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Teodora-Liliana Dinu, Department of Mathematics, "Fraţii Buzeşti" College, Bd. Ştirbei–Vodă No. 5, 200352 Craiova, Romania.
email: tldinu@gmail.com

VARIATIONAL METHODS IN THE STUDY OF INEQUALITY PROBLEMS FOR NONLINEAR ELLIPTIC SYSTEMS WITH LACK OF COMPACTNESS

Abstract

We establish the existence of an entire weak solution for a class of stationary Schrödinger systems with subcritical discontinuous nonlinearities and lower bounded potentials that blow-up at infinity. The proof relies on Chang's version of the Mountain Pass Lemma for locally Lipschitz functionals. Our result generalizes in a nonsmooth framework, a result of Rabinowitz [12] related to entire solutions of the Schrödinger equation.

1 Introduction and the Main Result.

In quantum mechanics, the Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics; that is, it predicts the future behavior of a dynamic system. The linear form of Schrödinger's equation is

$$\Delta\psi + \frac{8\pi^2 m}{\hbar^2}\left(E(x) - V(x)\right)\psi = 0$$

where ψ is the Schrödinger wave function, m is the mass, $\hbar$ denotes Planck's constant, E is the energy, and V stands for the potential energy. The structure of the nonlinear Schrödinger equation is much more complicated. This equation describes various phenomena arising: in self-channelling of a high-power ultra-short laser in matter, in the theory of Heisenberg ferromagnets

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and magnons, in dissipative quantum mechanics, in condensed matter theory, and in plasma physics (e.g., the Kurihara superfluid film equation). We refer to [8] for a modern overview, including applications.

Consider the model problem

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \qquad \text{in } \mathbb{R}^N \ (N \geq 2), \tag{1}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$. In the study of this equation Oh [11] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some a) introduced in Kato [9]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov-Schmidt type reduction, Oh [11] proved the existence of a standing wave solution of Problem (1), i.e. a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x). \tag{2}$$

Note that substituting (2) into (1) leads to

$$-\frac{\hbar^2}{2}\Delta u + (V(x) - E)\, u = |u|^{p-1}u.$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2\,(V_\hbar(x) - E)\, u = |u|^{p-1}u \text{ in } \mathbb{R}^N,$$

where $V_\hbar(x) = V(\hbar x)$.

In a celebrated paper, Rabinowitz [12] continued the study of standing wave solutions of nonlinear Schrödinger equations. After constructing a standing wave equation, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + b(x)u = f(x,u) \text{ in } \mathbb{R}^N,$$

under suitable conditions on b and assuming that f is smooth, super-linear and subcritical.

Inspired by Rabinowitz' paper, we consider the class of coupled elliptic systems in $\mathbb{R}^N$ $(N \geq 3)$

$$\begin{cases} -\Delta u_1 + a(x)u_1 = f(x,u_1,u_2) & \text{in } \mathbb{R}^N \\ -\Delta u_2 + b(x)u_2 = g(x,u_1,u_2) & \text{in } \mathbb{R}^N. \end{cases} \tag{3}$$

We point out that coupled nonlinear Schrödinger systems describe some physical phenomena such as the propagation in birefringent optical fibers or

Kerr-like photorefractive media in optics. Another motivation to the study of coupled Schrödinger systems arises from the Hartree-Fock theory for the double condensate, i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states, cf. [6].

Throughout this paper we assume that $a, b \in L^\infty_{\rm loc}(\mathbb{R}^N)$ and there exist $\underline{a}\,,\ \underline{b} > 0$ such that $a(x) \geq \underline{a}\,,\ \ b(x) \geq \underline{b}$ a.e. in $\mathbb{R}^N$, and $\mathrm{esslim}_{|x|\to\infty} a(x) = \mathrm{esslim}_{|x|\to\infty} b(x) = +\infty$. Our aim in this paper is to study the existence of solutions to the above problem in the case when f, g are not continuous functions. Our goal is to show how variational methods can be used to find existence results for stationary nonsmooth Schrödinger systems.

