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ON LUZIN'S (N)-PROPERTY OF THE SUM OF TWO FUNCTIONS

Abstract

We prove that, for any nonconstant continuous function f, there exists a continuous N-function g such that $f+g$ is not an N-function. This answers a query by F. S. Cater.

A function $f:[0,1]\to\mathbb{R}$ is said to have Luzin's (N)-property (or to be an N-function), if for every set $M\subset[0,1]$ of Lebesgue measure zero, the set $f(M)$ has Lebesgue measure zero as well. In [2] S. Mazurkiewicz found an N-function $g:[0,1]\to\mathbb{R}$ such that $f+g$ is not an N-function for any nonconstant linear function f. In reference [1], F. S. Cater posed this question. For any nonconstant continuous N-function f, must there exist a continuous N-function g, depending on f, such that $f+g$ is not an N-function? Using Mazurkiewicz's method, we will prove that the answer is positive.

We will use the following notation.

For a continuous function $f:[0,1]\to\mathbb{R}$, we define the mapping $\Phi_f: [0,1]\times\mathbb{R}\to[0,1]\times\mathbb{R}$ by $\Phi_f(x,y)=(x,y+f(x))$.

For a closed interval $A=[a,b]\times[c,d]\subset\mathbb{R}^2$, let $l_A=a$, $r_A=b$, $b_A=c$, $t_A=d$, and define the set $\Psi(A)\subset\mathbb{R}^2$ by

$$\Psi(A)=\{l_A\}\times[b_A,t_A]\cup\{r_A\}\times[b_A,t_A].$$

The one dimensional Lebesgue measure of a set M will be denoted by $|M|$, P_x and P_y we will use for the orthogonal projections on the axes.

Theorem. *Let $f:[0,1]\to\mathbb{R}$ be a nonconstant continuous function. Then there is a continuous N-function $g:[0,1]\to\mathbb{R}$ and a set M of Lebesgue measure zero, such that $(f+g)(M)$ contains an interval.*

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PROOF. Let $G \subset (0,1)$ be the maximal open set on which f is locally constant. Write $G = \bigcup_{i\in I}(\alpha_i, \beta_i)$, where I is countable and intervals (α_i, β_i) are pairwise disjoint. Put $G^* := \bigcup_{i\in I}[\alpha_i, \beta_i]$, $J := (0,1) \setminus G$ and $J^* := (0,1) \setminus G^*$. Then clearly J is nonempty and has no isolated points. It is not difficult to check that:

(I) J^* is dense in J.

(II) For each $\gamma > 0$ and $x \in J^*$ there are $x_l, x_r \in (x-\gamma, x+\gamma) \cap J^*$ such that $x_l < x < x_r$, $f(x_l) \neq f(x)$ and $f(x_r) \neq f(x)$.

We will construct inductively a sequence $\{Q_n\}_{n=1}^{\infty}$ of nonempty finite systems of closed intervals in $[0,1] \times \mathbb{R}$ such that following conditions hold:

(i) If $n > 1$, then $\bigcup Q_n \subset \bigcup Q_{n-1}$.

(ii) For any $A, B \in Q_n$, $A \neq B$ we have $P_y(A) = P_y(B)$ or $P_y(A) \cap P_y(B) = \emptyset$. $|P_y(A)| = |P_y(B)|$ (Denote this constant value by V_n.) and

$$P_x(A) \cap P_x(B) = \emptyset.$$

(iii) If $n > 1$, then

$$\Big|P_y\Big(\bigcup Q_n\Big)\Big| \leq \frac{2}{3}\Big|P_y\Big(\bigcup Q_{n-1}\Big)\Big| \text{ and } \Big|P_x\Big(\bigcup Q_n\Big)\Big| \leq \frac{1}{2}\Big|P_x\Big(\bigcup Q_{n-1}\Big)\Big|.$$

(iv) If $n > 1$, then

$$P_y\Big(\Phi_f\Big(\bigcup_{A\in Q_{n-1}} \Psi(A)\Big)\Big) \subset P_y\Big(\Phi_f\Big(\bigcup_{A\in Q_n} \Psi(A)\Big)\Big).$$

(v) For $A \in Q_n$ we have $l_A, r_A \in J^*$ and $f(l_A) \neq f(r_A)$.

By (II) we can choose $a, b \in J^*$, $a < b$, such that $f(a) \neq f(b)$. Then put

$$Q_1 = \{[a,b] \times [0,1]\}.$$

If we have defined the system Q_n, then Q_{n+1} will be obtained by the following construction.

