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INFINITE DIMENSIONAL BANACH SPACE OF BESICOVITCH FUNCTIONS

Abstract

Let $C([0,1])$ be the set of all continuous functions mapping the unit interval $[0,1]$ into $\mathbb{R}$. A function $f \in C([0,1])$ is called Besicovitch if it has nowhere one-sided derivative (finite or infinite). We construct a set $\mathcal{B}_{\text{sup}} \subset C([0,1])$ such that $(\mathcal{B}_{\text{sup}}, || \, ||_{\text{sup}})$ is an infinite dimensional Banach (sub)space in C([0,1]) and each nonzero element of $\mathcal{B}_{\text{sup}}$ is a Besicovitch function.

1 Introduction.

In this paper we continue our investigation of nowhere differentiable functions [3] and, in particular, Besicovitch functions [4] — real-valued functions of a real variable without *finite or infinite* one-sided derivatives. They were introduced many years ago by the classical work of Besicovitch [2]. In 1932 Saks [7] proved that the collection of all Besicovitch functions is of the first category in the space $C([0,1])$ of continuous functions mapping the unit interval $[0,1]$ into $\mathbb{R}$ equipped by the supremum norm $|| \, ||_{\text{sup}}$.

Recently, it has been proved in [1] (see also [9]) that there exists an infinite dimensional closed subspace of $C([0,1])$ such that each (not identically zero) function from this subspace has nowhere one-sided *finite* derivative.

In this work we show that an analogous assertion remains true also for the class of Besicovitch functions. More precisely, we construct a set $\mathcal{B}_{\text{sup}} \subset C([0,1])$, such that $(\mathcal{B}_{\text{sup}}, || \, ||_{\text{sup}})$ is an infinite dimensional Banach (sub)space in C([0,1]) and each nonzero element of $\mathcal{B}_{\text{sup}}$ is a Besicovitch function.

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2 Basic Pepper's Construction.

The following construction is due to Pepper [6]. For $a > 0$, let us construct in $[0, a]$ a discontinuum

$$E = [0, a] \setminus L, \text{where } L = \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{2^{m-1}} r_{m,p}, \tag{2.1}$$

and the open intervals $r_{m,p} = (a_{m,p}, b_{m,p})$ are constructed as follows:

- $d_{1,1} = [0, a]$, $r_{1,1} \subset d_{1,1}$ so that the center of $r_{1,1}$ coincides with that of $d_{1,1}$, $\lambda(r_{1,1}) = \frac{a}{4}$, where λ is the Lebesgue measure
- for $m > 1$, if $d_{m,1} \cdots d_{m,2^{m-1}}$ are (from the left to the right) the intervals of the set $[0, a] \setminus \bigcup_{q=1}^{m-1} \bigcup_{p=1}^{2^{q-1}} r_{q,p}$, then $r_{m,p} \subset d_{m,p}$ so that the center of $r_{m,p}$ coincides with that of $d_{m,p}$ and $\lambda(r_{m,p}) = \frac{a}{4^m}$.

We have $\lambda(E) = \frac{a}{2}$. For $b > 0$, let $\varphi : [0, a] \to [0, b]$ be a nondecreasing continuous function defined by

$$\varphi(x) = 2\frac{b}{a}\lambda(E \cap [0, x]) . \tag{2.2}$$

Then $\varphi(0) = 0$, $\varphi(a) = b$, φ is constant on every interval $r_{m,p}$ and

$$\varphi(r_{m,p}) = \frac{b(2p-1)}{2^m}, \ m \in \mathbb{N}, \ p = 1, \ldots, 2^{m-1}. \tag{2.3}$$

Hence, for each m, p (with respect to $[0, a]$),

$$0 \le \varphi(r_{m,p}) \pm \frac{b}{2^m} \le b. \tag{2.4}$$

Remark 2.1. For any segment $d_{m+1,p}$,

$$\lambda(d_{m+1,p}) = a\Big(\frac{1}{2^{m+1}} + \frac{1}{2 \cdot 4^m}\Big). \tag{2.5}$$

Since all parts of the graph of φ corresponding to the segments $d_{m+1,1}$, $d_{m+1,2}$, $\ldots$, $d_{m+1,2^m}$ are similar, for any $d_{m+1,p} = [u, v]$ we have

$$\varphi(v) - \varphi(u) = \frac{\varphi(a) - \varphi(0)}{2^m} = \frac{b}{2^m}.$$

One can verify that $b_{m,p} = \frac{a}{2^{m+1}} + \frac{3}{2 \cdot 4^m}\frac{a}{} + (2p-2)\frac{a}{2^m} \cdot \frac{4^m - 2^m - 2}{4^m - 2^{m+1}}$. Using (2.3) and the fact that $\dfrac{4^m - 2^m - 2}{4^m - 2^{m+1}} \in [1, 2]$ for each $m \in \mathbb{N}$, we get

$$\frac{b}{2a} \le \frac{\varphi(b_{m,p})}{b_{m,p}}. \tag{2.6}$$

Define a function $p : [0, 2a] \to [0, b]$ by

$$p(x) = \begin{cases} \varphi(x) & \text{for } x \in [0, a], \\ \varphi(2a - x) & \text{for } x \in [a, 2a]. \end{cases}$$

The function p and the interval $[0, 2a]$ form the (canonical) step-triangle with the base $[0, 2a]$ and the left side $\{(x, p(x));\ x \in [0, a]\}$, resp. right side $\{(x, p(x));\ x \in [a, 2a]\}$.