Throughout this paper we assume that $f(x,\cdot,\cdot)$, $g(x,\cdot,\cdot) \in L^\infty_{\rm loc}(\mathbb{R}^2)$. Let

$$\begin{aligned}
\underline{f}(x,t_1,t_2) &= \lim_{\delta\to 0} \mathrm{essinf}\{f(x,s_1,s_2); |t_i - s_i| \leq \delta; i = 1,2\}\\
\overline{f}(x,t_1,t_2) &= \lim_{\delta\to 0} \mathrm{esssup}\{f(x,s_1,s_2); |t_i - s_i| \leq \delta; i = 1,2\}\\
\underline{g}(x,t_1,t_2) &= \lim_{\delta\to 0} \mathrm{essinf}\{g(x,s_1,s_2); |t_i - s_i| \leq \delta; i = 1,2\}\\
\overline{g}(x,t_1,t_2) &= \lim_{\delta\to 0} \mathrm{esssup}\{g(x,s_1,s_2); |t_i - s_i| \leq \delta; i = 1,2\}.
\end{aligned}$$

Under these conditions we reformulate Problem (3) as

$$\begin{aligned}
&-\Delta u_1 + a(x)u_1 \in [\underline{f}(x,u_1(x),u_2(x)), \overline{f}(x,u_1(x),u_2(x))] \text{ a.e. } x \in \mathbb{R}^N\\
&-\Delta u_2 + b(x)u_2 \in [\underline{g}(x,u_1(x),u_2(x)), \overline{g}(x,u_1(x),u_2(x))] \text{ a.e. } x \in \mathbb{R}^N.
\end{aligned} \tag{4}$$

Let $H^1 = H(\mathbb{R}^N, \mathbb{R}^2)$ be the Sobolev space of all $U = (u_1,u_2) \in (L^2(\mathbb{R}^N))^2$ with weak derivatives $\dfrac{\partial u_1}{\partial x_j}, \dfrac{\partial u_2}{\partial x_j} (j = 1,\ldots,N)$ also in $L^2(\mathbb{R}^N)$, endowed with the usual norm

$$\|U\|^2_{H_1} = \int_{\mathbb{R}^N} (|\nabla U|^2 + |U|^2)\,dx = \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2 + u_1^2 + u_2^2)\,dx.$$

Given the functions $a, b : \mathbb{R}^N \to \mathbb{R}$ as above, define the subspace

$$E = \{U = (u_1,u_2) \in H^1; \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\,dx < +\infty\}.$$

Then the space E endowed with the norm

$$\|U\|^2_E = \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\,dx$$

becomes a Hilbert space.

Since $a(x) \geq \underline{a} > 0, b(x) \geq \underline{b} > 0$, we have the continuous embeddings $H^1 \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ for all $2 \leq q \leq 2^* = 2N/(N-2)$.

We assume throughout the paper that $f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are nontrivial measurable functions satisfying the hypotheses

$$\begin{cases} |f(x,t)| \leq C(|t| + |t|^p) & \text{for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2 \\ |g(x,t)| \leq C(|t| + |t|^p) & \text{for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2 \end{cases} \tag{5}$$

where $p < 2^*$;

$$\begin{cases} \lim\limits_{\delta \to 0} \operatorname{esssup} \left\{ \frac{|f(x,t)|}{|t|}; (x,t) \in \mathbb{R}^N \times (-\delta, +\delta)^2 \right\} = 0 \\ \lim\limits_{\delta \to 0} \operatorname{esssup} \left\{ \frac{|g(x,t)|}{|t|}; (x,t) \in \mathbb{R}^N \times (-\delta, +\delta)^2 \right\} = 0; \end{cases} \tag{6}$$

f and g are chosen so that the mapping $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, t_1, t_2) := \int_0^{t_1} f(x, \tau, t_2)\, d\tau + \int_0^{t_2} g(x, 0, \tau)\, d\tau$ satisfies

$$\begin{cases} F(x, t_1, t_2) = \int\limits_0^{t_2} g(x, t_1, \tau)\, d\tau + \int\limits_0^{t_1} f(x, \tau, 0)\, d\tau \\ \text{and } F(x, t_1, t_2) = 0 \text{ if and only if } t_1 = t_2 = 0; \end{cases} \tag{7}$$

and there exists $\mu > 2$ such that for any $x \in \mathbb{R}^N$

$$0 \leq \mu F(x, t_1, t_2) \leq \begin{cases} t_1 \underline{f}(x, t_1, t_2) + t_2 \underline{g}(x, t_1, t_2); & t_1, t_2 \in [0, +\infty) \\ t_1 \underline{f}(x, t_1, t_2) + t_2 \overline{g}(x, t_1, t_2); & t_1 \in [0, +\infty), t_2 \in (-\infty, 0] \\ t_1 \overline{f}(x, t_1, t_2) + t_2 \overline{g}(x, t_1, t_2); & t_1, t_2 \in (-\infty, 0] \\ t_1 \overline{f}(x, t_1, t_2) + t_2 \underline{g}(x, t_1, t_2); & t_1 \in (-\infty, 0], t_2 \in [0, +\infty). \end{cases} \tag{8}$$

