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Ewa Strońska, Mathematical Institute, Kazimierz Wielki University, Plac Weyssenhoffa 11, 85-071 Bydgoszcz, Poland. email: stronska@neostrada.pl

ON ALMOST CONTINUOUS DERIVATIONS

Abstract

It is proved that every derivation is the sum of two almost continuous (in Stallings' sense) derivations and the limit of a sequence (of a transfinite sequence) of almost continuous derivations.

A function $g : (a, b) \to \mathbb{R}$ is said to be almost continuous (in Stallings' sense [5]) if for every open set $D \subset \mathbb{R}^2$ containing the graph $\mathrm{Gr}(g)$ of the function g there is a continuous function $h : (a, b) \to \mathbb{R}$ with $\mathrm{Gr}(h) \subset D$.

A function $f : \mathbb{R} \to \mathbb{R}$ is called additive ([4]) if it satisfies Cauchy's equation

$$f(x + y) = f(x) + f(y), \text{ for all } x,\, y \in \mathbb{R}.$$

An additive function $f : \mathbb{R} \to \mathbb{R}$ is called a derivation if it satisfies the equation

$$f(xy) = xf(y) + yf(x), \text{ for all } x,\, y \in \mathbb{R}.$$

It is well known that there exists a discontinuous additive almost continuous function $f : \mathbb{R} \to \mathbb{R}$ ([2] and [3]) and that every additive function is the sum of two additive almost continuous functions and the limit of a sequence (of a transfinite sequence) of additive almost continuous functions ([1]). In this article I prove analogous theorems for derivations.

If $f : \mathbb{R} \to \mathbb{R}$ is a function, by a blocking set of f we mean a closed set $K \subset \mathbb{R}^2$ such that $\mathrm{Gr}(f) \cap K = \emptyset$ and $\mathrm{Gr}(g) \cap K \neq \emptyset$ for every continuous function $g : \mathbb{R} \to \mathbb{R}$. An irreducible blocking set (IBS) K of f is a blocking set of f such that no proper subset of K is a blocking set ([3]).

It is known that $f : \mathbb{R} \to \mathbb{R}$ is almost continuous if and only if it has no blocking set. Moreover, if f is not almost continuous, then there is an

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(IBS) K of f and the x-projection $\mathrm{pr}_x(K)$ of K is a non-degenerate interval ([3]). Let

$$K_0, K_1, \dots, K_\alpha, \dots,\ \alpha < \omega_c,$$

be a transfinite sequence of all irreducible blocking sets in $\mathbb{R}^2$, with $K_\alpha \neq K_\beta$ for $\alpha \neq \beta$, $\alpha, \beta < \omega_c$ and ω_c denoting the first ordinal of the cardinality of the continuum.

Let $F \subset K$ be a field. An element $a \in K$ is called algebraically dependent (or algebraic) over F if there exists a non-trivial ($\neq 0$) polynomial p with the coefficients from F such that $p(a) = 0$.

The algebraic closure of F (in K) is the set

$$\mathrm{algcl}(F) = \{a \in K : a \text{ is algebraic over } F\}.$$

It is known that $\mathbb{R} \neq \mathrm{algcl}(\mathbb{Q})$ and there exists an algebraic base of $\mathbb{R}$ over $\mathbb{Q}$ ([4, p. 102]).

In the proofs of the main theorems we use the following.

Theorem 1 ([4] Th. 1, p. 352)**.** *Let K be a field of characteristic zero, let F be a subfield of K, let X be an algebraic base of K over F, if it exists, and let $X = \emptyset$ otherwise. Let $f : F \to K$ be a derivation. Then, for every function $u : X \to K$ there exists a unique derivation $g : K \to K$ such that $g|_F = f$ and $g|_X = u$.*

Theorem 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a derivation, then there are two almost continuous derivations $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g + h$.*

PROOF. We apply transfinite induction. Since $\mathrm{pr}_x(K_0)$ is a non-degenerate interval, there are algebraically independent (over $\mathbb{Q}$) elements $u_0, v_0 \in \mathrm{pr}_x(K_0) \setminus \mathrm{algcl}(\mathbb{Q})$.