Set $R = \min_{A\in Q_n} |f(l_A) - f(r_A)|$. Thus $R > 0$ and there exists an odd l, $l > 2$, such that $\frac{2}{l}V_n < R$. Now fix $A \in Q_n$ and assume that $f(l_A) < f(r_A)$. Choose $N \in \mathbb{N}$ such that

$$\frac{f(r_A) - f(l_A)}{N} < \frac{1}{4l}V_n.$$

For $j = 0, \ldots, N-1$ set

$$d_j = \min\Big\{x \in [l_A, r_A] : f(x) = f(l_A) + \frac{j}{N}(f(r_A) - f(l_A))\Big\}$$

and put $d_N = r_A$. Then we have

$$l_A = d_0 < d_1 < \cdots < d_{N-1} < d_N = r_A,$$

$$d_j \in [l_A, r_A] \cap J \quad \text{for all} \quad j = 0, \ldots, N,$$

and

$$f(d_{j+1}) - f(d_j) = \frac{f(r_A) - f(l_A)}{N} < \frac{1}{4l} V_n.$$

Put $K = \frac{l+1}{2}$ and $\delta_1 = \frac{1}{3} \min_{j=1,\ldots,N-1}(d_{j+1} - d_j)$. Since f is continuous, we can find $\delta_2, \delta_3 > 0$ such that:

$$\text{if } x \in (d_j - \delta_2, d_j + \delta_2) \cap [l_A, r_A], \text{ then } |f(d_j) - f(x)| < \frac{1}{4l} V_n,$$

$$\text{if } x \in (l_A, l_A + \delta_3) \cap [l_A, r_A], \text{ then } |f(l_A) - f(x)| < \frac{1}{2}\big(R - \frac{2}{l} V_n\big),$$

$$\text{if } x \in (r_A - \delta_3, r_A) \cap [l_A, r_A], \text{ then } |f(l_A) - f(x)| < \frac{1}{2}\big(R - \frac{2}{l} V_n\big).$$

Set $\delta = \min(\delta_1, \delta_2, \delta_3)$ and by (I) find points

$$u_i^j \in (d_j - \delta, d_j + \delta) \cap [l_A, r_A] \cap J^*, \; i = 1, \ldots, K, \; j = 0, \ldots, N-1,$$

and points

$$v_i^N \in (r_A - \delta, r_A] \cap J^*, \; i = 1, \ldots, K,$$

such that $l_A = u_1^0$ and for each $j = 0, \ldots, N-1$, we have

$$u_1^j < u_2^j < \cdots < u_K^j \text{ and } v_1^N < v_2^N < \cdots < v_K^N = r_A.$$

Further choose

$$v_i^j \in [l_A, r_A] \cap J^*, \; i = 1, \ldots, K, \; j = 0, \ldots, N-1,$$
$$u_i^N \in [l_A, r_A] \cap J^*, \; i = 1, \ldots, K,$$

such that:

$$u_i^j < v_i^j \quad \text{for each } i, j,$$

$$\text{intervals } [u_i^j, v_i^j] \text{ are pairwise disjoint,} \tag{1}$$

$$\sum_{i,j} (v_i^j - u_i^j) \leq \frac{1}{2}(r_A - l_A) \tag{2}$$

and

$$0 < |f(u_i^j) - f(v_i^j)| < \frac{1}{2l} V_n. \tag{3}$$

To finish the construction put

$$A_i^j = [u_i^j, v_i^j] \times \Big[b_A + \frac{2i-2}{l}V_n, b_A + \frac{2i-1}{l}V_n\Big]$$

and

$$Q^A = \big\{A_i^j : i = 1, \dots, K, j = 0, \dots, N\big\}.$$

The second inequality in (3) implies that each set $P_y(\Phi_f(\Psi(A_i^j)))$ is connected. For any $i = 1, \dots, K$ and any $j = 0, \dots, N-1$ we have

$$\begin{aligned} |f(u_i^j) - f(u_i^{j+1})| \leq & |f(u_i^j) - f(d_j)| \\ & + |f(d_j) - f(d_{j+1})| + |f(d_{j+1}) - f(u_i^{j+1})| \leq \frac{3}{4l}V_n, \end{aligned}$$

so $P_y(\Phi_f(\bigcup_{j=0}^N \Psi(A_i^j)))$ is connected as well. Since

$$\begin{aligned} &\Big(f(v_i^N) + b_A + \frac{2i-1}{l}V_n\Big) - \Big(f(v_{i+1}^1) + b_A + \frac{2(i+1)-2}{l}V_n\Big) \\ =& f(v_i^j) - f(v_{i+1}^1) - \frac{1}{l}V_n \\ \geq& f(r_A) - \big(R - \frac{2}{l}V_n\big) - f(l_A) - \big(R - \frac{2}{l}V_n\big) \geq \frac{1}{l}V_n > 0, \end{aligned}$$

the set $P_y\big(\Phi_f\big(\bigcup_{B\in Q^A} \Psi(B)\big)\big)$ is connected. Due to the fact that

$$f(l_A) + b_A \in P_y(\Phi_f(\Psi(A_1^0))) \text{ and } f(r_A) + t_A \in P_y(\Phi_f(\Psi(A_K^N))),$$

we have

$$P_y\big(\Phi_f\big(\bigcup_{B\in Q^A} \Psi(B)\big)\big) \supset P_y(\Phi_f(\Psi(A))). \tag{4}$$

In the case $f(l_A) > f(r_A)$ we use the above construction for

$$A' = [l_A, r_A] \times [-t_A, -b_A]$$

and the function $-f$. Denote the system constructed in this way by $Q^{A'}$ and put

$$Q^A = \big\{[l_B, r_B] \times [-t_B, -b_B] : B \in Q^{A'}\big\}.$$

Finally set $Q_{n+1} = \bigcup_{A\in Q_n} Q^A$.

