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## COLLECTIONS OF COMPACT SETS AND FUNCTIONS HAVING $G_\delta$ -GRAPHS

### Abstract

We continue the investigation began in [1] of the connection between the structure of a function  $f$  defined on a subset of a space  $X$  and the Borel complexity of the set  $\mathcal{C}(f) = \{C \in \mathbf{J}(X) : f|_C \text{ is continuous}\}$  where  $\mathbf{J}(X)$  denotes the nonempty compact subsets of  $X$  with the Hausdorff metric. Two hierarchies of functions with  $G_\delta$ -graphs are defined. We conjecture that they coincide.

### 1 Introduction.

Given a Polish space  $X$  let  $\mathbf{J}(X)$  denote the collection of nonempty compact subsets of  $X$  with the Hausdorff metric. We investigate the connection between the structure of a function  $f$  defined on a subset of a space  $X$  and the Borel complexity of the set  $\mathcal{C}(f) = \{C \in \mathbf{J}(X) : f|_C \text{ is continuous}\}$  in  $\mathbf{J}(X)$ .

The following proposition allows restricting our consideration to functions with  $G_\delta$ -graph.

**Proposition 1** ([1], Theorem 5). *Suppose  $X$  and  $Y$  are Polish spaces and  $f: X \rightarrow Y$  is Borel.  $f$  has  $G_\delta$ -graph in  $X \times Y$  if and only if  $\mathcal{C}(f)$  is Borel in  $\mathbf{J}(X)$ .*

Hence, functions with  $G_\delta$ -graph may be considered as having a hierarchy of complexity based on the Borel complexity of the collection compact sets

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upon which their restrictions are continuous. These collections of compact sets are rather abstract which in turn would mean that the classification of  $f$  could be difficult. Our goal here is to attempt to connect this hierarchy of functions with  $G_\delta$ -graphs, with a hierarchy based on the structure of the graph that is much more accessible. There is a gap that remains to be filled. The results we do have seem to suggest that the connection between the hierarchies is legitimate. We can verify the connection for functions which are not too complex. We also investigate some interesting properties of functions such that  $\mathcal{C}(f)$  is an  $F_{\sigma\delta}$ -set.

## 2 Preliminaries.

Given a function  $f: X \rightarrow Y$  and  $A \subseteq X$  we let  $f|A$  denote the restriction of  $f$  to  $A$ . Given a product of two sets  $X \times Y$  we let  $\pi_X$  and  $\pi_Y$  denote the usual projections onto  $X$  or  $Y$ , respectively.

Suppose  $X$  is a Polish space with metric  $d$ . For a set  $A \subseteq X$  we write  $\text{cl}_X(A)$ ,  $\text{int}_X(A)$ ,  $\text{bd}_X(A)$  for the closure, interior, and boundary of  $A$  in  $X$ , respectively. When it is understood what space we are referring to the subscript will be dropped. Given sets  $A, B \subseteq X$  we define  $d(A, B) = \inf(\{d(x, y) : x \in A \ \& \ y \in B\})$ . Given sets  $A, B \subseteq X$  we define the Hausdorff distance between  $A$  and  $B$  to be  $H_d(A, B) = \max(\sup(\{d(\{x\}, B) : x \in A\}), \sup(\{d(A, \{y\}) : y \in B\}))$ . When  $H_d$  is restricted to the compact subsets of  $X$  it is a metric known as the Hausdorff metric. The diameter of a nonempty set  $A \subseteq X$  is defined by  $\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$ . If  $A = \emptyset$ , we let  $\text{diam}(A) = 0$ . It is known that if  $X$  is Polish, then  $J(X)$  is Polish as well [2, 4.25]. By a Cantor set we mean a compact totally disconnected metric space with no isolated points.

