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CONTINUITY OF DARBOUX FUNCTIONS WITH NICE FINITE ITERATIONS

Abstract

A function that maps intervals into intervals is called a Darboux function. We prove that if g is a continuous function that is non-constant on every non-empty open interval, and f is a Darboux function such that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer n_x, and the set of all such n_x is bounded, then f is continuous. In the above statement, the hypothesis "the set of all such n_x is bounded" cannot be dropped. We also show that if g is a continuous function that takes a constant value k on some non-empty open interval I and $k \in I$, then there exists a discontinuous Darboux function $f : \mathbb{R} \to \mathbb{R}$ with the property that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer $n_x \leq 2$. In the previous statement, if $k \notin I$, then no conclusion can be drawn about the function f.

1 Introduction.

It is shown in [4] that if $f : \mathbb{R} \to \mathbb{R}$ is a surjective Darboux function and $g : \mathbb{R} \to \mathbb{R}$ is a function such that $g \circ f$ is continuous, then g is continuous. It is also shown that "continuous" and "Darboux" can be interchanged in the above statement. A special case of the above result is that if the n^{th} iterate of a surjective Darboux function f is continuous for some positive integer n, then f is continuous. If f is a Darboux function and every real number is a periodic point (that is, $f^{n_x}(x) = x$), then $f^2(x) = x$ for all x, and f is continuous (see [6]). It is natural to ask if $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function and $g : \mathbb{R} \to \mathbb{R}$ is a

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continuous function such that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer n_x, what can be said about the function f? In [5], we showed that there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ that is non-constant on every non-empty open interval and a discontinuous Darboux function $f : \mathbb{R} \to \mathbb{R}$ such that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer n_x. In this paper, we prove that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function that is non-constant on every non-empty open interval, $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function, and m is a positive integer such that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer $n_x \leq m$, then f is continuous. We also show that if "continuous" and "Darboux" are interchanged in the hypotheses of the above statement, then g is continuous. In the above statements, g is non-constant on every non-empty open interval cannot be dropped.

Definition 1. A real-valued function f on the set of all real numbers is called a Darboux function if a and b are real numbers and $f(a) \neq f(b)$, then for any real number y between $f(a)$ and $f(b)$, there exists a real number x between a and b such that $y = f(x)$; that is, the image of every interval is an interval.

It is well-known that every continuous function on $\mathbb{R}$ is Darboux. However, not every Darboux function is continuous [1]. Recall that a function f is an n-to-1 (respectively, finite-to-1) function if $\left|f^{-1}(y)\right| = n$ (respectively, $f^{-1}(y)$ is finite) for every real number y in the range of f. It is proved in [2] that a continuous n-to-1 function from $\mathbb{R}$ into $\mathbb{R}$ exists if and only if n is an odd integer. A classical result states that if $f : \mathbb{R} \to \mathbb{R}$ is Darboux and $f^{-1}(y)$ is a closed set for every real number y, then f is continuous. This implies that any n-to-1 Darboux function is continuous.

2 Theorems and Examples.

The following simple proposition is used repeatedly in this paper.

Proposition 1. *The following conditions are equivalent for a Darboux function $f : \mathbb{R} \to \mathbb{R}$.*

(i) f is discontinuous at a real number a.

(ii) There exists a positive real number ϵ such that $a \in \overline{f^{-1}(y)}$ for every $y \in (f(a),\, f(a) + \epsilon)$ or $a \in \overline{f^{-1}(y)}$ for every $y \in (f(a) - \epsilon,\, f(a))$.

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Darboux function. If $f^{-1}(y)$ is a closed set for every y in an everywhere dense subset of $\mathbb{R}$, then f is continuous. In particular, a finite-to-1 Darboux function is continuous.*

Corollary 2. *If $g : \mathbb{R} \to \mathbb{R}$ is a finite-to-1 function, $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function, and m is a positive integer such that, for every real number x, there exists some positive integer $n_x \leq m$ with the property that $f^{n_x}(x) = g(x)$, then f is continuous. In particular, if the n^{th} iterate of a Darboux function is a non-constant polynomial function for some positive integer n, then the Darboux function is continuous.*

Proof. First, we prove that $\forall y \in \mathbb{R}$, $f^{-1}(y)$ is finite. Assume, to the contrary, that $f^{-1}(y)$ is infinite. For $x \in f^{-1}(y)$, let n_x be a positive integer such that $n_x \leq m$ and $f^{n_x}(x) = g(x)$. Consequently, for infinitely many values of x in $f^{-1}(y)$, n_x is same and $g(x)$ is same. This contradicts that g is finite-to-1. So, $f^{-1}(y)$ is finite, and the result follows from Corollary 1. □

Proposition 2 ([4], Theorem 1)**.** *Let f and g be real-valued functions on the reals, and let f be surjective.*

(i) If $g \circ f$, the composition of g with f, is continuous and f is Darboux, then g is continuous.