Definition 1. *A function $U = (u_1, u_2) \in E$ is called solution to the problem (4) if there exists a function $W = (w_1, w_2) \in L^2(\mathbb{R}^N, \mathbb{R}^2)$ such that*

(i) $\underline{f}(x, u_1(x), u_2(x)) \leq w_1(x) \leq \overline{f}(x, u_1(x), u_2(x))$ *a.e. x in $\mathbb{R}^N$;*
$\underline{g}(x, u_1(x), u_2(x)) \leq w_2(x) \leq \overline{g}(x, u_1(x), u_2(x))$ *a.e. x in $\mathbb{R}^N$;*

(ii) $\int\limits_{\mathbb{R}^N} (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x) u_1 v_1 + b(x) u_2 v_2)\, dx = \int\limits_{\mathbb{R}^N} (w_1 v_1 + w_2 v_2)\, dx,$
for all $(v_1, v_2) \in E$.

Our main result is the following.

Theorem 1. *Assume that conditions (5) - (8) are fulfilled. Then Problem (4) has at least a nontrivial solution in E.*

2 Auxiliary Results.

We first recall some basic notions from the Clarke gradient theory for locally Lipschitz functionals (see [4, 5] for more details). Let E be a real Banach space and assume that $I : E \to \mathbb{R}$ is a locally Lipschitz functional. Then the Clarke generalized gradient is defined by

$$\partial I(u) = \{\xi \in E^*; I^0(u,v) \geq \langle \xi, v\rangle, \text{for all } v \in E\},$$

where $I^0(u,v)$ stands for the directional derivative of I at u in the direction v; i.e.

$$I^0(u,v) = \limsup_{\substack{w\to u \\ \lambda \searrow 0}} \frac{I(w+\lambda v) - I(w)}{\lambda}.$$

Let Ω be an arbitrary domain in $\mathbb{R}^N$. Let E_Ω be

$$\left\{U = (u_1,u_2) \in H^1(\Omega;\mathbb{R}^2); \int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\,dx < +\infty\right\}$$

which is endowed with the norm

$$\|U\|^2_{E_\Omega} = \int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2)\,dx.$$

Then E_Ω becomes a Hilbert space.

Lemma 1. *The functional $\Psi_\Omega : E_\Omega \to \mathbb{R}, \Psi_\Omega(U) = \int_\Omega F(x,U)\,dx$ is locally Lipschitz on E_Ω.*

PROOF. We first observe that

$$\begin{aligned} F(x,U) = F(x,u_1,u_2) &= \int_0^{u_1} f(x,\tau,u_2)\,d\tau + \int_0^{u_2} g(x,0,\tau)\,d\tau \\ &= \int_0^{u_2} g(x,u_1,\tau)\,d\tau + \int_0^{u_1} f(x,\tau,0)\,d\tau \end{aligned}$$

is a Carathéodory functional which is locally Lipschitz with respect to the second variable. Indeed, by (5)

$$\begin{aligned} |F(x,t_1,t) - F(x,s_1,t)| = \left|\int_{s_1}^{t_1} f(x,\tau,t)\,d\tau\right| &\leq \left|\int_{s_1}^{t_1} C(|\tau,t| + |\tau,t|^p)\,d\tau\right| \\ &\leq k(t_1,s_1,t)|t_1 - s_1|. \end{aligned}$$

Similarly

$$|F(x,t,t_2)-F(x,t,s_2)|\le k(t_2,s_2,t)|t_2-s_2|.$$

Therefore

$$\begin{aligned}|F(x,t_1,t_2)-F(x,s_1,s_2)| &\le |F(x,t_1,t_2)-F(x,s_1,t_2)|\\ &\quad+|F(x,t_1,s_2)-F(x,s_1,s_2)|\\ &\le k(V)|(t_2,s_2)-(t_1,s_1)|\end{aligned}$$

where V is a neighborhood of $(t_1,t_2),(s_1,s_2)$.