Now we construct a Besicovitch function $f\colon\ [0, 2a] \to [0, b]$, see [6].

0th step: Call the segment $(0, 2a)$ an L-segment of the zero category.

1st step: Construct a (canonical) step-triangle with the base $[0, 2a]$ and height b. Call this step-triangle "triangle of the first category", the segments of $L \subset [0, a]$ (see (2.1)) and symmetric segments in $[a, 2a]$ "L-segments of the first category" and the corresponding segments on the sides of the canonical step-triangle "M-segments of the first category".

The sides of the step-triangle define a function f_1 $(= p)$.

***n*th step:** On each of those M-segments of $n - 1$st category, construct in a similar way, a step-triangle directed inside the step-triangle of $n - 1$st category on whose side the triangle has its base. On equal segments construct equal triangles and the height of the triangle constructed on $r_{m,p}$ (with respect to its base) is to be equal to $\frac{b'}{2^m}$, where b' is a height of the bigger triangle of the $n - 1$st category on whose side the triangle has its base. Call these triangles "triangles of the nth category", new L-segments "L-segments of the nth category" and the corresponding segments on the sides of new triangles "M-segments of the nth category".

The union of sides of all triangles constructed so far defines a function f_n. Since for each $n \in \mathbb{N}$, f_n is continuous and $||f_{n+1} - f_n||_{\sup} = \frac{b}{2^n}$, the continuous map $f = \lim_{n\to\infty} f_n$is well defined.

A point $x \in [0, 2a]$ outside of L-segments of the first category will be called a point of the first category. A point which belongs to an L-segment of the first category but not to that of the second category will be called a point of the second category, etc. Any point which belongs to L-segments of all categories will be called a point of infinite category.

3 The Space $\mathcal{B}_\infty$.

Our aim is to construct a set $\mathcal{B}_\infty \subset C([0, 2a])$ in which any nonzero element is a Besicovitch function and which is an infinite dimensional linear (sub)space in $C([0, 2a])$.

We start by recalling one notion from symbolic dynamics [5]. A sequence $(x(n))_{n=1}^{\infty}$ of symbols is called a Toeplitz sequence provided that $\mathbb{N}$ can be decomposed into arithmetic progressions such that $x(n)$ is constant on each arithmetic progression. We will use two sequences $\eta = (\eta(n))_{n=1}^{\infty} \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$ and $\theta = (\theta(n))_{n=0}^{\infty} \in (\mathbb{N}\cup\{0\})^{\mathbb{N}\cup\{0\}}$. The sequence η is chosen as any Toeplitz sequence which is *onto*; i.e., $\eta(\mathbb{N}) = \mathbb{N}\cup\{0\}$. The sequence θ is defined by $\theta(2n) = \theta(2n+1) = \eta(n+1)$, for n greater than or equal to 0. Thus,

$$\theta = \eta(1)\eta(1)\eta(2)\eta(2)\eta(3)\eta(3)\eta(4)\eta(4)\eta(5)\eta(5)\eta(6)\eta(6)\eta(7)\eta(7)\dots$$

Clearly also, the sequence θ is Toeplitz (but indexed from 0). For $n \geq 0$ we denote by m_n the infinite vector $(m_{n0}\ m_{n1}\ \dots\ m_{nj}\ \dots)$ satisfying

$$m_{nj} = \begin{cases} 0 & \text{for } j < n, \\ 1 & \text{for } j \geq n. \end{cases}$$

Finally, we define the infinite matrix $A = (a_{nj})_{n,j=0}^{\infty}$ with the rows $m_{\theta(n)}$, i.e., the matrix satisfying $a_{nj} = m_{\theta(n)j}$ for each n, j.

Definition 3.1. Let $h \in C([0,2a])$, $I = (c,d) \subset [0,2a]$ and $h(c) = h(d)$. We say that $h|I$ is positively, resp. negatively oriented (on I) if

$$h((c+d)/2) > h(c), \text{ resp. } h((c+d)/2) < h(c).$$

We put

$$o\langle h, I\rangle = \begin{cases} 1 & \text{if } h|I \text{ is positively oriented,} \\ -1 & \text{if } h|I \text{ is negatively oriented,} \\ 0 & \text{otherwise.} \end{cases}$$

The number $|h((c+d)/2) - h(c)|$ will be called the height of h on I; we denote it $v\langle h, I\rangle$.

By $\mathfrak{L}^n$, $n \in \mathbb{N}\cup\{0\}$ we denote the set of all L-segments of the nth category. In particular, $\mathfrak{L}^0 = \{(0,2a)\}$.