(ii) If $g \circ f$ is Darboux and f is continuous, then g is Darboux.

Corollary 3. *If $f : \mathbb{R} \to \mathbb{R}$ is a surjective Darboux function and f^n is continuous for some positive integer n, then f is continuous.*

Note that Corollary 3 is not true if the condition "surjective" is dropped. For, let $f(x) = |\sin(\frac{1}{x})|$ whenever $x < 0$, and $f(x) = 1$ otherwise. Then f is Darboux and $f^2(x) = 1$ for all x, but f is discontinuous at 0. However, we prove the following theorem, which directly implies that, in Corollary 3, "surjective Darboux function" can be replaced by "Darboux function that is non-constant on every non-empty open interval."

Theorem 1. *Let g be a continuous function that is non-constant on every non-empty open interval. If $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function and m is a positive integer such that, for every real number x, there exists a positive integer $n_x \leq m$ with the property that $f^{n_x}(x) = g(x)$, then*

(i) for every $1 \leq n \leq m$, the restriction $f^n \restriction D$ of f^n is non-constant on every somewhere dense set D,

(ii) f is continuous,

(iii) g has a fixed point if and only if f has a fixed point.

PROOF OF *(i).* To prove, let us assume the opposite, that is, that p is the smallest positive integer such that $f^p \restriction D$ is constant for some somewhere dense set D. For each $1 \leq n \leq m$, let $D_n = \{d \in D : f^n(d) = g(d)\}$. First, we prove that D_n is nowhere dense for every integer n with $p \leq n \leq m$. For, since $f^p \restriction D$ is constant and $n \geq p$, $f^n \restriction D$ is constant. $g \restriction D_n$ is constant because $f^n \restriction D$ is constant, $D_n \subseteq D$, and $g \restriction D_n = f^n \restriction D_n$. Since g is continuous and $g \restriction D_n$ is constant, g is constant on $\overline{D_n}$. If D_n is somewhere dense, then $\overline{D_n}$ contains a non-empty open interval. Then g is constant on some non-empty open interval, which contradicts the definition of g. So, D_n is nowhere dense for every integer n with $p \leq n \leq m$. Note that $D = \cup_{1 \leq n \leq m} D_n$, where D is somewhere dense and D_n is nowhere dense for every integer n with $p \leq n \leq m$. Since a finite union of nowhere dense sets is nowhere dense, we have $1 < p$, and D_k is somewhere dense for some positive integer $k < p$. We know that $g \restriction D_k = f^k \restriction D_k$. Hence, $(f^{p-k} \circ g) \restriction D_k = f^p \restriction D_k$ is constant, and D_k is somewhere dense. Since $\overline{D}_k$ contains a non-empty open interval and g is a continuous function that is non-constant on any open interval, $g(\overline{D}_k)$ contains a non-empty open interval. $g(D_k)$ is somewhere dense because $\overline{g(D_k)} \supseteq g(\overline{D}_k)$. Denote the set $g(D_k)$ by S. Then, by (*), $f^{p-k} \restriction S$ is constant, S is somewhere dense, and $p-k$ is a positive integer smaller than p. This contradicts the choice of p. Thus, the statement (i) is true.

PROOF OF (ii). Let $y \in f(\mathbb{R})$. For each $x \in f^{-1}(y)$, there exists a positive integer $n \leq m$ such that $g(x) = f^n(x) = f^{n-1}(f(x)) = f^{n-1}(y)$ (for $n = 1$, $f^{n-1}(y)$ is defined to be y). Hence, $f^{-1}(y) \subseteq \cup_{1 \leq n \leq m} g^{-1}(f^{n-1}(y)) = g^{-1}(\{f^{n-1}(y) : 1 \leq n \leq m\})$. Since g is continuous and $\{f^{n-1}(y) : 1 \leq n \leq m\}$ is a closed set, $g^{-1}(\{f^{n-1}(y) : 1 \leq n \leq m\})$ is a closed set. Consequently,

$$\overline{f^{-1}(y)} \subseteq \overline{g^{-1}(\{f^{n-1}(y) : 1 \leq n \leq m\})} = \cup_{1 \leq n \leq m} g^{-1}(f^{n-1}(y)) \qquad (**)$$