Let  $X$  be a separable metric space. For  $0 < \alpha < \omega_1$  let  $\Sigma_\alpha^0(X)$  and  $\Pi_\alpha^0(X)$  stand for the subclasses of Borel sets defined as in [2, 11.B] (e.g.,  $\Pi_2^0$  is  $G_\delta$  and  $\Sigma_2^0$  is  $F_\sigma$ ). A set  $A \subseteq X$  is said to be  $\Sigma_\alpha^0$ -hard provided that for any zero-dimensional Polish space  $Y$  and  $B \in \Sigma_\alpha^0(Y)$  there is a continuous function  $f: Y \rightarrow X$  such that  $f^{-1}(A) = B$ . To say that  $A$  is  $\Sigma_\alpha^0$ -hard is essentially saying that  $A$  is at least as complex as any  $\Sigma_\alpha^0$ -set. In particular, if  $A \subseteq X$  is  $\Sigma_\alpha^0$ -hard, then  $A \notin \Pi_\alpha^0(X)$ . If  $A$  is  $\Sigma_\alpha^0$ -hard and  $A \in \Sigma_\alpha^0(X)$ , then we say  $A$  is  $\Sigma_\alpha^0$ -complete. The notions of  $\Pi_\alpha^0$ -hard and  $\Pi_\alpha^0$ -complete are defined analogously.

Let  $W, Y, Z \subseteq X$ . We say that  $Y$  separates  $W$  from  $Z$  provided that  $W \subseteq Y$  and  $Y \cap Z = \emptyset$ .

For a function  $f: X \rightarrow Y$  and  $S \subseteq X$  we define the oscillation of  $f$  on a set  $S$  by  $\text{osc}(f, S) = \sup\{d(f(x), f(y)) : x, y \in S\}$ . We define the oscillation of  $f|S$  at the point  $x \in S$  by  $\text{osc}(f, S, x) = \inf_{\delta > 0} \text{osc}(f, (S \cap B_\delta(x)))$  where

$B_\delta(x)$  is the set of all points of distance less than  $\delta$  from  $x$ . We say a function  $f: S \rightarrow Y$  is  $\epsilon$ -continuous provided that  $\text{osc}(f, S, x) \leq \epsilon$  for all  $x \in S$ .

We say  $f: X \rightarrow Y$  is of Borel class  $\alpha$  and write  $\mathcal{B}_\alpha$  provided that  $f^{-1}(U) \in \Sigma_{\alpha+1}^0(X)$  for every open set  $U \subseteq Y$ . Indeed, when we mention the Borel complexity of a function we mean the minimal value of  $\alpha$  such that  $f \in \mathcal{B}_\alpha$ .

We say a function  $f: X \rightarrow \mathbb{R}$  is lower (upper) semicontinuous in the first sense provided that  $f$  has the property that  $f^{-1}(-\infty, r)$  ( $f^{-1}(r, +\infty)$ ) is open for all  $r \in \mathbb{R}$ . By semicontinuity in the second sense we mean lower or upper semicontinuity as defined for set valued functions. Recall that a function  $f: X \rightarrow \mathcal{A}$  where  $\mathcal{A}$  is a collection of compact subsets of a metric space endowed with the Hausdorff metric is said to be upper semicontinuous provided that  $\lim x_n = x$  implies that  $\lim f(x_n) \subseteq f(x)$ . Similarly, we say  $g: X \rightarrow \mathcal{A}$  is lower semicontinuous provided that  $\lim x_n = x$  implies that  $\lim f(x_n) \supseteq f(x)$ .

### 3 Results.

We begin with some definitions describing the structure of graphs of functions. Complete characterizations of these structures in terms of the Borel classification of  $\mathcal{C}(f)$  for very small values of  $\alpha < \omega_1$  are given. However, the results and their proofs strongly suggest extensions for all  $\alpha$ . For this reason we include the general definitions. It is not known if any of the results presented here for small values of  $\alpha$  can be extended to higher values of  $\alpha$ .

Let  $2 \leq \alpha < \omega_1$ . Let  $X$  and  $Y$  be metric spaces. If a function  $f: X \rightarrow Y$  has the property that for every  $x \in X$  and  $\epsilon > 0$  there exist open sets  $U \subseteq X$  and  $V \subseteq Y$  such that  $x \in f^{-1}(V) \cap U$ , and a  $G \in \Pi_\beta^0(X)$  where  $\beta < \alpha$  such that  $f^{-1}(V) \cap U \subseteq G$  and oscillation  $f|G$  not more than  $\epsilon$ , then we say that  $f \in \mathcal{E}_\alpha$ . The following proposition is straight forward to verify.

