-integrable), if there exists an approximately continuous VBG-function (resp. $[VBG]$-function) $F\colon [a,b] \to \mathbb{R}$ satisfying Lusin's $\mathcal{N}$ condition, such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. We say f is $\mathcal{F}_3$-integrable ($\mathcal{F}_1$-integrable), if there exists an approximately continuous ACG-function (resp. $[ACG]$-function) $F\colon [a,b] \to \mathbb{R}$ such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. In each case the integral of f is defined as $F(b) - F(a)$.

The $\mathcal{F}_4$-integral is $AK_{\mathcal{N}}$-integral of Gordon [6], while $\mathcal{F}_2$ is equivalent to Sarkhel's $T_{\mathrm{ap}}D$-integral [9], $\mathcal{F}_1$ is known as Kubota integral [7].

In present paper we shall deal with the property $\mathrm{A}(N)$ and the class $\mathfrak{F}$. These notions originate from [4].

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Definition 1.2. Let $E \subset D \subset \mathbb{R}$, $F\colon D \to \mathbb{R}$, and let N be an integer. We say that F is $\mathrm{A}(N)$ on E if for each $\varepsilon > 0$ there exists a number $\delta > 0$ with the property that for any sequence $(J_j)_j$ of nonoverlapping intervals meeting E with $\sum_j |J_j| < \delta$ and for each j, the image $F(E \cap J_j)$ is contained in N intervals, $I_{j1}, \dots, I_{jN}$ such that $\sum_j \sum_{i=1}^{N} |I_{ji}| < \varepsilon$.

We say that a function $G\colon [a,b] \to \mathbb{R}$ belongs to the class $\mathfrak{F}$, if there is an $[a,b]$-form $\{E_n\}_{n=1}^{\infty}$ with a suitable sequence of integers $\{N_n\}_{n=1}^{\infty}$ such that for each n, G is $\mathrm{A}(N_n)$ on E_n. We do not assume that members of $\mathfrak{F}$ are continuous as was done in [4].

Definition 1.3. [4]. We say that an $f\colon [a,b] \to \mathbb{R}$ is integrable in the sense of Foran, $f \in F$ for short, if there exists a continuous function $G \in \mathfrak{F}$ such that $G'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. The integral of f is defined to be $G(b) - G(a)$.

With the aid of, for instance, O'Malley's monotonicity lemma the correctness (uniqueness) of the above definition remains valid if approximately continuous primitives are considered; see [5]. This observation gave rise to an integral encompassing both the Foran integral and the Kubota integral ($\mathcal{F}_1$-integral).

Definition 1.4. [5]. We say that an $f\colon [a,b] \to \mathbb{R}$ is integrable in the approximate sense of Foran, $f \in AF$ for short, if there exists an approximately continuous function $G \in \mathfrak{F}$ such that $G'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. The integral of f is defined to be $G(b) - G(a)$.

It is easy to see that a G is *AC* on E iff it is $\mathrm{A}(1)$ on E. So, the *AF*-integral is more general than the $\mathcal{F}_3$-integral, not only the $\mathcal{F}_1$-integral. However, its relation to $\mathcal{F}_2$- and $\mathcal{F}_4$-integrals is not as clear and, as it seems, so far has not been considered. It is the main concern of our note (Sections 2&3). Some results of the paper have already been announced at Walla Walla symposium [11].

2 Relation to $\mathcal{F}_2$-Integral.

Lemma 2.1. [4, p. 361]. *If F and G are respectively $\mathrm{A}(N_1)$ and $\mathrm{A}(N_2)$ on E, then $F + G$ is $\mathrm{A}(N_1 N_2)$ on E.*

Lemma 2.2. *Let E be closed. If a VB-function $F\colon E \to \mathbb{R}$ satisfies $\mathcal{N}$, then it is $\mathrm{A}(3)$ (on E).*

PROOF. Take the Lebesgue decomposition of F: $F = F_1 + F_2$, F_1 a continuous *VB*-function, F_2 a jump function. Since F and F_2 satisfy $\mathcal{N}$, so does F_1 (see