To complete the proof, assume, to the contrary, that f is discontinuous at a real number a. Then, by Proposition 1, there exists an $\epsilon > 0$ such that $a \in \overline{f^{-1}(y)}$ for every $y \in (f(a), f(a) + \epsilon)$ or $a \in \overline{f^{-1}(y)}$ for every $y \in (f(a) - \epsilon, f(a))$. Consider the case $a \in \overline{f^{-1}(y)}$ for every $y \in (f(a), f(a) + \epsilon)$. The other case is similar. By (**), $g(a) = f^{n-1}(y)$ for some positive integer $n \leq m$. For each integer n, let $Y_n = \{y \in (f(a), f(a) + \epsilon) : g(a) = f^{n-1}(y)\}$. Then $(f(a), f(a) + \epsilon) = \cup_{1 \leq n \leq m} Y_n$ and $|Y_1| \leq 1$. Hence, Y_j is somewhere dense for some integer j with $2 \leq j \leq m$, and, by the definition of Y_j, $f^{j-1} \restriction Y_j$ is constant. This contradicts part (i) of the theorem. Thus, f is continuous on $\mathbb{R}$.

PROOF OF (iii). It is easy to see that every fixed point of f is a fixed point of g. Conversely, suppose that g has a fixed point. If $x < f(x)$ for all x, then $x < f(x) < f^2(x) \cdots < f^n(x) = g(x)$, which contradicts that g has a fixed point. This shows that $a \geq f(a)$ for some real number a. Similarly, $b \leq f(b)$ for some real number b. Consequently, since f is continuous, $f(x) - x = 0$ for some x. Thus, f has a fixed point. $\square$

It is worth mentioning here that, under the hypotheses of the above theorem, if g has a point of period 2, then f has a point of period 2. This is a consequence of famous Sarkovskii's Theorem. However, if $f(x) = -x + 1$ and $g(x) = x$, then every point except $\frac{1}{2}$ is a periodic point of f with period 2, but g has no point of period 2.

It is interesting to compare the following corollary with Corollary 2.

Corollary 4. *If $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function, $g : \mathbb{R} \to \mathbb{R}$ is a countable-to-1 continuous function, and m is a positive integer such that, for every real number x, there exists some positive integer $n_x \leq m$ with the property that $f^{n_x}(x) = g(x)$, then f is continuous.*

Corollary 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Darboux function. For some positive integer n, if f^n is continuous and non-constant on any non-empty interval, then f is continuous. In particular, if the n^{th} iterate of a Darboux function is a polynomial, sine, or cosine function, then the Darboux function is continuous.*

The following theorem shows that in Theorem 1 if the condition "g is non-constant on every non-empty open interval" is dropped, then the function f need not be continuous.

Theorem 2. *Let g be a continuous function that takes a constant value k on some non-empty open interval I and $k \in I$. Then there exists a discontinuous Darboux function $f : \mathbb{R} \to \mathbb{R}$ with the property that for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer $n_x \leq 2$.*

Proof. Choose a and b in I such that $a < k < b$. Pick an interval (p, q) containing k such that $(p, q) \subseteq (a, b)$. Let c be a real number in the interval (a, p). Define $f : \mathbb{R} \to \mathbb{R}$ as follows. For $x \in (a, c]$, let $f(x) = k + \epsilon \sin(\frac{1}{x-a})$, where $\epsilon = \frac{1}{2}\min\{k - p,\, q - k\}$. For $x \in (c,\, p)$, let the graph of f be the line segment joining the points $(c, f(c))$ and (p, k); i.e., $f(x) = \frac{k-f(c)}{p-c}(x-c)+f(c)$. For $x \notin (a,\, p)$, let $f(x) = g(x)$. Clearly, f is continuous at all points except the point a. It is easy to see that f maps intervals into intervals, and hence, f is Darboux. For $x \in (a, c]$, $f(x) \in [k - \epsilon, k + \epsilon] \subseteq (k - 2\epsilon, k + 2\epsilon) \subseteq (p, q)$. Since the graph of f over the interval (c, p) is the line segment joining the points $(c, f(c))$ and (p, k), and both $f(c)$ and k belong to (p, q), we have $f(x) \in (p, q)$ for $x \in (c, p)$. Consequently, for $x \in (a, p)$, $f(x) \in (p, q)$. Note that $(p,q)\cap(a,p)=\varnothing$, $f=g$ on $\mathbb{R}\setminus(a,p)$, and $f((p,q))=g((p,q))=\{k\}$. This implies that, for $x\in(a,p)$, $f^2(x)=k=g(x)$. Thus, for every x, $f^{n_x}(x)=g(x)$ for some positive integer $n_x\leq 2$, but f is discontinuous at a. □

The following two examples show that in Theorem 2, if $k\notin I$, then no conclusion can be drawn about the function f.