Definition 3.2. Let $A = (a_{nj})_{n,j=0}^{\infty}$ be as above. We introduce functions $F_0, \dots, F_k, \dots$ from $C([0,2a])$ constructed analogously as the function f in Section 1 and such that for any L-segment $I \in \mathfrak{L}^n$ of the nth category, $n \in \mathbb{N}\cup\{0\}$,

$$v\langle F_k, I\rangle = a_{nk} \cdot v\langle f, I\rangle, \text{ and } o\langle F_k, I\rangle = \operatorname{sign}(a_{nk}) \cdot o\langle f, I\rangle.$$

Let $\mathcal{S}$ be the set of all real convergent series. For any element $\mu = \sum_{k=0}^{\infty} \mu_k \in \mathcal{S}$ let $||\mu||_{\mathcal{S}} = \sup_{n \geq 0} |\sum_{k=n}^{\infty} \mu_k|$. In [8] the authors showed that $(\mathcal{S}, || \, ||_{\mathcal{S}})$ is a Banach space. Now put

$$\mathcal{B}_\infty = \Big\{ \sum_{k=0}^{\infty} \mu_k F_k \colon \ \sum_{k=0}^{\infty} \mu_k \in \mathcal{S} \Big\}.$$

We finish this section with two lemmas.

Lemma 3.3. *The following statements are true.*

(i) Any element $H \in \mathcal{B}_\infty$ is a continuous function from $C([0, 2a])$.

(ii) Let $\mu(\ell) := \sum_{k=0}^{\infty} \mu_{k,\ell}$, $\ell \geq 1$ and $\mu := \sum_{k=0}^{\infty} \mu_k$ be series from $\mathcal{S}$, $H_\ell = \sum_{k=0}^{\infty} \mu_{k,\ell} F_k$ and $H = \sum_{k=0}^{\infty} \mu_k F_k$. Then

$$||\mu(\ell) - \mu||_{\mathcal{S}} \to_\ell 0 \quad \Longrightarrow \quad ||H_\ell - H||_{\sup} \to_\ell 0.$$

Proof. (i) It is clear when $||\mu||_{\mathcal{S}} = 0$. Let

$$H = \sum_{k=0}^{\infty} \mu_k F_k \tag{3.1}$$

for some nonzero $\mu \in \mathcal{S}$. By Definition 3.2, for any L-segment $I \in \mathfrak{L}^n$ of the nth category

$$\begin{aligned} o\langle H, I\rangle \ v\langle H, I\rangle &= \sum_{k=0}^{\infty} \mu_k \ o\langle F_k, I\rangle \ v\langle F_k, I\rangle \\ &= \sum_{k=0}^{\infty} \mu_k \ \operatorname{sign}(a_{nk}) \ a_{nk} \ o\langle f, I\rangle \ v\langle f, I\rangle \\ &= o\langle f, I\rangle \ v\langle f, I\rangle \sum_{k=\theta(n)}^{\infty} \mu_k. \end{aligned} \tag{3.2}$$

Hence

$$v\langle H, I\rangle \leq ||\mu||_{\mathcal{S}} \ v\langle f, I\rangle.$$

The last inequality together with construction of f imply that $H \in C([0, 2a])$.

Let us prove (ii). From (3.2) and our assumption $||\mu(\ell) - \mu||_{\mathcal{S}} \to_\ell 0$ we get for each $n \in \mathbb{N} \cup \{0\}$ and $I \in \mathfrak{L}^n$,

$$o\langle H, I\rangle \ v\langle H, I\rangle = o\langle f, I\rangle \ v\langle f, I\rangle \sum_{k=\theta(n)}^{\infty} \mu_k$$

$$= \lim_{\ell\to\infty} o\langle f, I\rangle\ v\langle f, I\rangle \sum_{k=\theta(n)}^{\infty} \mu_{k,\ell} = \lim_{\ell\to\infty} o\langle H_\ell, I\rangle\ v\langle H_\ell, I\rangle.$$

Applying the above equality consequently on L-segments of the nth category, $n = 0, 1, \dots$, we get the conclusion (ii). □

Lemma 3.4. *Let $\mu \in \mathcal{S}$ be nonzero. Then the function $\sum_{k=0}^{\infty} \mu_k F_k$ is nonzero on any subinterval of $[0, 2a]$.*

Proof. Let H be given by (3.1). Since μ is nonzero and $\theta\colon \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$ is onto, there exists an $n \geq 0$ such that $\sum_{k=\theta(n)}^{\infty} \mu_k \neq 0$. By our construction, $o\langle f, I\rangle\ v\langle f, I\rangle \neq 0$ for any L-segment I. Then (3.2) implies that $v\langle H, I\rangle$ is nonzero for any $I \in \mathfrak{L}^n$. Thus, the function H is nonzero on any subinterval of $[0, 2a]$. □

4 Besicovitch Functions in $\mathcal{B}_\infty$.

Let f be a function defined on a (one-sided) neighborhood of x. The derived numbers $D^+f(x)$, $D_+f(x)$ of f at x are equal to

$$D^+f(x) = \limsup_{h\to 0_+} \frac{f(x+h) - f(x)}{h},\ \ D_+f(x) = \liminf_{h\to 0_+} \frac{f(x+h) - f(x)}{h},$$

and the analogous limits from the left define $D^-f(x),\ D_-f(x)$. Obviously, f has a one-sided derivative at a point x if and only if either $D^+f(x) = D_+f(x)$ or $D^-f(x) = D_-f(x)$.

The main result of this Section is the following.

