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A FUNCTIONAL ANALYSIS POINT OF VIEW ON THE ARZELA-ASCOLI THEOREM

Abstract

We discuss the Arzela-Ascoli pre-compactness Theorem from the point of view of Functional Analysis, using compactness in ℓ^∞ and its dual.

The Arzela-Ascoli Theorem is a very important technical result, used in many branches of mathematics. Aside from its numerous applications to Partial Differential Equations, the Arzela-Ascoli Theorem is also used as a tool in obtaining Functional Analysis results, such as the compactness for duals of compact operators, as presented for example [1]. The purpose of this note is to offer a new perspective on the Arzela-Ascoli Theorem based on a functional analytic proof.

The theorem of Arzela and Ascoli deals with (relative) compactness in the Banach space $C(K)$ of complex valued continuous functions on a compact Hausdorff space K. One helpful characterization of pre-compactness for sets in a complete metric space is the following well-known criterion, which we state without proof.

Proposition 1. *Let $(\mathcal{Y}, d)$ be a complete metric space. For a subset $\mathcal{M} \subset \mathcal{Y}$, the following are equivalent:*

(i) $\mathcal{M}$ is relatively compact in $\mathcal{Y}$; i.e., its closure $\overline{\mathcal{M}}$ in $\mathcal{Y}$ is compact;

(ii) $\mathcal{M}$ contains no infinite subsets $\mathcal{T}$, satisfying

$$\inf \big\{ d(x, y) : x, y \in \mathcal{T},\ x \neq y \big\} > 0.$$

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Proposition 2. *Let $\mathcal{X}$ be a normed vector space, let $\mathcal{S} \subset \mathcal{X}$ be a compact subset, and let $\mathcal{B}$ be the unit ball in $\mathcal{X}^*$—the (topological) dual of $\mathcal{X}$—equipped with the w^*-topology. If we consider the Banach algebra $\mathcal{A} = C(\mathcal{S})$, equipped with the uniform topology, then the restriction map $\Theta : (\mathcal{B}, w^*) \ni \phi \longmapsto \phi\big|_{\mathcal{S}} \in (\mathcal{A}, \|\,.\,\|)$ is continuous. In particular, the set $\Theta(\mathcal{B})$ is compact in $\mathcal{A}$.*

PROOF. To prove continuity, we start with a net $(\phi_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{B}$, that converges to some $\phi \in \mathcal{B}$ in the w^*-topology, and let us show that the net $\big(\phi_\lambda\big|_{\mathcal{S}}\big)_{\lambda\in\Lambda}$ converges to $\phi\big|_{\mathcal{S}}$ uniformly on $\mathcal{S}$. Fix some $\varepsilon > 0$, and (use compactness of $\mathcal{S}$) choose points $s_1, \dots, s_n \in \mathcal{S}$, such that

(∗) *for every $s \in \mathcal{S}$, there exists $k \in \{1, \dots, n\}$, with $\|s - s_k\| < \varepsilon/3$.*

Using the condition $\phi_\lambda \xrightarrow{w^*} \phi$, there exists $\lambda_\varepsilon \in \Lambda$, such that

$$|\phi_\lambda(s_k) - \phi(s_k)| < \varepsilon/3,\ \forall\, \lambda \geq \lambda_\varepsilon,\ k \in \{1, \dots, n\}. \tag{1}$$

Now we are done, since if we start with some arbitrary $s \in \mathcal{S}$, and we choose $k \in \{1, \dots, n\}$, such that $\|s - s_k\| < \varepsilon/3$, then using (1) we get

$$\begin{aligned} |\phi_\lambda(s) - \phi(s)| &\leq |\phi_\lambda(s) - \phi_\lambda(s_k)| + |\phi_\lambda(s_k) - \phi(s_k)| + |\phi(s_k) - \phi(s)| \\ &\leq \|\phi_\lambda\| \cdot \|s - s_k\| + |\phi_\lambda(s_k) - \phi(s_k)| + \|\phi\| \cdot \|s - s_k\| \\ &\leq 2\|s - s_k\| + |\phi_\lambda(s_k) - \phi(s_k)| \leq \varepsilon,\ \forall\, \lambda \geq \lambda_\varepsilon. \end{aligned}$$