For all $x\in\Omega$ let $\chi_1(x)=\max\{u_1(x),v_1(x)\}$ and $\chi_2(x)=\max\{u_2(x),v_2(x)\}$. It is obvious that if $U=(u_1,u_2),V=(v_1,v_2)$ belong to E_Ω, then $(\chi_1,\chi_2)\in E_\Omega$. So, by Hölder's inequality and the continuous embedding $E_\Omega\subset L^p(\Omega;\mathbb{R}^2)$,

$$|\Psi_\Omega(U)-\Psi_\Omega(V)|\le C(\|\chi_1,\chi_2\|_{E_\Omega})\|U-V\|_{E_\Omega},$$

which concludes the proof. □

The following result is a generalization of Lemma 6 in [10].

Lemma 2. *Let Ω be an arbitrary domain in $\mathbb{R}^N$ and let $f:\Omega\times\mathbb{R}^2\to\mathbb{R}$ be a Borel function such that $f(x,\cdot)\in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Then $\underline{f}$ and $\overline{f}$ are Borel functions.*

PROOF. Since the requirement is local, we may suppose that f is bounded by M and it is nonnegative. Let

$$f_{m,n}(x,t_1,t_2)=\Bigg(\int_{t_1-\frac1n}^{t_1+\frac1n}\int_{t_2-\frac1n}^{t_2+\frac1n}|f(x,s_1,s_2)|^m\,ds_1\,ds_2\Bigg)^{\frac1m}.$$

Since $\overline{f}(x,t_1,t_2)=\lim\limits_{\delta\to0}\text{esssup}\{f(x,s_1,s_2);|t_i-s_i|\le\delta;i=1,2\}$, we deduce that for every $\varepsilon>0$, there exists $n\in\mathbb{N}^*$ such that for $|t_i-s_i|<\frac1n(i=1,2)$ we have $|\text{esssup}f(x,s_1,s_2)-\overline{f}(x,t_1,t_2)|<\varepsilon$ or, equivalently,

$$\overline{f}(x,t_1,t_2)-\varepsilon<\text{esssup}f(x,s_1,s_2)<\overline{f}(x,t_1,t_2)+\varepsilon. \tag{9}$$

By the second inequality in (9) we obtain $f(x,s_1,s_2)\le\overline{f}(x,t_1,t_2)+\varepsilon$ a.e. $x\in\Omega$ for $|t_i-s_i|<\frac1n(i=1,2)$ which yields

$$f_{m,n}(x,t_1,t_2)\le(\overline{f}(x,t_1,t_2)+\varepsilon)\Big(\sqrt{4/n^2}\Big)^{\frac1m}. \tag{10}$$

Let

$$A = \left\{ (s_1, s_2) \in \mathbb{R}^2; |t_i - s_i| < \frac{1}{n}(i = 1, 2); \overline{f}(x, t_1, t_2) - \varepsilon \le f(x, s_1, s_2) \right\}.$$

By the first inequality in (9) and the definition of the essential supremum we obtain that $|A| > 0$ and

$$f_{m,n} \le \left(\int\int_A (f(x, s_1, s_2))^m ds_1 ds_2 \right)^{\frac{1}{m}} \ge (\overline{f}(x, s_1, s_2) - \varepsilon)|A|^{1/m}. \quad (11)$$

Since (10) and (11) imply $\overline{f}(x, t_1, t_2) = \lim_{n\to\infty} \lim_{m\to\infty} f_{m,n}(x, t_1, t_2)$, it suffices to prove that $f_{m,n}$ is Borel. Let $\mathcal{M} = \{f : \Omega \times \mathbb{R}^2 \to \mathbb{R}; |f| \le M$ and f is a Borel function$\}$ and $\mathcal{N} = \{f \in \mathcal{M}; f_{m,n}$ is a Borel function$\}$. Cf. [2, p.178], $\mathcal{M}$ is the smallest set of functions having the properties:

(i) $\{f \in C(\Omega \times \mathbb{R}^2; \mathbb{R}); |f| \le M\} \subset \mathcal{M}$;

(ii) $f^{(k)} \in \mathcal{M}$ and $f^{(k)} \xrightarrow{k} f$ imply $f \in \mathcal{M}$.