Corollary 3.2 in the sequel). Since E is closed, by the Banach-Zarecki theorem we get F_1 is *AC*; i.e., A(1). In view of Lemma 2.1 it is enough to check if F_2 is A(3). Let $\{x_n\}_n$ be the set of jumps of F_2. We have

$$V(F_2;E) = \sum_n \big(|F_2(x_n-) - F_2(x_n)| + |F_2(x_n+) - F_2(x_n)|\big) < \infty,$$

so for a given $\varepsilon > 0$ there is an N with

$$\sum_{n>N} \big(|F_2(x_n-) - F_2(x_n)| + |F_2(x_n+) - F_2(x_n)|\big) < \varepsilon. \tag{1}$$

Consider a number $\delta > 0$ such that no interval with length less than δ contains more than one point among $x_1, \dots, x_N$. Take any sequence $(J_j)_j$ of intervals meeting E with $\sum_j |J_j| < \delta$. Each J_j contains at most one point x_n, $n \le N$. If $x_n \in J_j$, $n \le N$, put

$$I_{j1} = \big[\inf F_2(E_j^-), \sup F_2(E_j^-)\big],\ I_{j2} = \big[\inf F_2(E_j^+), \sup F_2(E_j^+)\big]$$

where $E_j^- = E \cap J_j \cap (-\infty, x_n)$, $E_j^+ = E \cap J_j \cap (x_n, \infty)$, and $I_{j3} = [F_2(x_n) - \frac{\varepsilon}{N}, F_2(x_n) + \frac{\varepsilon}{N}]$. In the opposite case put just

$$I_{j1} = \big[\inf F_2(E \cap J_j), \sup F_2(E \cap J_j)\big],\ I_{j2} = I_{j3} = \emptyset.$$

From (1) follows

$$\sum_j (|I_{j1}| + |I_{j2}| + |I_{j3}|) = \sum_j (|I_{j1}| + |I_{j2}|) + \sum_j |I_{j3}| < \varepsilon + 2N\frac{\varepsilon}{N} = 3\varepsilon.$$

Since clearly for each j, $I_{j1} \cup I_{j2} \cup I_{j3} \supset F_2(E \cap J_j)$, F_2 is indeed A(3). □

Remark 2.3. It is seen that if the F above is one-sided continuous, then it is A(2) (We don't need I_{j3}'s.), but it need not be so without this assumption (what we overlooked on page 26 in [11]). However, in the case from Lemma 2.2 one can always split E into a sequence of sets on which F is A(2) (by considering each discontinuity point separately).

Corollary 2.4. *The $\mathcal{F}_2$-integral is covered by the approximate Foran integral.*

3 Relation to $\mathcal{F}_4$-Integral.

Ene [3] gave various variational characterizations for Lusin's $\mathcal{N}$ condition in case of a *VB*-function (on *arbitrary* sets). We will use one of them (Lemma 3.1).

Let $E \subset D \subset \mathbb{R}$, $F\colon D \to \mathbb{R}$, $r > 0$. Let

$$V(F;E;r) = \sup \sum_{i=1}^{n} |F(b_i) - F(a_i)|, \quad V(F;E;0) = \inf_{r>0} V(F;E;r), \quad [9]$$

$$\mu_F(E) = \inf \sum_{m} V(F;E_m;0),$$

where sup ranges over all families of nonoverlapping closed intervals $\{[a_i,b_i]\}_{i=1}^{n}$ with both endpoints in E, such that $\sum_{i=1}^{n}(b_i - a_i) < r$. The second inf ranges over all E-forms $\{E_m\}_m$.

Note that an F is AC on a set E iff $V(F;E;0) = 0$. The lemma below follows from *Lemma 10* and *Theorem 4*, (iii)⇒(i), of [3].

Lemma 3.1. *A VB-function $F\colon E \to \mathbb{R}$ satisfies $\mathcal{N}$ iff $\mu_F(E) = 0$.*

As a consequence, the following can be deduced [3, Corollary 3].