Theorem 4.1. *Each nonzero function from $\mathcal{B}_\infty$ is a Besicovitch function.*

Proof. Let H given by (3.1) be nonzero. If we put

$$\nu_n = \sum_{k=n}^{\infty} \mu_k,\ \ n = 0, 1, \dots,$$

then $||\mu||_{\mathcal{S}} = \sup_{n\geq 0} |\nu_n| > 0$.

By Definition 3.2, for any L-segment $I \in \mathfrak{L}^n$,

$$o\langle H, I\rangle = \operatorname{sign}(\nu_{\theta(n)}) \cdot o\langle f, I\rangle. \tag{4.1}$$

We use L-segments (with corresponding categories) and intervals taken with respect to $[0, 2a]$. In order to simplify our notation, in the first part of this proof we denote them $r_{m,p}$, resp. $d_{m,p}$ as L-segments of the 1st category, resp. intervals taken with respect to $[c,(c+d)/2]$ (instead of $[0,a]$). Analogously to (2.1), we put

$$E' = [c,(c+d)/2] \setminus \bigcup_{m,p} r_{m,p}. \tag{4.2}$$

Point of finite category. Assume that x is a point of the nth category, $n \in \mathbb{N}$, contained in an interval $[c,d]$, where $(c,d) \in \mathfrak{L}^{n-1}$ is of the $n-1$st category.

By the symmetry, w.l.o.g. we can assume that

- $x \in [c,(c+d)/2]$,
- $o\langle f,(c,d)\rangle = 1$,
- $\nu_{\theta(n-1)} \geq 0$.

Fix $h > 0$, and let

- $r_{m,p} = (\alpha,\beta)$ the maximal L-segment of the nth category contained in $(x,x+h)$, resp.* $(x-h,x)$;
- Δ the least positive integer for which $\nu_{\theta(n-1+\Delta)} \neq 0$ (such a Δ exists since θ is Toeplitz and onto the set $\mathbb{N}\cup\{0\}$);
- $J = (\alpha',\beta')$ the maximal L-segment of the $(n-1+\Delta)$th category contained in (α,β); obviously,

$$o\langle f,(\alpha',\beta')\rangle = (-1)^{\Delta}. \tag{4.3}$$

In particular, if $\Delta = 1$, then $J = (\alpha,\beta)$. Note that since $r_{m,p} \subset d_{m,p}$ and $r_{m,p}$ is maximal, the point x has to be from $[\delta,\alpha)$, resp.* $(\beta,\varepsilon]$, where $[\delta,\alpha] = d_{m+1,2p-1}$, resp.* $[\beta,\varepsilon] = d_{m+1,2p}$. Put $\gamma' = (\alpha'+\beta')/2$.

(+) Assume that $x \neq (c+d)/2$ is not the left end of any L-segment of the nth category and show that $H'_+(x)$ does not exist. In this case $r_{m,p} \subset (x,x+h)$. From (2.2), (4.2) and our assumption $\nu_{\theta(n-1)} \geq 0$, we get

$$\begin{aligned} 0 \leq \frac{H(\alpha)-H(x)}{\alpha-x} &= \sum_{k=0}^{\infty} \mu_k \frac{[F_k(\alpha)-F_k(x)]}{\alpha-x} \\ &= \sum_{k=0}^{\infty} \mu_k a_{(n-1)k} \frac{2v\langle f,(c,d)\rangle}{\frac{d-c}{2}} \cdot \frac{\lambda(E'\cap[x,\alpha])}{\alpha-x} \\ &\leq \nu_{\theta(n-1)} \frac{4v\langle f,(c,d)\rangle}{d-c} \end{aligned} \tag{4.4}$$

for each h. Hence

$$D_+H(x) \leq \nu_{\theta(n-1)} \frac{4v\langle f,(c,d)\rangle}{d-c} \text{ and } D^+f(x) \geq 0. \tag{4.5}$$

(I+) Either $\nu_{\theta(n-1+\Delta)} < 0$ and Δ is odd, or $\nu_{\theta(n-1+\Delta)} > 0$ and Δ is even. From (4.1) and (4.3) we obtain $o\langle H, J\rangle = 1$. It implies that $H(x) \leq H(\alpha') = H(\beta') < H(\gamma')$ for $x < \alpha' < \gamma' < \beta'$. Using (2.5) we get

$$\begin{aligned}
&\frac{H(\gamma') - H(x)}{\gamma' - x} - \frac{H(\beta') - H(x)}{\beta' - x} \geq \frac{H(\gamma') - H(\beta')}{\gamma' - x} \\
>&\frac{o\langle H, J\rangle \; v\langle H, J\rangle}{\beta - \delta} = \frac{\nu_{\theta(n-1+\Delta)} \; o\langle f, J\rangle \; v\langle f, J\rangle}{\frac{d-c}{2}[\frac{1}{2^{m+1}} + \frac{1}{2\cdot 4^m} + \frac{1}{4^m}]} \\
=&\frac{|\nu_{\theta(n-1+\Delta)}|}{(d-c)[\frac{1}{2^{m+2}} + \frac{1}{2^2\cdot 4^m} + \frac{1}{2\cdot 4^m}]} \cdot \frac{v\langle f, (c,d)\rangle}{2^{m+\Delta-1}} \\
>&\frac{|\nu_{\theta(n-1+\Delta)}| \; v\langle f, (c,d)\rangle}{2^{\Delta}(d-c)} > 0
\end{aligned}$$

independent of h.