Example 1. *There exist a continuous function $g:\mathbb{R}\to\mathbb{R}$ taking a constant value k on some non-empty open interval I with $k\notin I$ and a positive integer m such that if f is a Darboux function with the property that, for every real number x, $f^{n_x}(x)=g(x)$ for some positive integer $n_x\leq m$, then f is continuous.*

CONSTRUCTION. Let

$$g(x)=\begin{cases}x+1 & \text{whenever } x\geq 0\\ 1 & \text{otherwise.}\end{cases}$$

First, we prove the following results.

(1) $f(x)>0$ for all $x\leq 0$.

(2) $f\restriction(0,\infty)$ is finite-to-1, and f is continuous at every real number $x>0$.

(3) $f(x)>x>0$ for all $x>0$.

(4) For every positive integer n, $f^n\restriction(0,\infty)$ is finite-to-1.

PROOF OF (1). For each $x\leq 0$, there exists a positive integer $n_x\leq m$ such that $f^{n_x}(x)=g(x)=1$. Let $n=\max\{n_x : x\leq 0\}$. Then $f^n(a)=1$ for some $a\leq 0$, and $\{f^n(x):x\leq 0\}\subseteq\{1,f^{n-1}(1),f^{n-2}(1),\dots,f(1)\}$. Since any Darboux function maps every interval onto an interval, f^n is Darboux, and $f^n(x)=1$ for some $x\leq 0$; f^n takes the constant value 1 on the interval $(-\infty,0]$. To prove (1), assume, to the contrary, that $f(c)\leq 0$ for some $c\leq 0$. Denote $f(c)$ by b. Then $f^{n-1}(b)=f^{n-1}(f(c))=f^n(c)=1$. Hence, $f^n(b)=f(1)$. Because $b\leq 0$, $f^n(b)=1$. Consequently, $f(1)=1$. So, 1 is a fixed point of both f and g. By the construction, g has no fixed point. Thus, $f(x)>0$ for all $x\leq 0$.

PROOF OF (2). Suppose that $f\restriction(0,\infty)$ is not finite-to-1. Then, for some infinite subset D of $(0,\infty)$, $f\restriction D$ is constant. Hence, there exists a positive integer $n\leq m$ such that $g(d)=f^n(d)$ for all d in some infinite subset D_1 of D.

This implies that $g \restriction D_1$ is constant, which contradicts that $g \restriction (0, \infty)$ is one-to-one. Thus, $f \restriction (0, \infty)$ is finite-to-1, and, by Proposition 1, f is continuous at every real number $x > 0$.

PROOF OF (3). Note that every fixed point of f is a fixed point of g. Since g has no fixed point and f is continuous on $(0, \infty)$, either $f(x) > x$ for all $x > 0$ or $f(x) < x$ for all $x > 0$. By (1), $f(0) > 0$. If $f(x) < x$ for all $x > 0$, then $f((0, \frac{f(0)}{2})) \subseteq (-\infty, \frac{f(0)}{2})$ and $f(0) \notin (-\infty, \frac{f(0)}{2}]$. Hence, $f([0, \frac{f(0)}{2}))$ is not an interval. This contradicts that f is Darboux. Thus, $f(x) > x > 0$ for all $x > 0$.

PROOF OF (4). Suppose not. Let n be the smallest positive integer such that $f^n \restriction (0, \infty)$ is not finite-to-1. Then there exists an infinite subset D of $(0, \infty)$ such that $f^n \restriction D$ is constant. Since D is infinite and $f \restriction (0, \infty)$ is finite-to-1, $f(D)$ is infinite. By (3), we have $f(D) \subseteq (0, \infty)$. By the definition of n, $f^{n-1} \restriction (0, \infty)$ is finite-to-1. Hence, $f^{n-1}(f(D))$ is infinite, which contradicts that $f^n \restriction D$ is constant. This completes the proof of (4). □

By (2), f is continuous on $(0, \infty)$. To show that f is continuous on $\mathbb{R}$, assume, to the contrary, that f is discontinuous at a real number $a \leq 0$. By (1), $f(a) > 0$. Then, by Proposition 1 and by (**) in the proof of the second part of Theorem 1, there exist $\epsilon > 0$ and a positive integer i such that $f^i(y)$ is same for infinitely many values of y in $(f(a) - \epsilon, f(a) + \epsilon)$. Without loss of generality, we may assume that $\epsilon < f(a)$. Then $(f(a) - \epsilon, f(a) + \epsilon) \subseteq (0, \infty)$, and $f^i(0, \infty)$ is not finite-to-1. This contradicts (4). Thus, f is continuous on $\mathbb{R}$.