Corollary 3.2. *For any $E \subset \mathbb{R}$, the class of VB-functions satisfying $\mathcal{N}$ defined on E, is an algebra.*

Let $V(F;D)$ stand for the ordinary variation of F on $D \subset E$; i.e., $V(F;D) = V(F;D;\infty)$. For a *VB*-function F on a bounded set E define the function $\mathcal{V}$ on E by

$$\mathcal{V}(x) = V(F;[a,x] \cap E), \tag{2}$$

where $a = \inf E$.

Lemma 3.3. *Let $E \subset \mathbb{R}$ be bounded. If a continuous VB-function $F\colon E \to \mathbb{R}$ satisfies $\mathcal{N}$, then so does $\mathcal{V}$.*

Proof. Our purpose is to prove $\mu_{\mathcal{V}}(E) = 0$. Take arbitrary $\varepsilon > 0$ and let an E-form $\{E_m\}_m$ be such that $\sum_m V(F;E_m;0) < \varepsilon$. For each m one can find a number $r_m > 0$ so that $\sum_m V(F;E_m;r_m) < \varepsilon$. Fix an m. Let (a_n,b_n), $n = 1,2,\dots$, be intervals contiguous to the closure of E_m. There is an N with

$$\sum_{n>N} (\mathcal{V}(b_n+) - \mathcal{V}(a_n-)) < \frac{\varepsilon}{2^m}.$$

Here and in what follows we agree that $\mathcal{V}(b_n+)$ ($\mathcal{V}(a_n-)$) means $\mathcal{V}(b_n)$ (resp. $\mathcal{V}(a_n)$) if b_n (resp. a_n) is right (resp. left) isolated in E. Take $s_m > 0$ less than r_m and $b_1 - a_1, \dots, b_N - a_N$. Take arbitrary nonoverlapping intervals $[c_1,d_1], \dots, [c_q,d_q]$ with both endpoints in E_m, such that $\sum_{i=1}^{q}(d_i - c_i) < s_m$. Notice that none of them overlaps with any among $[a_1,b_1], \dots, [a_N,b_N]$. Fix i and put

$$\mathcal{I}_i = \{n : \ (a_n,b_n) \subset [c_i,d_i]\}.$$

Clearly, $n > N$ for each $n \in \mathcal{I}_i$. Choose points $c_i = c_{i0} < c_{i1} < \cdots < c_{il} = d_i$ in E with

$$\mathcal{V}(d_i) - \mathcal{V}(c_i) = V(F; [c_i, d_i] \cap E) < \sum_{j=1}^{l} |F(c_{ij}) - F(c_{i,j-1})| + \frac{\varepsilon}{q2^m}. \tag{3}$$

Let $\mathcal{J}_i$ be the collection of all $n \in \mathcal{I}_i$ with some c_{ij} in (a_n, b_n). To simplify notation we assume that $b_{n'} \leq a_{n''}$ if $n' < n''$ and $n', n'' \in \mathcal{J}_i$. Since $\mathcal{V}$ is continuous, we can assume that if $c_{ij} \in (a_n, b_n)$ for some $n > N$ and j, then for the last j with $c_{ij} \leq a_n$, say $\alpha(n)$, and the first j with $c_{ij} \geq b_n$, say $\omega(n)$, we have $\mathcal{V}(a_n-) - \mathcal{V}(c_{i\alpha}) < \frac{\varepsilon}{lq2^{m+1}}$ and $\mathcal{V}(c_{i\omega}) - \mathcal{V}(b_n+) < \frac{\varepsilon}{lq2^{m+1}}$ (it is possible despite α, ω depend on l, since we would be to append here at most twice that many c_{ij}'s as members of $\mathcal{J}_i$). Notice that for an $n \in \mathcal{J}_i$

$$\sum_{j=\alpha+1}^{\omega} |F(c_{ij}) - F(c_{i,j-1})| < \mathcal{V}(b_n+) - \mathcal{V}(a_n-) + \frac{\varepsilon}{lq2^m}.$$