(II+) Either $\nu_{\theta(n-1+\Delta)} < 0$ and Δ is even, or $\nu_{\theta(n-1+\Delta)} > 0$ and Δ is odd. From (4.1) and (4.3) we get $o\langle H, J\rangle = -1$. Then $H(x) \leq H(\alpha')$ and $H(\gamma') < H(\alpha') = H(\beta')$ for $x < \alpha' < \gamma' < \beta'$. We get analogously as above,

$$\begin{aligned}
&\frac{H(\alpha') - H(x)}{\alpha' - x} - \frac{H(\gamma') - H(x)}{\gamma' - x} \geq \frac{H(\alpha') - H(\gamma')}{\gamma' - x} \\
>&\frac{-o\langle H, J\rangle \; v\langle H, J\rangle}{\beta - \delta} = \frac{\nu_{\theta(n-1+\Delta)} \; (-o\langle f, J\rangle) \; v\langle f, J\rangle}{\frac{d-c}{2}[\frac{1}{2^{m+1}} + \frac{1}{2\cdot 4^m} + \frac{1}{4^m}]} \\
>&\frac{|\nu_{\theta(n-1+\Delta)}| \; v\langle f, (c,d)\rangle}{2^{\Delta}(d-c)} > 0
\end{aligned}$$

independent of h. Thus, (I+) and (II+), together with (4.5), imply that $H'_+(x)$ does not exist.

(-) Assume that $x \neq c$ is not the right end of any L-segment of the nth category and show that $H'_-(x)$ does not exist. In this case $r_{m,p} \subset (x-h, x)^*$. Since the situation is completely analogous to the previous one, we can be rather brief.

Similarly as in (4.4), we get

$$D_-H(x) \leq \nu_{\theta(n-1)} \frac{4v\langle f, (c,d)\rangle}{d-c}, \; D^-f(x) \geq 0. \tag{4.6}$$

(I-) If either $\nu_{\theta(n-1+\Delta)} < 0$ and Δ is odd, or $\nu_{\theta(n-1+\Delta)} > 0$ and Δ is even, then $o\langle H, J\rangle = 1$. Similarly as in (I+),

$$\frac{H(x) - H(\beta')}{x - \beta'} - \frac{H(x) - H(\gamma')}{x - \gamma'} > \frac{|\nu_{\theta(n-1+\Delta)}| \; v\langle f, (c,d)\rangle}{2^{\Delta}(d-c)} > 0$$

independent of h.

(II-) If either $\nu_{\theta(n-1+\Delta)} < 0$ and Δ is even, or $\nu_{\theta(n-1+\Delta)} > 0$ and Δ is odd, then $o\langle H, J\rangle = -1$. Similarly as in (II+), we get

$$\frac{H(x)-H(\gamma')}{x-\gamma'} - \frac{H(x)-H(\alpha')}{x-\alpha'} > \frac{|\nu_{\theta(n-1+\Delta)}|\ v\langle f,(c,d)\rangle}{2^{\Delta}(d-c)} > 0$$

independent of h.

Summarizing, (I-),(II-) and (4.6) imply that $H'_-(x)$ does not exist. We have proved the following.

Proposition 4.2. *At any point of finite category no one sided derivative of nonzero H given by (3.1) exists.*

Point of infinite category. Recall that it is a point which belongs to L-segments of all categories.

Let us consider nonzero H given by (3.1). Fix a point x of infinite category and the corresponding nested sequence of L-segments, i.e., the sequence

$$(\alpha_1,\beta_1) \supset (\alpha_2,\beta_2) \supset \cdots,\ (\alpha_n,\beta_n) \in \mathfrak{L}^n,\ \{x\} = \bigcap_n (\alpha_n,\beta_n).$$

Fix $h > 0$. From some number onwards, the segments (α_n,β_n) are included in the interval $(x-h, x+h)$. By our construction of the functions $F_0,\ldots,F_k,\ldots$ (mainly since θ is Toeplitz and onto the set $\mathbb{N}\cup\{0\}$), there exists a positive integer σ such that

- $(\alpha_\sigma,\beta_\sigma) \subset (x-h, x+h)$,
- $\theta(\sigma) = \theta(\sigma+1)$,
- $|\nu_{\theta(\sigma)}| = ||\mu||_{\mathcal{S}} \geq |\nu_{\theta(n)}|$, $n \geq \sigma$ ($n \geq 0$ in fact).

W.l.o.g., we can assume that

- $x \in (\alpha_\sigma, (\alpha_\sigma+\beta_\sigma)/2)$,
- $o\langle f, (\alpha_\sigma,\beta_\sigma)\rangle = 1$,
- $\nu_{\theta(\sigma)} > 0$.