Having proven the continuity of Θ, the second assertion follows from Alaoglu's Theorem (which states that $\mathcal{B}$ is compact in the w^*-topology). □

Theorem (Arzela-Ascoli)**.** *Let K be a compact Hausdorff space, and let $\mathcal{M} \subset C(K)$ be a set which is*

- ***pointwise bounded:*** $\sup\big\{|f(p)| : f \in \mathcal{M}\big\} < \infty$, $\forall\, p \in K$;
- ***equicontinuous:*** *for every $p \in K$ and $\varepsilon > 0$, there exists a neighborhood $N_{p,\varepsilon}$ of p in K, such that* $\sup_{f\in\mathcal{M}} |f(q) - f(p)| \leq \varepsilon$, $\forall\, q \in N_{p,\varepsilon}$.

Then $\mathcal{M}$ is relatively compact in $C(K)$ in the uniform topology.

PROOF. The main observation is that, using pointwise boundedness, any set $\mathcal{T} \subset \mathcal{M}$ gives rise to a map

$$\Phi_{\mathcal{T}} : K \ni p \longmapsto \big[f(p)\big]_{f\in\mathcal{T}} \in \ell^\infty(\mathcal{T}).$$

(Here $\ell^\infty(\mathcal{T})$ denotes the Banach space of all bounded functions from $\mathcal{T}$ to $\mathbb{C}$.) Furthermore, by equicontinuity the map $\Phi_{\mathcal{T}}$ is in fact continuous, when $\ell^\infty(\mathcal{T})$ is equipped with the norm topology. Since K is compact, so is the set

$$\mathcal{S}_{\mathcal{T}} = \Phi_{\mathcal{T}}(K) \subset \ell^\infty(\mathcal{T}).$$

We are going to argue by contradiction (see Proposition 1), assuming the existence of some $\rho > 0$, and of an infinite set $\mathcal{T} \subset \mathcal{M}$, such that

$$\|f - g\| > \rho,\ \forall\, f, g \in \mathcal{T},\ f \neq g. \tag{2}$$

Consider now the unit ball $\mathcal{B}$ in the dual space $\ell^\infty(\mathcal{T})^*$, and use Proposition 2 to conclude that the set

$$\Theta_{\mathcal{T}} = \left\{ \phi\big|_{\mathcal{S}_{\mathcal{T}}} : \phi \in \mathcal{B} \right\} \subset C(\mathcal{S}_{\mathcal{T}})$$

is compact in $C(\mathcal{S}_{\mathcal{T}})$ in the norm topology. Consider now the coordinate maps $e_f : \ell^\infty(\mathcal{T}) \to \mathbb{C}$, $f \in \mathcal{T}$, and their restrictions $\theta_f = e_f\big|_{\mathcal{S}_{\mathcal{T}}} \in C(\mathcal{S}_{\mathcal{T}})$, which satisfy

$$\theta_f\big(\Phi_{\mathcal{T}}(p)\big) = f(p),\ \forall\, p \in K,\ f \in \mathcal{T}. \tag{3}$$

Now, if we start with $f, g \in \mathcal{T}$, $f \neq g$, then using (2) there exists $p \in K$, such that $|f(p) - g(p)| > \rho$, so by (3) we also get

$$\Big|\theta_f\big(\Phi_{\mathcal{T}}(p)\big) - \theta_g\big(\Phi_{\mathcal{T}}(p)\big)\Big| > \rho.$$

This way we have shown that

$$\|\theta_f - \theta_g\|_{C(\mathcal{S}_{\mathcal{T}})} > \rho,\ \forall\, f, g \in \mathcal{T},\ f \neq g,$$

and therefore the set $\Theta_{\mathcal{T}}$, which contains all the θ_f's, cannot be compact in the norm topology. □

References

[1] W. Rudin, *Functional Analysis*, McGraw-Hill, Springer-Verlag, New York, 1973