If $c_{ij} \notin (a_n, b_n)$ for any n, then $c_{ij} \in \operatorname{cl} E_m$. Up to continuity of F, there is a $\gamma_{ij} \in E_m \cap [c_i, d_i]$ such that

$$|F(c_{ij}) - F(\gamma_{ij})| < \frac{\varepsilon}{lq2^m}$$

and $\gamma_{ij'} < \gamma_{ij''}$ for any $j' < j''$ with $c_{ij'}, c_{ij''} \in \operatorname{cl} E_m$. Estimate (there is at most $l-1$ members of $\mathcal{J}_i$)

$$\begin{aligned}
&\sum_{j=1}^{l} |F(c_{ij}) - F(c_{i,j-1})| \\
&= \Big(\sum_{n \in \mathcal{J}_i} \sum_{\alpha(n)+1}^{\omega(n)} + \sum_{n \in \mathcal{J}_i} \sum_{\omega(n-1)+1}^{\alpha(n)} + \sum_{\omega(\max \mathcal{J}_i)+1}^{l} \Big) |F(c_{ij}) - F(c_{i,j-1})| \\
&\leq \sum_{n \in \mathcal{J}_i} \Big(\mathcal{V}(b_n+) - \mathcal{V}(a_n-) + \frac{\varepsilon}{lq2^m} \Big) \\
&\quad + \Big(\sum_{n \in \mathcal{J}_i} \sum_{\omega(n-1)+1}^{\alpha(n)} + \sum_{\omega(\max \mathcal{J}_i)+1}^{l} \Big) \big(|F(c_{ij}) - F(\gamma_{ij})| \\
&\quad + |F(\gamma_{ij}) - F(\gamma_{i,j-1})| + |F(c_{i,j-1}) - F(\gamma_{i,j-1})| \big) \\
&< \sum_{n \in \mathcal{J}_i} (\mathcal{V}(b_n+) - \mathcal{V}(a_n-)) + \frac{\varepsilon}{q2^m} + 2\frac{\varepsilon}{q2^m}
\end{aligned}$$

$$+\Big(\sum_{n\in\mathcal{J}_i}\sum_{\omega(n-1)+1}^{\alpha(n)}+\sum_{\omega(\max\mathcal{J}_i)+1}^{l}\Big)|F(\gamma_{ij})-F(\gamma_{i,j-1})|.$$

We agree that $\omega(\min\mathcal{J}_i-1)=0$. From (3), since $\gamma_{ij}\in E_m\cap[c_i,d_i]$ and $s_m\le r_m$, we get

$$\sum_{i=1}^{q}(\mathcal{V}(d_i)-\mathcal{V}(c_i))<\sum_{n>N}(\mathcal{V}(b_n+)-\mathcal{V}(a_n-))+3\frac{\varepsilon}{2^m}+V(F;E_m;r_m).$$

That means,

$$V(\mathcal{V};E_m;s_m)<4\frac{\varepsilon}{2^m}+V(F;E_m;r_m),$$

where from

$$\mu_{\mathcal{V}}(E)\le\sum_m V(\mathcal{V};E_m;0)\le\sum_m V(\mathcal{V};E_m;s_m)<5\varepsilon.$$

By Lemma 3.1, $\mathcal{V}\colon E\to\mathbb{R}$ satisfies $\mathcal{N}$. □

Remark 3.4. One could have dropped the continuity assumption in Lemma 3.3.