Using above assumptions, the fact that $\nu_{\theta(\sigma)}$ has a maximal absolute value and repeatedly applying (2.4) with respect to (α_n,β_n), $n \geq \sigma+1$, we get

$$H(\alpha_\sigma) \leq H(x) \leq H(\alpha_{\sigma+1}). \tag{4.7}$$

Hence $D^+f(x) \geq 0$, $D^-f(x) \geq 0$ and $D_+f(x) \leq 0$, $D_-f(x) \leq 0$. Put $\gamma = (\alpha_\sigma+\beta_\sigma)/2$ and $J = (\alpha_\sigma,\beta_\sigma)$. By virtue of (4.7), (4.1) and (3.2),

$$\begin{aligned}\frac{H(\gamma)-H(x)}{\gamma-x} - \frac{H(\beta_\sigma)-H(x)}{\beta_\sigma-x} &\geq \frac{H(\gamma)-H(\beta_\sigma)}{\beta_\sigma-x}\\ > \frac{o\langle H,J\rangle\ v\langle H,J\rangle}{\beta_\sigma-\alpha_\sigma} = \frac{\nu_{\theta(\sigma)}\ o\langle f,J\rangle\ v\langle f,J\rangle}{\beta_\sigma-\alpha_\sigma} &= \nu_{\theta(\sigma)}\frac{b}{2a}\end{aligned}$$

independent of h. Thus, $H'_+(x)$ does not exist. Similarly, with the help of (2.6),

$$\frac{H(x)-H(\alpha_\sigma)}{x-\alpha_\sigma} - \frac{H(x)-H(\alpha_{\sigma+1})}{x-\alpha_{\sigma+1}} \geq \frac{H(\alpha_{\sigma+1})-H(\alpha_\sigma)}{x-\alpha_\sigma}$$
$$> \frac{H(\alpha_{\sigma+1})-H(\alpha_\sigma)}{\beta_{\sigma+1}-\alpha_\sigma} = \frac{\nu_{\theta(\sigma)}\ [f(\alpha_{\sigma+1})-f(\alpha_\sigma)]}{\beta_{\sigma+1}-\alpha_\sigma} > \nu_{\theta(\sigma)}\frac{b}{2a}$$

independent of h. Thus, $H'_-(x)$ does not exist. We have proved the next assertion.

Proposition 4.3. *At any point of infinite category no one sided derivative of nonzero H given by (3.1) exists.*

This finishes the proof of Theorem 4.1. □

By Lemma 3.4, there is an isomorphism $\iota\colon\ \mathcal{B}_\infty \to \mathcal{S}$ given by

$$\iota\Big(\sum_{k=0}^{\infty}\mu_k F_k\Big) = \sum_{k=0}^{\infty}\mu_k.$$

Thus, we can equip the set $\mathcal{B}_\infty$ by the norm defined as

$$||H||_{\mathcal{B}_\infty} = ||\iota(H)||_{\mathcal{S}}.$$

We get the following theorem.

Theorem 4.4. *The space $(\mathcal{B}_\infty, ||\cdot||_{\mathcal{B}_\infty})$ is a Banach space.*

PROOF. It is an easy consequence of our definitions and the fact that $(\mathcal{S}, ||\ ||_{\mathcal{S}})$ is a Banach space. □

5 The Space $\mathcal{B}_{\mathrm{sup}}$.

In this section we define the set $\mathcal{B}_{\mathrm{sup}}$ announced in our Introduction. Similarly as $\mathcal{B}_\infty$, also the set $\mathcal{B}_{\mathrm{sup}}$ will be defined as a linear hull of countably many linearly independent functions $G_0, \dots, G_k, \dots$ from $C([0, 2a])$, where each function G_k will be obtained from F_k by a suitable perturbation.

Definition 5.1. Let $h \in C([0, 2a])$, $I_j = (c_j, d_j) \subset [0, 2a]$ be pairwise disjoint intervals, $h(c_j) = h(d_j)$ and $\nu_j \in \mathbb{R}$. A map $g \in C([0, 2a])$ is a $\lfloor (I_j)_j \oplus (\nu_j)_j \rfloor$-perturbation of h if g satisfies

$$g(x) = \begin{cases} h(x) & \text{if } x \notin \bigcup_j I_j, \\ \nu_j(h(x) - h(c_j)) + h(c_j) & \text{if } x \in I_j. \end{cases}$$

Recall that L-segments $r_{m,p}$ were introduced in Section 2.

Definition 5.2. Let us consider the sequence $(r_{2m-1,1})_{m=1}^{\infty}$ of L-segments of the first category. We define a sequence $(G_k)_{k=0}^{\infty}$ of functions from $C([0,2a])$ by

(i) $G_0 = F_0$,

(ii) for $k \in \mathbb{N}$, G_k is defined as a $\lfloor (r_{2m-1,1})_{m=1}^{k} \oplus (1-2^{2m-1})_{m=1}^{k} \rfloor$-perturbation of F_k.

Finally, let us define $\mathcal{B}_{\sup} \subset C([0,2a])$ as follows: $H \in \mathcal{B}_{\sup}$ if and only if

$$\exists\ (\nu_k)_{k=0}^{\infty}\ \forall\ x \in [0,2a]\colon\ \ H(x) = \sum_{k=0}^{\infty} \nu_k G_k(x). \tag{5.1}$$

In this case we say that H is given by the sequence $(\nu_k)_{k=0}^{\infty}$ and we write $H = \sum_{k=0}^{\infty} \nu_k G_k$. Note that by our definition, $B_{\sup} \subset C([0,2a])$.