Since $\mathcal{N}$ obviously contains the continuous functions and (ii) is also true for $\mathcal{N}$, by the Lebesgue dominated convergence theorem, we obtain that $\mathcal{M} = \mathcal{N}$. For $\underline{f}$ we note that $\underline{f} = -(-\overline{f}))$ and the proof of Lemma 2 is complete. □

Let us now assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain. By the continuous embedding $L^{p+1}(\Omega; \mathbb{R}^2) \hookrightarrow L^2(\Omega; \mathbb{R}^2)$, we may define the locally Lipschitz functional $\Psi_\Omega : L^{p+1}(\Omega; \mathbb{R}^2) \to \mathbb{R}$ by $\Psi_\Omega(U) = \int_\Omega F(x, U)\, dx$.

Lemma 3. *Under the above assumptions and for any $U \in L^{p+1}(\Omega; \mathbb{R}^2)$, we have*

$$\partial\Psi_\Omega(U)(x) \subset [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))] \text{ a.e. } x \text{ in } \Omega,$$

in the sense that if $W = (w_1, w_2) \in \partial\Psi_\Omega(U) \subset L^{p+1}(\Omega; \mathbb{R}^2)$ then

$$\underline{f}(x, U(x)) \le w_1(x) \le \overline{f}(x, U(x)) \text{ a.e. } x \text{ in } \Omega \quad (12)$$

$$\underline{g}(x, U(x)) \le w_2(x) \le \overline{g}(x, U(x)) \text{ a.e. } x \text{ in } \Omega. \quad (13)$$

PROOF. By the definition of the Clarke gradient we have

$$\int_\Omega (w_1 v_1 + w_2 v_2)\, dx \le \Psi_\Omega^0(U, V) \text{ for all } V = (v_1, v_2) \in L^{p+1}(\Omega; \mathbb{R}^2).$$

Choose $V = (v, 0)$ such that $v \in L^{p+1}(\Omega)$, $v \geq 0$ a.e. in Ω. Thus, by Lemma 2,

$$\begin{aligned}
\int_\Omega w_1 v &\leq \limsup_{\substack{(h_1,h_2)\to U \\ \lambda \searrow 0}} \frac{\int_\Omega \left(\int_{h_1(x)}^{h_1(x)+\lambda v(x)} f(x,\tau,h_2(x))\,d\tau \right) dx}{\lambda} \\
&\leq \int_\Omega \left(\limsup_{\substack{(h_1,h_2)\to U \\ \lambda \searrow 0}} \frac{1}{\lambda} \int_{h_1(x)}^{h_1(x)+\lambda v(x)} f(x,\tau,h_2(x))\,d\tau \right) dx \qquad (14) \\
&\leq \int_\Omega \overline{f}(x,u_1(x),u_2(x))v(x)\,dx.
\end{aligned}$$

Analogously we obtain

$$\int_\Omega \underline{f}(x,u_1(x),u_2(x))v(x)\,dx \leq \int_\Omega w_1 v\,dx \text{ for all } v \geq 0 \text{ in } \Omega.$$

Arguing by contradiction, suppose that (12) is false. Then there exist $\varepsilon > 0$, a set $A \subset \Omega$ with $|A| > 0$ and w_1 as above such that in A

$$w_1(x) > \overline{f}(x,U(x)) + \varepsilon. \qquad (15)$$

Taking $v = \mathbf{1}_A$ in (14) we obtain

$$\int_\Omega w_1 v\,dx = \int_A w_1\,dx \leq \int_A \overline{f}(x,U(x))\,dx,$$

which contradicts (15). Proceeding in the same way we obtain the corresponding result for g in (13). $\square$

By Lemma 3, Lemma 2.1 in Chang [3] and the embedding $E_\Omega \hookrightarrow L^{p+1}(\Omega,\mathbb{R}^2)$ we also obtain that for $\Psi_\Omega : E_\Omega \to \mathbb{R}$, $\Psi_\Omega(U) = \int_\Omega F(x,U)\,dx$ we have

$$\partial\Psi_\Omega(U)(x) \subset [\underline{f}(x,U(x)),\overline{f}(x,U(x))] \times [\underline{g}(x,U(x)),\overline{g}(x,U(x))] \text{ a.e. } x \in \Omega.$$

Let $V \in E_\Omega$. Then $\tilde{V} \in E$, where $\tilde{V} : \mathbb{R}^N \to \mathbb{R}^2$ is defined by

$$\tilde{V} = \begin{cases} V(x) & x \text{ in } \Omega \\ 0 & \text{otherwise.} \end{cases}$$

For $W \in E^*$ we consider $W_\Omega \in E_\Omega^*$ such that $\langle W_\Omega, V\rangle = \langle W, \tilde{V}\rangle$ for all V in E_Ω. Set $\Psi : E \to \mathbb{R}$, $\Psi(U) = \int_{\mathbb{R}^N} F(x, U)$.