Lemma 3.5. *Let $F_1,F_2\colon E\to\mathbb{R}$ be monotone functions satisfying $\mathcal{N}$, and let $G\colon E\to\mathbb{R}$ be* A(N) *(on E). Then, the sum F_1+F_2+G satisfies $\mathcal{N}$ too.*

PROOF. Let a $D\subset E$ be of measure zero. Fix $\varepsilon>0$ and $l=1,2$. There is an open subset $O_l\supset F_l(D)$ with measure less than ε. Let $\{I_i^l\}_i$ be the family of its components. Put

$$K_i^l=\Big(\inf F_l^{-1}(I_i^l),\sup F_l^{-1}(I_i^l)\Big)\text{ and }K_{ij}=K_i^1\cap K_j^2.$$

Let δ be appropriate for ε in the sense of Definition 1.2. As $\bigcup_i K_i^l\supset D$, one can cover D with a sequence of open intervals $(M_k)_k$, so that $\sum_k|M_k|<\delta$ and so that each M_k is contained in some K_{ij}. For each k there are intervals $\Delta_{k1},\dots,\Delta_{kN}$ such that $G(D\cap M_k)\subset\bigcup_{j=1}^N\Delta_{kj}$ and $\sum_k\sum_{j=1}^N|\Delta_{kj}|<\varepsilon$. For a k and $l=1,2$ put

$$M_k^l=\Big[\inf F_l(D\cap M_k),\sup F_l(D\cap M_k)\Big].$$

Since F_l is monotone, M_k^l's are nonoverlapping, $l=1,2$. Moreover, for each k there are i and j with $M_k^1\subset I_i^1$, $M_k^2\subset I_j^2$. Thus, $\sum_k|M_k^l|<\varepsilon$, $l=1,2$. We have $F_l(D\cap M_k)\subset M_k^l$. Hence $(F_1+F_2)(D\cap M_k)$ is a subset of the interval $M_k^1+M_k^2$ with the length $|M_k^1|+|M_k^2|$. Consequently,

$$(F_1+F_2+G)(D\cap M_k)\subset\bigcup_{j=1}^N\big(\Delta_{kj}+M_k^1+M_k^2\big)$$

and so

$$\big|(F_1+F_2+G)(D\cap M_k)\big| \le \sum_{j=1}^{N}\big(|\Delta_{kj}|+|M_k^1|+|M_k^2|\big).$$

Thus the measure of $(F_1+F_2+G)(D)$ does not exceed

$$\sum_k\sum_{j=1}^{N}\big(|\Delta_{kj}|+|M_k^1|+|M_k^2|\big) \le N\sum_k\sum_{l=1}^{2}|M_k^l| + \sum_k\sum_{j=1}^{N}|\Delta_{kj}| < 2N\varepsilon+\varepsilon.$$

Hence, $|(F_1+F_2+G)(D)|=0$. $\square$

Corollary 3.6. *Let $F\colon E\to\mathbb{R}$ be a continuous VB-function with $\mathcal{N}$, $G\colon E\to\mathbb{R}$ a function with the* A(N) *property (on E). Then, their sum $F+G$ satisfies $\mathcal{N}$ too.*

Proof. By Corollary 3.2 and Lemma 3.3, the functions $\mathcal{V}^+=\frac{1}{2}(\mathcal{V}+F)$, $\mathcal{V}^-=\frac{1}{2}(\mathcal{V}-F)$ satisfy $\mathcal{N}$; $\mathcal{V}$ is given by (2). Since $\mathcal{V}^+,\mathcal{V}^-$ are nondecreasing, from Lemma 3.5 it follows that also $F+G=\mathcal{V}^+-\mathcal{V}^-+G$ satisfies $\mathcal{N}$. $\square$

For the proof of compatibility of $\mathcal{F}_4$- and AF-integrals we can employ the aforementioned O'Malley's monotonicity theorem [8]. Let us quote it.

Lemma 3.7. *Suppose that a Baire one $H\colon\mathbb{R}\to\mathbb{R}$ satisfies the following two conditions:*

(i) At each $x\in\mathbb{R}$

$$\overline{\lim\operatorname{ap}}_{t\to x^-} H(t) \le H(x) \le \overline{\lim\operatorname{ap}}_{t\to x^+} H(t).$$

(ii) The image

$$H\Big(\{x:\ \textit{upper right approximate derivative of } H \textit{ at } x \textit{ is not positive}\}\Big)$$

has void interior.

Then H is nondecreasing.

Theorem 3.8 together with Example 3.9 are the main result of this note.