Next, we fix an ordinal $\alpha > 0$ with $\alpha < \omega_c$ and assume that for each ordinal $\beta < \alpha$ we have defined elements $u_\beta, v_\beta \in \mathrm{pr}_x(K_\beta)$, such that the set

$$S_\alpha = \{u_\beta, v_\beta :\ \beta < \alpha\}$$

is algebraically independent (over $\mathbb{Q}$) and $(u_{\beta_1}, v_{\beta_1}) \neq (u_{\beta_2}, v_{\beta_2})$ for $\beta_1 < \beta_2 \leq \beta$.

Finally, there are algebraically independent (over $\mathbb{Q}$) elements $u_\alpha, v_\alpha \in \mathrm{pr}_x(K_\alpha) \setminus \mathrm{algcl}(\mathbb{Q} \cup S_\alpha)$.

Observe that the set $S = A \cup B$, where $A = \{u_\alpha :\ \alpha < \omega_c\}$ and $B = \{v_\alpha :\ \alpha < \omega_c\}$ are algebraically independent (over $\mathbb{Q}$). Consequently, there is an algebraic base $X \supset S$ in $\mathbb{R}$ (over $\mathbb{Q}$).

For every $\alpha < \omega_c$, let $t_\alpha, z_\alpha \in \mathbb{R}$ be points such that

$$(u_\alpha, t_\alpha) \in K_\alpha \text{ and } (v_\alpha, z_\alpha) \in K_\alpha.$$

Let

$$g_1(x) = \begin{cases} t_\alpha & \text{if } x = u_\alpha,\ \alpha < \omega_c, \\ f(x) - z_\alpha & \text{if } x = v_\alpha,\ \alpha < \omega_c, \\ 0 & \text{otherwise in } X, \end{cases}$$

and let

$$h_1(x) = \begin{cases} f(x) - t_\alpha & \text{if } x = u_\alpha,\ \alpha < \omega_c, \\ z_\alpha & \text{if } x = v_\alpha,\ \alpha < \omega_c, \\ f(x) & \text{otherwise in } X, \end{cases}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be an extension of g_1 to some derivation. Since $f - g$ is a derivation, the function $h : \mathbb{R} \to \mathbb{R}$ such that $h|_X = h_1$ and $h(x) = f(x) - g(x)$ for $x \in \mathbb{R} \setminus X$ is an extension of h_1 to some derivation.

Observe that for every $\alpha < \omega_c$,

$$(u_\alpha, g(u_\alpha)) = (u_\alpha, t_\alpha) \in K_\alpha \text{ and } (v_\alpha, h(v_\alpha)) = (v_\alpha, z_\alpha) \in K_\alpha.$$

So, the functions g, h are almost continuous and evidently $f = g + h$. □

The next remark follows from Theorem 2.

Remark 1. *There are almost continuous derivations $f : \mathbb{R} \to \mathbb{R}$ which are discontinuous.*

Proof. It suffices to find a discontinuous derivation $\phi : \mathbb{R} \to \mathbb{R}$ ([4, Th. 2, p. 352]) and two almost continuous derivations $g, h : \mathbb{R} \to \mathbb{R}$ with $\phi = g + h$. Then, at least one derivation g or h is discontinuous. □

Theorem 3. *If $f : \mathbb{R} \to \mathbb{R}$ is a derivation, then there is a sequence of almost continuous derivations $f_n : \mathbb{R} \to \mathbb{R},\ \ n \geq 1$, such that $f = \lim_{n\to\infty} f_n$.*