Lemma 4. *Let $W \in \partial\Psi(U)$, where $U \in E$. Then $W_\Omega \in \partial\Psi_\Omega(U)$, in the sense that $W_\Omega \in \partial\Psi_\Omega(U|_\Omega)$.*

Proof. By the definition of the Clarke gradient we deduce that $\langle W, \tilde{V}\rangle \le \Psi^0(U, \tilde{V})$ for all V in E_Ω

$$\begin{aligned}\Psi^0(U,\tilde{V}) &= \limsup_{\substack{H\to U,\, H\in E\\ \lambda\to 0}} \frac{\Psi(H+\lambda\tilde{V}) - \Psi(H)}{\lambda}\\ &= \limsup_{\substack{H\to U,\, H\in E\\ \lambda\to 0}} \frac{\int_{\mathbb{R}^N} (F(x, H+\lambda\tilde{V}) - F(x,H))\,dx}{\lambda}\\ &= \limsup_{\substack{H\to U,\, H\in E\\ \lambda\to 0}} \frac{\int_{\Omega} (F(x, H+\lambda\tilde{V}) - F(x,H))\,dx}{\lambda}\\ &= \limsup_{\substack{H\to U,\, H\in E_\Omega\\ \lambda\to 0}} \frac{\int_{\Omega} (F(x, H+\lambda\tilde{V}) - F(x,H))\,dx}{\lambda} = \Psi^0_\Omega(U,V).\end{aligned}$$

Hence $\langle W_\Omega, V\rangle \le \Psi^0_\Omega(U,V)$ which implies $W_\Omega \in \partial\Psi^0_\Omega(U)$. □

By Lemmas 3 and 4 we obtain that for any $W \in \partial\Psi(U)$ (with $U \in E$), W_Ω satisfies (12) and (13). We also observe that for Ω_1, $\Omega_2 \subset \mathbb{R}^N$ we have $W_{\Omega_1}|_{\Omega_1\cap\Omega_2} = W_{\Omega_2}|_{\Omega_1\cap\Omega_2}$.

Let $W_0 : \mathbb{R}^N \to \mathbb{R}$, where $W_0(x) = W_\Omega(x)$ if $x \in \Omega$. Then W_0 is well defined and

$$W_0(x) \in [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))] \text{ a.e. } x \in \mathbb{R}^N$$

and, for all $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^2)$, $\langle W, \varphi\rangle = \int_{\mathbb{R}^N} W_0\varphi$. By density of $C_c^\infty(\mathbb{R}^N, \mathbb{R}^2)$ in E we deduce that $\langle W, V\rangle = \int_{\mathbb{R}^N} W_0 V\,dx$ for all V in E. Hence, for a.e. $x \in \mathbb{R}^N$

$$W(x) = W_0(x) \in [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))]. \quad (16)$$

3 Proof of Theorem 1.

Define the energy functional $I : E \to \mathbb{R}$ as

$$I(U) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2 \right) dx - \int_{\mathbb{R}^N} F(x, U)\, dx$$
$$= \frac{1}{2} \|U\|_E^2 - \Psi(U).$$

The existence of solutions to problem (4) will be justified by a nonsmooth variant of the Mountain-Pass Theorem (see [3]) applied to the functional I, even if the PS condition is not fulfilled. More precisely, we check the following geometric hypotheses.

$$I(0) = 0 \text{ and there exists } V \in E \text{ such that } I(V) \leq 0; \tag{17}$$

$$\text{there exist } \beta, \rho > 0 \text{ such that } I \geq \beta \text{ on } \{U \in E; \|U\|_E = \rho\}. \tag{18}$$

VERIFICATION OF (17). It is obvious that $I(0) = 0$. For the second assertion we need the following lemma.