Theorem 3.8. *Let a function $f\colon[a,b]\to\mathbb{R}$ be $\mathcal{F}_4$- and AF-integrable. Then both values of integral are equal. In other words, $\mathcal{F}_4$- and AF-integrals are compatible.*

PROOF. Let $F \in \mathcal{F}_4$ and let $G \in \mathfrak{F}$ be primitives for f. We apply Lemma 3.7. Take arbitrary $\varepsilon > 0$ and consider an auxiliary function H defined by

$$H(x) = \begin{cases} F(x) - G(x) + \varepsilon x & \text{for } x \in [a, b] \\ H(b) & \text{for } x > b \\ H(a) & \text{for } x < a. \end{cases}$$

Clearly, H is approximately continuous (hence Baire one), so (i) holds. We have

$$H'_{\mathrm{ap}}(x) = \varepsilon > 0 \text{ almost everywhere on } \mathbb{R}. \tag{4}$$

There is an $[a, b]$-form $\{D_n\}_n$ such that F is *VB* and G is $\mathrm{A}(N_n)$ on each D_n. By Lemma 2.1, $x \mapsto \varepsilon x - G(x)$ is $\mathrm{A}(N_n)$ on D_n. We can assume that F restricted to D_n is continuous. By Corollary 3.6, H satisfies $\mathcal{N}$ on each D_n; hence so on all of $\mathbb{R}$. Together with (4) this gives (ii). From Lemma 3.7 we get $H(b) - H(a) \geq 0$. The arbitrariness of ε implies $F(b) - F(a) \geq G(b) - G(a)$. The converse inequality can be justified in a similar way. $\square$

Example 3.9. *For $\mathcal{F}_4$-integral and the approximate Foran integral there are functions integrable in one sense and not in the other.*

PROOF. The function F from [10, Example 3.5] is an approximately continuous *VBG*-function with $\mathcal{N}$. However, it is not a primitive for *AF*-integral since it coincides with the Cantor's ternary function on a $\mathcal{G}_\delta$ dense subset of $\mathbb{C}$. By Theorem 3.8 its derivative is not *AF*-integrable. On the other hand, there are primitives for *AF*-integral (even for the ordinary Foran integral) without *VBG*, see [2]; their approximate derivatives cannot be $\mathcal{F}_4$-integrable (Theorem 3.8 again). $\square$

$$\begin{array}{ccccccccc}
 & & & \subsetneq & \mathcal{F}_2 & \subsetneq & & \subsetneq & \mathcal{F}_4 \\
\mathcal{F}_1 & \subsetneq & \mathcal{F}_2 \cap \mathcal{F}_3 & & \not\subset \;\; \not\supset & & \mathcal{F}_2 + \mathcal{F}_3 & & \not\subset \;\; \not\supset \\
 & & & \subsetneq & \mathcal{F}_3 & \subsetneq & & \subsetneq & AF
\end{array}$$

Question 3.10. *Is the inclusion $\mathcal{F}_2 + \mathcal{F}_3 \subset \mathcal{F}_4 \cap AF$, strict?*

4 $\mathcal{F}_3$-Integral Versus $[AF]$-Integral.

In order to make the chart above more self-complete, we would like to fill it also with the integral given by the class of primitives that are defined with the property $\mathrm{A}(N)$ and *closed* $[a,b]$-forms. We shall not however, extend the chart further graphically, as the result produced would not be too neat. We will only explain how this should be done.

Definitions 4.1. We say that a function $G\colon [a,b] \to \mathbb{R}$ belongs to the class $[\mathfrak{F}]$, if there is a closed $[a,b]$-form $\{E_n\}_{n=1}^{\infty}$ with a suitable sequence of integers $\{N_n\}_{n=1}^{\infty}$ such that for each n, G is $\mathrm{A}(N_n)$ on E_n. We say that an $f\colon [a,b] \to \mathbb{R}$ is $[AF]$-integrable, $f \in [AF]$ for short, if there exists an approximately continuous function $G \in [\mathfrak{F}]$ such that $G'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. The integral of f is defined to be $G(b) - G(a)$.