Proof. As in the proof of Theorem 2, for every $\alpha < \omega_c$ we find a sequence of points

$$x_{\alpha,n} \in \mathrm{pr}_x(K_\alpha),\ n = 1, 2, \ldots,$$

such that the set

$$S = \{x_{\alpha,n} : \alpha < \omega_c, \quad n \geq 1\}$$

is algebraically independent over $\mathbb{Q}$. Let $X \supset S$ be an algebraic basis (over $\mathbb{Q}$) in $\mathbb{R}$. For each point $x_{\alpha,n}$ there is a point $y_{\alpha,n}$ such that

$$(x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha,\ \alpha < \omega_c, n \geq 1.$$

For $n = 1, 2, \dots$, let

$$g_n(x) = \begin{cases} y_{\alpha,k} & \text{if } x = x_{\alpha,k},\ \alpha < \omega_c,\ \ k \geq n, \\ f(x) & \text{otherwise in } X, \end{cases}$$

and let f_n be an extension of g_n to a derivation on $\mathbb{R}$. Since

$$(x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha \cap \mathrm{Gr}(f_n) \text{ for } \alpha < \omega_c \text{ and } n \geq 1,$$

all functions f_n are almost continuous. Moreover, if $x = x_{\alpha,k}$, where $\alpha < \omega_c$, and $k \geq 1$, then $f_n(x) = f(x)$ for $n > k$ and if $x \in X$ and $x \neq x_{\alpha,k}$ for all $\alpha < \omega_c$ and $k \geq 1$, then $f_n(x) = f(x)$ for all $n \geq 1$. So, $f = \lim_{n\to\infty} f_n$ on X and consequently on $\mathbb{R}$. Thus, the proof is completed. □

Now we will consider the transfinite convergence. Recall that a transfinite sequence of functions $f_\alpha : \mathbb{R} \to \mathbb{R}$, where $\alpha < \omega_1$ (ω_1 denoting the first uncountable ordinal), converges to a function $f : \mathbb{R} \to \mathbb{R}$ (then we write $\lim_\alpha f_\alpha = f$) if for each point $x \in \mathbb{R}$ there is a countable ordinal $\beta(x)$ such that for each countable ordinal $\alpha > \beta(x)$ the equality $f_\alpha(x) = f(x)$ holds ([6]).

Theorem 4. *Assume that $\omega_1 = \omega_c$. If $f : \mathbb{R} \to \mathbb{R}$ is a derivation, then there is a transfinite sequence of almost continuous derivations $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha < \omega_1$, such that* $\lim_\alpha f_\alpha = f$.

PROOF. As above we find pairwise disjoint sets $T_\alpha, \alpha < \omega_1 = \omega_c$, such that every set

$$\mathrm{pr}_x(K_\alpha) \cap T_\alpha,\ \alpha < \omega_1,$$

is uncountable, and the union $\bigcup_{\alpha<\omega_1} \mathrm{pr}_x(K_\alpha) \cap T_\alpha$ is algebraically independent over $\mathbb{Q}$ in $\mathbb{R}$. For each $\alpha < \omega_1$, let $(x_{\alpha,\beta})_{\beta<\omega_1}$ be a transfinite sequence of all points of the set $\mathrm{pr}_x(K_\alpha) \cap T_\alpha$, and let

$$g_\alpha(x) = \begin{cases} y_{\alpha,\beta} & \text{if } x = x_{\alpha,\beta},\ \ \omega_1 > \beta \geq \alpha, \\ f(x) & \text{otherwise in } X, \end{cases}$$

where $y_{\alpha,\beta}$ are points such that

$$(x_{\alpha,\beta}, y_{\alpha,\beta}) \in K_\alpha, \quad \alpha, \beta < \omega_1,$$

and let f_α be an extension g_α to a derivation on $\mathbb{R}$. Analogously, as in the proof of Theorem 3 we can observe that all functions f_α are almost continuous and

$$\lim_\alpha f_\alpha = f.$$

This completes the proof. □

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