6 Basic Properties of $\mathcal{B}_{\sup}$.

Put $\gamma_0 = a$, and for each $m \in \mathbb{N}$, let γ_m be the center of $r_{2m-1,1}$.

Lemma 6.1. *The following assertions are true.*

(i)

$$G_k(\gamma_m) = \begin{cases} b & \text{if } m \le k, \\ 0 & \text{if } m > k. \end{cases}$$

(ii) If $H = \sum_{k=0}^{\infty} \nu_k G_k \in \mathcal{B}_{\sup}$, then $\sum_{k=0}^{\infty} \nu_k$ is a convergent series ($\sum_{k=0}^{\infty} \nu_k \in \mathcal{S}$).

(iii) $H \in \mathcal{B}_{\sup}$ given by the sequence $(\nu_k)_{k=0}^{\infty}$ is the zero function if and only if $\nu_k = 0$ for each k. In particular, the set

$$\{G_k\colon\ k \in \mathbb{N}_0\} \subset \mathcal{B}_{\sup}$$

is linearly independent.

Proof. (i) This is an easy consequence of Definition 5.1 and the construction of the functions $F_0, F_1, \dots, F_k, \dots$.

(ii) From (5.1) we get $H(\gamma_0) = b \sum_{k=0}^{\infty} \nu_k \in \mathbb{R}$.

(iii) By (ii) and Definition 5.1, the functions F_k, G_k, $k \ge 0$, coincide on the interval $[a, 2a]$. Then, (iii) follows from Lemma 3.4. □

Theorem 6.2. $(\mathcal{B}_{\sup}, \|\ \|_{\sup})$ *is an infinite dimensional Banach space.*

PROOF. By (5.1), $\mathcal{B}_{\sup}$ is a linear (sub)space in $C([0,2a])$. From Lemma 6.1(iii) we get that $\mathcal{B}_{\sup}$ is infinite dimensional. Thus it is sufficient to show that the set $\mathcal{B}_{\sup}$ is closed with respect to the topology induced by $|| \, ||_{\sup}$. Consider a Cauchy sequence

$$\Big(H_\ell = \sum_{k=0}^{\infty} \nu_{k,\ell} G_k\Big)_{\ell=1}^{\infty} \subset \mathcal{B}_{\sup}.$$

Since $(C([0,2a]), || \, ||_{\sup})$ is a Banach space, there is a map $H \in C([0,2a])$ such that $\lim_\ell ||H - H_\ell||_{\sup} = 0$.

For any $\varepsilon > 0$, we have $||H_\ell - H_{\ell'}||_{\sup} < \varepsilon$ whenever ℓ, ℓ' are sufficiently large. For such ℓ, ℓ', from Lemma 6.1(i), we obtain for each $m \in \mathbb{N}_0$,

$$\varepsilon > ||H_\ell - H_{\ell'}||_{\sup} \geq |H_\ell(\gamma_m) - H_{\ell'}(\gamma_m)| = b \; |\sum_{k=m}^{\infty} \nu_{k,\ell} - \nu_{k,\ell'}|;$$

i.e., the sequence $(\sum_{k=0}^{\infty} \nu_{k,\ell})_{\ell=1}^{\infty} \subset \mathcal{S}$ is Cauchy in the space $(\mathcal{S}, || \, ||_{\mathcal{S}})$. Denote $\nu(\ell) := \sum_{k=0}^{\infty} \nu_{k,\ell}$ and let

$$\nu := \sum_{k=0}^{\infty} \nu_k = || \, ||_{\mathcal{S}} - \lim_\ell \nu(\ell). \tag{6.1}$$

To finish our proof, it is sufficient to show that for each $x \in [0,2a]$,

$$\lim_\ell H_\ell(x) = \lim_\ell \sum_{k=0}^{\infty} \nu_{k,\ell} G_k(x) = \sum_{k=0}^{\infty} \nu_k G_k(x), \tag{6.2}$$

since then $H = \sum_{k=0}^{\infty} \nu_k G_k \in \mathcal{B}_{\sup}$.

The last equality is clear when $x \in [0,2a] \setminus \bigcup_{m \geq 1} r_{2m-1,1}$. On this set $F_k = G_k$ for each k and (6.2) follows immediately from (6.1) and Lemma 3.3(ii).

If $x \in r_{2m-1,1} = (\alpha, \beta)$, then from Definitions 5.1 and 5.2, we get

$$\begin{aligned}\sum_{k=0}^{\infty} \nu_{k,\ell} G_k(x) &= \sum_{k=0}^{m-1} \nu_{k,\ell} F_k(x) \\ &+ \sum_{k=m}^{\infty} \nu_{k,\ell} \; [(1 - 2^{2m-1})(F_k(x) - F_k(\alpha)) + F_k(\alpha)].\end{aligned}$$

Hence, again by (6.1), Lemma 3.3(ii) and Definition 5.2,

$$\begin{aligned}\lim_{\ell} H_\ell(x) &= \sum_{k=0}^{m-1} \nu_k F_k(x) + \sum_{k=m}^{\infty} \nu_k \ [(1-2^{2m-1})(F_k(x)-F_k(\alpha)) + F_k(\alpha)] \\ &= \sum_{k=0}^{\infty} \nu_k G_k(x).\end{aligned}$$

This proves the lemma. □

7 Besicovitch Functions in $\mathcal{B}_{\text{sup}}$.

Theorem 7.1. *Each nonzero function from $\mathcal{B}_{\text{sup}}$ is a Besicovitch function.*

PROOF. Let $H = \sum_{k=0}^{\infty} \nu_k G_k \in \mathcal{B}_{\text{sup}}$ be nonzero. Clearly, it means that $\nu_k \neq 0$ for some k. From Lemma 6.1(ii), we know that $\sum_{k=0}^{\infty} \nu_k$ is a convergent series.