The immediate relations are: $\mathcal{F}_1 \subset \mathcal{F}_2 \cap \mathcal{F}_3 \subset \mathcal{F}_2 \subsetneq [AF]$(Lemma 2.2), $[AF] \subset AF$, $[AF] \not\subset \mathcal{F}_4 \supset \mathcal{F}_2 + \mathcal{F}_3 \supset \mathcal{F}_3$, $[AF] \not\supset \mathcal{F}_4$. The rest depends on the second relation between $\mathcal{F}_3$ and $[AF]$.

Theorem 4.2. *We have* $\mathcal{F}_3 \not\subset [AF]$.

PROOF. We will consider the function $F\colon [0,1] \to \mathbb{R}$ from [10, Example 3.1]. (The notation used below comes from therein.) It has been shown in [10] that F is an approximately continuous ACG-function, but it is not $[VBG]$. In view of Theorem 3.8, it is enough to show $F \notin [\mathfrak{F}]$.

Suppose not. By the Baire Category Theorem, there is a portion $J_l^{(m)} \cap \mathbb{C}$, $l \in \{1, \dots, 2^m\}$, of $\mathbb{C}$ such that F is $\mathrm{A}(N)$ on $J_l^{(m)} \cap \mathbb{C}$ for some N. Take arbitrary $\delta > 0$ and pick an $r \in \mathbb{N}$ with

$$\sum_{k=2^r(l-1)+1}^{2^r l} \left|J_k^{(m+r)}\right| < \delta.$$

Note that $J_l^{(m)} \cap \mathbb{C} = \bigcup_{k=2^r(l-1)+1}^{2^r l} J_k^{(m+r)} \cap \mathbb{C}$. Let $(I_i^k)_{i=1}^N$, $k = 2^r(l-1) + 1, \dots, 2^r l$, be intervals satisfying

$$\bigcup_{i=1}^{N} I_i^k \supset F\big(J_k^{(m+r)} \cap \mathbb{C}\big).$$

According to the definition of F, for each $k = 2^r(l-1)+1, \dots, 2^r l$ and for each $s \geq m+r$, $\frac{1}{s} \in F\big(J_k^{(m+r)} \cap \mathbb{C}\big)$. So, for each k either

$$\frac{1}{t}, \frac{1}{t-1} \in I_{i_k}^k \tag{5}$$

for some $t \in \{m+r+1, \dots, m+r+N\}$ and i_k, or

$$\left[0, \frac{1}{m+r+N+1}\right] \subset I_{i_k}^k, \tag{6}$$

for some i_k. In case (5) we have

$$\sum_{i=1}^{N} |I_i^k| \geq \left|I_{i_k}^k\right| \geq \frac{1}{t-1} - \frac{1}{t} = \frac{1}{t(t-1)} \geq \frac{1}{(m+r+N)(m+r+N-1)},$$

while in (6), $\sum_{i=1}^{N} |I_i^k| \geq \left|I_{i_k}^k\right| \geq \frac{1}{m+r+N+1}$. So,

$$\sum_{k=2^r(l-1)+1}^{2^r l} \sum_{i=1}^{N} |I_i^k| \geq \frac{2^r}{(m+r+N)(m+r+N-1)}.$$

As r could have been taken arbitrarily large, m and N are fixed, we arrived at a contradiction. The function F does not belong to $[\mathcal{F}]$. $\square$

Corollary 4.3. *$[AF] \subsetneq AF$, $[AF] \not\supset \mathcal{F}_2 + \mathcal{F}_3$.*

5 $\mathcal{D}_F$ is Nowhere Dense.

We conclude the note with a continuity property of primitives for AF-integral. The result is analogous to that for $\mathcal{F}_4$-integral [10, Theorem 4.4].

Definition 5.1. We say that a real function F is quasi-continuous if the set $F \restriction \mathcal{C}_F$ is dense in F (in the sense of graphs).

The following lemma we borrow from [1].