The condition (i) is clear. To verify condition (ii) choose $C, D \in Q_{n+1}$. There exist $A, B \in Q_n$ such that $C \subset A$ and $D \subset B$. By the induction hypothesis, we have $P_y(A) = P_y(B)$ or $P_y(A) \cap P_y(B) = \emptyset$. In the second case we have

$$P_y(C) \subset P_y(A) \text{ and } P_y(D) \subset P_y(B).$$

In the first case we have

$$P_y(C) = [b_C, t_C] = \Big[b_A + \frac{2i-2}{l}V_n, b_A + \frac{2i-1}{l}V_n\Big]$$

and

$$P_y(D) = [b_D, t_D] = \Big[b_A + \frac{2i'-2}{l}V_n, b_A + \frac{2i'-1}{l}V_n\Big]$$

for some $1 \le i, i' \le K$. Obviously, if $i = i'$, then $P_y(C) = P_y(D)$. In the case $i \ne i'$ we have $P_y(C) \cap P_y(D) = \emptyset$, because of the fact that

$$|b_C - b_D| \ge 2\frac{V_n}{l} \text{ and } |t_C - b_C| = |t_D - b_D| = \frac{V_n}{l} = V_{n+1}.$$

The last part of (ii) follows from the induction hypothesis and the fact that

$$P_x(C) \subset P_x(A) \text{ and } P_x(D) \subset P_x(B) \text{ if } A \ne B.$$

If $A = B$, it follows from (1). The first part of (iii) holds, since

$$\Big|P_y\big(\bigcup Q_{n+1}\big)\Big| = \Big|\bigcup_{A\in Q_{n+1}} P_y(A)\Big| = \frac{l+1}{2l}\Big|\bigcup_{A\in Q_n} P_y(A)\Big| \le \frac{2}{3}\Big|P_y\big(\bigcup_{A\in Q_n} A\big)\Big|.$$

The second part of (iii) follows from (ii) and (2). Finally (iv) follows from (4) and (v) from (3).

Define $L_n = \bigcup Q_n$. Using (ii) we have that $L = \bigcap_{n=1}^{\infty} L_n$ is nonempty compact. Moreover, due to the fact that $V_n \to 0$, we see that L is a graph of a continuous function h defined on $P_x(L)$. Now extend h linearly on the components of $[0,1] \setminus P_x(L)$ to a continuous function g defined on $[0,1]$. From (iii) we have $|P_x(L)| = |P_y(L)| = 0$. By this and the fact, that linear functions have Luzin's property (N), we have for any set $M \subset [0,1]$ of Lebesgue measure zero

$$\begin{aligned} |f(M)| \le& |f(M \cap P_x(L))| + |f(M \setminus P_x(L))| \\ \le& |P_y(L)| + |f(M \setminus P_x(L))| = 0, \end{aligned}$$

so g is an N-function as well. On the other hand, due to the compactness of the sets L_n and by (iv) we have

$$(f+g)(L) \supset P_y\big(\Phi_f\big(\bigcup_{A\in Q_1} \Psi(A)\big)\big).$$

To complete the proof it is sufficient to observe that the set on the right side contains interval $[f(l_A), f(l_A)+1]$, where A is the interval in Q_1. □

Remark. a) Note that if we start the construction with a system $Q_1 = \{[a,b] \times [0,\tau]\}$ for some $\tau > 0$, we can construct g such that $|g| \leq \tau$ on $[a,b]$.

b) Let $\mathcal{F}$ be a system of nonconstant continuous functions on $[0,1]$. We can ask, whether there exists a continuous N-function h such that $f+h$ is not an N-function for any $f \in \mathcal{F}$.

Suppose that there are $c, C > 0$ such that for any $f, g \in \mathcal{F}$ and any $x, y \in [0,1]$ satisfying $g(x) \neq g(y)$ we have

$$c \leq \frac{f(x)-f(y)}{g(x)-g(y)} \leq C.$$

By changing some details in the construction described above, we can obtain that in this case the answer is positive. (This condition implies that the set J^* is identical for all functions in $\mathcal{F}$, the numbers l and N from the construction can be chosen uniformly for all functions in $\mathcal{F}$, depending only on c and C respectively.) Moreover, using a), it is not difficult to show that the answer is also positive, if $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and each $\mathcal{F}_i$ has the above property on the interval $[\frac{1}{2i+1}, \frac{1}{2i}]$, $i \in \mathbb{N}$. In particular, if $\mathcal{F}$ is the system of all nonconstant linear functions, this gives the original Mazurkiewicz's result.

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References

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