Lemma 5. *There exist two positive constants C_1 and C_2 such that*

$$f(x, s, 0) \geq C_1 s^{\mu - 1} - C_2 \textit{ for a.e. } x \in \mathbb{R}^N;\ s \in [0, +\infty).$$

PROOF. We first observe that (8) implies

$$0 \leq \mu F(x, s, 0) \leq \begin{cases} s\underline{f}(x, s, 0) & s \in [0, +\infty) \\ s\overline{f}(x, s, 0) & s \in (-\infty, 0], \end{cases}$$

which places us in the conditions of Lemma 5 in [10].

VERIFICATION OF (17) CONTINUED. Choose $v \in C_c^\infty(\mathbb{R}^N) - \{0\}$ so that $v \geq 0$ in $\mathbb{R}^N$. We have $\int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2 < \infty$; hence $t(v, 0) \in E$ for all $t \in \mathbb{R}$. Thus by Lemma 5 we obtain

$$I(t(v, 0)) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2\, dx - \int_{\mathbb{R}^N} \int_0^{tv} f(x, \tau, 0)\, d\tau$$
$$\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2\, dx - \int_{\mathbb{R}^N} \int_0^{tv} (C_1 \tau^{\mu - 1} - C_2)\, d\tau$$
$$= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + a(x)v^2\, dx + C_2 t \int_{\mathbb{R}^N} v\, dx - C_1' t^\mu \int_{\mathbb{R}^N} v^\mu\, dx < 0$$

for $t > 0$ large enough.

VERIFICATION OF (18). We observe that (6), (7) and (8) imply that, for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that

$$\left.\begin{array}{l} |f(x,s)| \le \varepsilon|s| + A_\varepsilon |s|^p \\ |g(x,s)| \le \varepsilon|s| + A_\varepsilon |s|^p \end{array}\right. \quad \text{for a.e. } (x,s) \in \mathbb{R}^N \times \mathbb{R}^2 . \tag{19}$$

By (19) and Sobolev's embedding theorem we have, for any $U \in E$,

$$\begin{aligned} |\Psi(U)| = |\Psi(u_1,u_2)| &\le \int_{\mathbb{R}^N} \int_0^{|u_1|} ||f(x,\tau,u_2)|\, d\tau + \int_{\mathbb{R}^N} \int_0^{u_2} |g(x,0,\tau)|\, d\tau \\ &\le \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2} |(u_1,u_2)|^2 + \frac{A_\varepsilon}{p+1} |(u_1,u_2|^{p+1} \right) dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2} |u_2|^2 + \frac{A_\varepsilon}{p+1} |u_2|^{p+1} \right) dx \\ &\le \varepsilon \|U\|_{L^2}^2 + \frac{2A_\varepsilon}{p+1} \|U\|_{L^{p+1}}^{p+1} \le \varepsilon C_3 \|U\|_E^2 + C_4 \|U\|_E^{p+1} \end{aligned}$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus

$$I(U) = \frac{1}{2}\|U\|_E^2 - \Psi(U) \ge \frac{1}{2}\|U\|_E^2 - \varepsilon C_3 \|U\|_E^2 - C_4 \|U\|_E^{p+1} \ge \beta > 0,$$

for $\|U\|_E = \rho$, with ρ, ε and β sufficiently small positive constants.

Let $\mathcal{P} = \{\gamma \in C([0,1],E); \gamma(0) = 0, \gamma(1) \ne 0 \text{ and } I(\gamma(1)) \le 0\}$ and $c = \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} I(\gamma(t))$. Set $\lambda_I(U) = \min_{\xi \in \partial I(U)} \|\xi\|_{E^*}$. Thus, by the nonsmooth version of the Mountain Pass Lemma [3], there exists a sequence $\{U_M\} \subset E$ such that

$$I(U_m) \to c \text{ and } \lambda_I(U_m) \to 0. \tag{20}$$

So, there exists a sequence $\{W_m\} \subset \partial\Psi(U_m)$; $W_m = (w_m^1, w_m^2)$ such that

$$(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + a(x)u_m^2 - w_m^2) \to 0 \quad \text{in } E^*. \tag{21}$$

Note that, by (8),

$$\begin{aligned} \Psi(U) \le \frac{1}{\mu}\Big(&\int_{u_1 \ge 0} u_1(x)\underline{f}(x,U)\, dx + \int_{u_1 \le 0} u_1(x)\overline{f}(x,U)\, dx \\ &+ \int_{u_2 \ge 0} u_1(x)\underline{g}(x,U)\, dx + \int_{u_2 \le 0} u_2(x)\overline{g}(x,U)\, dx \Big). \end{aligned}$$