Lemma 5.2. *Every Darboux Baire one function (on an interval) which satisfies $\mathcal{N}$ is quasi-continuous.*

Theorem 5.3. *Let an $F \in \mathcal{F}$ be approximately continuous. Then, the set $\mathcal{D}_F$ (of discontinuity points of F) is nowhere dense.*

PROOF. Suppose not. By the Baire Category Theorem there is an interval I such that F is $\mathrm{A}(N)$ on a dense subset E of $\mathcal{C}_F \cap I$, and such that $\mathcal{D}_F \cap \operatorname{int} I \neq \emptyset$. Take an $x \in \mathcal{D}_F \cap \operatorname{int} I$ and pick a sequence of points $x_n^0 \in I$ with $x_n^0 \to x$ and $|F(x_n^0) - F(x)| \geq M > 0$ for all n. We may assume $(x_n^0)_{n=1}^{\infty}$ is decreasing and also $F(x_n^0) - F(x) \geq M$ for all n. (If not, we would just pass to a subsequence or consider $-F$ in place of F.) Since F is Darboux, the right cluster set of F at x contains the interval $[F(x), F(x)+M]$; i.e., each member of this interval is a right limit point of F at x. Hence, for $i = 1, \ldots, N$ there is a sequence $(x_n^i)_{n=1}^{\infty}$, $x_n^i \searrow x$, with

$$\begin{aligned} F(x) \leq F(x) + M\frac{2N-2i}{2N} &\leq F(x_n^i) \\ &\leq F(x) + M\frac{2N-2i+1}{2N} < F(x) + M \end{aligned} \tag{7}$$

for each n. Sequences $(x_n^i)_n$, $i = 0, \dots, N$, may be found step by step so that

$$x_n^0 > \dots > x_n^N > x_{n+1}^0$$

for each $n \geq 1$.

By Lemma 5.2, F is quasi-continuous. Set $z_0^N = \max I$. Since E is dense in $\mathcal{C}_F \cap I$, from quasi-continuity of F it follows that for each $n = 1, 2, \dots$ there are points $z_n^0 \in E \cap (x_n^1, z_{n-1}^N)$, $z_n^i \in E \cap (x_n^{i+1}, z_n^{i-1})$, $i = 1, \dots, N-1$, $z_n^N \in E \cap (x_{n+1}^0, z_n^{N-1})$, such that $|F(z_n^i) - F(x_n^i)| < \frac{M}{8N}$, $i = 0, \dots, N$. Fix an n. If $i \neq j$, by (7) we have

$$|F(x_n^i) - F(x_n^j)| \geq \frac{M}{2N},$$

whence

$$\begin{aligned} |F(z_n^i) - F(z_n^j)| &\geq |F(x_n^i) - F(x_n^j)| - |F(x_n^i) - F(z_n^i)| \\ &\quad - |F(x_n^j) - F(z_n^j)| \geq \frac{M}{2N} - 2\frac{M}{8N} = \frac{M}{4N}. \end{aligned} \tag{8}$$

Choose δ for $\varepsilon = 1$ according to Definition 1.2 (for F being A(N) on E). There is an n_0 with $z_{n_0}^0 - x < \delta$. Clearly, $\sum_{n=n_0}^{\infty}(z_n^0 - z_n^N) < \delta$. For each $n \geq n_0$ consider any intervals $I_{n1}, \dots, I_{nN}$ covering $F(E \cap [z_n^N, z_n^0])$. At least one of them, say I_{ni_n}, must contain two points among $F(z_n^0), \dots, F(z_n^N)$. Thus by (8) we have $|I_{ni_n}| \geq \frac{M}{4N}$. Summing this up over all $n \geq n_0$, we obtain

$$\sum_{n=n_0}^{\infty} \sum_{i=1}^{N} |I_{ni}| \geq \sum_{n=n_0}^{\infty} |I_{ni_n}| \geq \sum_{n=n_0}^{\infty} \frac{M}{4N} = \infty > 1,$$

a contradiction. □

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