Example 2. *There exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ taking a constant value k on some non-empty open interval I with $k \notin I$ and a discontinuous Darboux function f such that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer $n_x \leq 3$.*

CONSTRUCTION. Let

$$g(x) = \begin{cases} -2x - 1 & \text{for } x \leq 0 \\ -1 & \text{otherwise.} \end{cases}$$

Let $h : (0, 1) \to (-\infty, -1)$ be a function that maps every non-empty open interval onto $(-\infty, -1)$. Such a function can be easily constructed by transfinite induction. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} h(x) & \text{whenever } x \in (0, 1) \\ g(x) & \text{otherwise.} \end{cases}$$

For $x \in (0,1)$, $f(x) < -1$, $f^2(x) = f(f(x)) = -2f(x) - 1 > 1$, and $f^3(x) = f(f^2(x)) = -1 = g(x)$. By the construction, $f(x) = g(x)$ whenever $x \notin (0,1)$. So, $f^{n_x}(x) = g(x)$ for some positive integer $n_x \leq 3$. Since f maps every interval of $\mathbb{R}$ onto an interval, f is Darboux. Clearly, g is continuous on $\mathbb{R}$ and f is discontinuous on $(0,1)$. □

By interchanging "Darboux" and "continuous" in the hypotheses of Theorem 1, we can now prove the following.

Proposition 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and m be a positive integer. Suppose g is a Darboux function that is non-constant on every non-empty open interval and, for every real number x, there exists a positive integer n_x such that $n_x \leq m$ and $f^{n_x}(x) = g(x)$. Then g is continuous.*

PROOF. Assume, to the contrary, that g is discontinuous at a real number a. Then, by Proposition 1, there exists $\epsilon > 0$ such that $a \in \overline{g^{-1}(y)}$ for every $y \in (g(a), g(a) + \epsilon)$ or $a \in \overline{g^{-1}(y)}$ for every $y \in (g(a) - \epsilon, g(a))$. Consider the case $a \in \overline{g^{-1}(y)}$ for every $y \in (g(a), g(a) + \epsilon)$. The other case is similar. It is easy to see that $g^{-1}(y) \subseteq \cup_{1 \leq n \leq m} (f^n)^{-1}(y)$. Since

$$a \in \overline{g^{-1}(y)} \subseteq \overline{\cup_{1 \leq n \leq m} (f^n)^{-1}(y)} = \cup_{1 \leq n \leq m} \overline{(f^n)^{-1}(y)} = \cup_{1 \leq n \leq m} (f^n)^{-1}(y)$$

for each $y \in (g(a), g(a) + \epsilon)$, we have $f^n(a) = y$ for some $n \leq m$. This is impossible because the set $\{f^n(a) : 1 \leq n \leq m\}$ is finite. □

The following example shows that the hypothesis "$n_x \leq m$" is necessary in the statement of the above theorem.

Example 3. *There exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a discontinuous Darboux function $g : \mathbb{R} \to \mathbb{R}$ that is non-constant on every non-empty open interval such that, for every real number x, $f^{n_x}(x) = g(x)$ for some positive integer n_x.*

CONSTRUCTION. Let $f(x) = |\frac{\sin(\frac{1}{x})}{2}|^{2^n}$ for $x \in [-\frac{1}{n\pi}, -\frac{1}{(n+1)\pi})$, where $n \in \mathbb{N}$,

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0 \\ x + \frac{1}{\pi} & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} \frac{\sin^2(\frac{1}{x})}{4} & \text{for } x \in [-\frac{1}{\pi}, 0) \\ f(x) & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f^n(x) = g(x)$ for $x \in [-\frac{1}{n\pi}, -\frac{1}{(n+1)\pi})$ and $n \in \mathbb{N}$. Clearly, f is continuous on $\mathbb{R}$, g is Darboux, and for every real number x, there exists a positive integer n_x such that $f^{n_x}(x) = g(x)$, but g is discontinuous at 0. □

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