Therefore, by (16),

$$\Psi(U) \leq \frac{1}{\mu} \int_{\mathbb{R}^N} U(x)W(x)\,dx = \frac{1}{\mu} \int_{\mathbb{R}^N} (u_1 w_1 + u_2 w_2)\,dx,$$

for every $U \in E$ and $W \in \partial\Psi(U)$. Hence, if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E^* and E, we have

$$\begin{aligned} I(U_m) =& \frac{\mu - 2}{2\mu} \int_{\mathbb{R}^N} (|\nabla u_m^1|^2 + |\nabla u_m|^2 + a(x)|u_m|^1 + b(x)|u_m|^2)\,dx \\ &+ \frac{1}{\mu} \langle (-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m \rangle \\ &+ \frac{1}{\mu} \langle W_m, U_m \rangle - \Psi(U_m) \\ \geq& \frac{\mu - 2}{2\mu} \int_{\mathbb{R}^N} (|\nabla u_m^1|^2 + |\nabla u_m^2|^2 + a(x)|u_m^1|^2 + b(x)|u_m^2|^2)\,dx \\ &+ \frac{1}{\mu} \langle (-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m \rangle \\ \geq& \frac{\mu - 2}{2\mu} \|U_m\|_E^2 - o(1)\|U_m\|_E. \end{aligned}$$

This relation in conjunction with (20) implies that the Palais-Smale sequence $\{U_m\}$ is bounded in E. Thus, it converges weakly (up to a subsequence) in E and strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$ to some U. Taking into account that $W_m \in \partial\Psi(U_m)$ and $U_m \rightharpoonup U$ in E, we deduce from (21) that there exists $W \in E^*$ such that $W_m \rightharpoonup W$ in E^* (up to a subsequence). Since the mapping $U \longmapsto F(x, U)$ is compact from E to L^1, it follows that $W \in \partial\Psi(U)$. Therefore

$$W(x) \in [\underline{f}(x, U(x)), \overline{f}(x, U(x))] \times [\underline{g}(x, U(x)), \overline{g}(x, U(x))] \text{ a.e. } x \text{ in } \mathbb{R}^N$$

and $(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2) = 0$, or equivalently

$$\int_{\mathbb{R}^N} (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x)u_1 v_1 + b(x)u_2 v_2)\,dx = \int_{\mathbb{R}^N} (w_1 v_1 + w_2 v_2)\,dx$$

for all $(v_1, v_2) \in E$. These last two relations show that U is a solution of the problem (4).

It remains to prove that $U \not\equiv 0$. If $\{W_m\}$ is as in (21), then by (8), (16),

(20) and for large m

$$\begin{aligned}\frac{c}{2} \leq & I(U_m) - \frac{1}{2}\langle(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m\rangle \\ = & \frac{1}{2}\langle W_m, U_m\rangle - \int_{\mathbb{R}^N} F(x, U_m)\,dx \\ \leq & \frac{1}{2}\Big(\int_{u_1\geq 0} u_1(x)\underline{f}(x,U)\,dx + \int_{u_1\leq 0} u_1(x)\overline{f}(x,U)\,dx \\ & + \int_{u_2\geq 0} u_1(x)\underline{g}(x,U)\,dx + \int_{u_2\leq 0} u_2(x)\overline{g}(x,U)\,dx\Big).\end{aligned} \tag{22}$$

Now, taking into account the definition of $\overline{f}$, $\underline{f}$,$\overline{g}$, $\underline{g}$ we deduce that $\overline{f}$, $\underline{f}$, $\overline{g}$, $\underline{g}$ verify (17), too. So by (22) we obtain

$$\frac{c}{2} \leq \int_{\mathbb{R}^N} (\varepsilon|U_m|^2 + A_\varepsilon|u_m|^{p+1}) = \varepsilon\|U_m\|_{L^2}^2 + A_\varepsilon\|U_m\|_{L^{p+1}}^{p+1}.$$

So, $\{U_m\}$ does not converge strongly to 0 in $L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$. With the same arguments as in the proof of Theorem 1 in [7], we deduce that $U \not\equiv 0$, which concludes our proof. □

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