By Definition 5.2, the functions G_k, F_k coincide on the set

$$C = [0, 2a] \setminus \bigcup_{m \geq 1} r_{2m-1,1},$$

where $r_{2m-1,1} = (a_{2m-1,1}, b_{2m-1,1})$ is the leftmost L-segment with the length $a/4^{2m-1}$; i.e.,

$$H(x) = \sum_{k=0}^{\infty} \nu_k F_k(x), \ x \in C. \tag{7.1}$$

The fact that the real series $\sum_{k=0}^{\infty} \nu_k$ converges, (7.1) and Theorem 4.1 imply that

- $H'_+(x), H'_-(x)$ does not exist at any $x \in \operatorname{int} C$
- $H'_+(x)$ does not exist at any $x \in \{0\} \cup \{b_{2m-1,1} \colon\ m \geq 1\}$(in order to show that $H'_+(0)$ does not exist we can use the intervals $r_{2m,1}$, $m \geq 1$ as the maximal L-segments of the first category contained in $(0, h)$)
- $H'_-(x)$ does not exist at any $x \in \{a_{2m-1,1} \colon\ m \geq 1\}$

Thus, it remains to show that $H'_-(x)$, resp. $H'_+(x)$ does not exist at any point $x \in \{b_{2m-1,1} \colon\ m \geq 1\}$, resp. $x \in \{a_{2m-1,1} \colon\ m \geq 1\}$. By symmetry, we will only prove the latter case.

Fix $a = a_{2m-1,1}$ and show that $H'_+(a)$ does not exist. For each $x \in r_{2m-1,1}$

we get from Definitions 5.1 and 5.2,

$$\begin{aligned} H(x) &= \sum_{k=0}^{\infty} \nu_k G_k(x) \\ &= \sum_{k=0}^{m-1} \nu_k F_k(x) + \sum_{k=m}^{\infty} \nu_k \; [(1-2^{2m-1})(F_k(x)-F_k(a)) + F_k(a)] \qquad (7.2) \\ &= (1-2^{2m-1}) \left[\sum_{k=0}^{m-1} \frac{\nu_k}{1-2^{2m-1}} F_k(x) + \sum_{k=m}^{\infty} \nu_k F_k(x) \right] + 2^{2m-1} \sum_{k=m}^{\infty} \nu_k F_k(a). \end{aligned}$$

Since, by Theorem 4.1, the function

$$G(x) = \sum_{k=0}^{m-1} \frac{\nu_k}{1-2^{2m-1}} F_k(x) + \sum_{k=m}^{\infty} \nu_k F_k(x)$$

is Besicovitch, $G'_+(a)$ does not exist. Hence, by (7.2), also $H'_+(a)$ does not exist. This proves the theorem. □

Thus, for the value $a = 1/2$ we get the following.

Theorem 7.2. $(\mathcal{B}_{\sup}, || \, ||_{\sup})$ *is an infinite dimensional Banach (sub)space in C([0,1]) and each nonzero element of* $\mathcal{B}_{\sup}$ *is a Besicovitch function.*

PROOF. It is an immediate consequence of Theorem 6.2 and Theorem 7.1. □

References

[1] E. I. Berezhnoĭ, *The Subspace of* $C[0,1]$ *Consisting of Functions Having No Finite One-sided Derivatives at Any Point*, Mat. Zametki, **73(3)** (2003), 348–354.

[2] A. S. Besicovitch, *Diskussion der stetigen Funktionen im Zusammenhang mit der Frage über ihre Differentierbarkeit*, Bulletin de l'Académie des Sciences de Russie, **19** (1925), 527–540.

[3] J. Bobok, *On Non-differentiable Measure-preserving Functions*, Real Anal. Exchange, **16** (1990-91), 119–129.

[4] ———, *On a Space of Besicovitch Functions*, Real Anal. Exchange, **30(1)** (2004-2005), 173–182.

[5] D. Lind and B. Marcus, *An introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995.

[6] E. D. Pepper, *On Continuous Functions Without a Derivative*, Fund. Math., **12** (1928), 244–253.

[7] S. Saks, *On the Functions of Besicovitch in the Space of Continuous Functions*, Fund. Math., **19** (1932), 211–219.

[8] G. S. De Souza and G. O. Golightly, *On Some Spaces of Summable Sequences and Their Duals*, Internat. J. Math. Math. Sci., **9(1)** (1986), 71–79.

[9] V. P. Fonf, V. I. Gurariy and M. I. Kadets, *An Infinite Dimensional Subspace of $C[0,1]$ Consisting of Nowhere Differentiable Functions*, C. R. Acad. Bulgare Sci., **52(11-12)** (1999), 13–16.