**Proposition 2.** *For  $n \in \omega \setminus \{0, 1\}$  we have  $\mathcal{B}_{n-2} \subseteq \mathcal{E}_n \subseteq \mathcal{B}_{n-1}$ . For a limit ordinal  $\alpha$  we have  $\bigcup_{\beta < \alpha} \mathcal{B}_\beta \subseteq \mathcal{E}_\alpha \subseteq \mathcal{B}_\alpha$ . For a successor ordinal  $\omega < \alpha$  we have  $\mathcal{B}_{\alpha-1} \subseteq \mathcal{E}_\alpha \subseteq \mathcal{B}_{\alpha+1}$ .*

If a function  $f: X \rightarrow Y$  has the property that for every  $x \in X$  there exist open sets  $U \subseteq X$  and  $V \subseteq Y$  such that  $x \in U$ ,  $f(x) \in V$ , and  $f|(f^{-1}(V) \cap U)$  is continuous, then we say  $f \in \mathcal{F}_2$ . The class  $\mathcal{F}_2$  is what was called  $T_1$  in [1]. In the case when  $3 \leq \alpha < \omega_1$  we say  $f \in \mathcal{F}_\alpha$  provided that for every  $x \in X$  there exist open sets  $U \subseteq X$  and  $V \subseteq Y$  such that  $x \in f^{-1}(V) \cap U$ , and  $f|(f^{-1}(V) \cap U)$  is  $\mathcal{E}_\beta$  for some  $\beta < \alpha$ . Notice that a function in  $\mathcal{F}_2$  does not have to be measurable or have the Baire property since every characteristic function is in  $\mathcal{F}_2$ . Of course, if  $f \in \mathcal{F}_\alpha$  and has  $G_\delta$ -graph, we might expect to get control over the Borel complexity of  $f$ . How much control? Again we

know the answer for small values of  $\alpha$  but not in general. This is our first problem.

**Problem 1.** *If  $f \in \mathcal{F}_\alpha$  and has  $G_\delta$ -graph in  $X \times Y$ , what is the Borel complexity of  $f$ ?*

The next two theorems relate the classes  $\mathcal{E}_\alpha$  and  $\mathcal{F}_\alpha$  to the Borel complexity of  $\mathcal{C}(f)$  for general  $\alpha$ .

**Theorem 3.** *Suppose  $X$  and  $Y$  are Polish,  $M \subseteq X$  is Borel,  $f: M \rightarrow Y$  is Borel, and that  $2 \leq \alpha < \omega_1$ . Each of the following statements implies the next.*

- (i)  $\mathcal{C}(f) \in \Pi_{\alpha+1}^0(\mathcal{J}(X))$
- (ii)  $\mathcal{C}(f|K)$  is not  $\Sigma_{\alpha+1}^0$ -hard for all compact  $K \subseteq M$ .
- (iii)  $f \in \mathcal{E}_\alpha$ .

**Theorem 4.** *Suppose  $X$  and  $Y$  are Polish,  $M \subseteq X$  is Borel,  $f: M \rightarrow Y$  is Borel, and that  $2 \leq \alpha < \omega_1$ .*

- (i) *If  $\alpha$  is a successor ordinal and  $\mathcal{C}(f) \in \Sigma_{\alpha+1}^0(\mathcal{J}(X))$ , then  $f \in \mathcal{F}_\alpha$ .*
- (ii) *If  $\alpha$  is a limit ordinal and  $\mathcal{C}(f) \in \Sigma_\alpha^0(\mathcal{J}(X))$ , then  $f \in \mathcal{F}_\alpha$ .*

If the converses of these theorems could be established for functions with  $G_\delta$ -graph in  $X \times Y$ , then we would have a good correspondence between the types of complexity.

The following three results verify the converses of Theorem 3 and Theorem 4 for small values of  $\alpha$ .

**Theorem 5.** *Let  $X$  and  $Y$  be Polish,  $A \subseteq X$ , and  $\alpha \in \{2, 3\}$ . A function  $f: A \rightarrow Y$  with  $G_\delta$ -graph in  $X \times Y$  is in  $\mathcal{F}_\alpha$  if and only if  $\mathcal{C}(f) \in \Sigma_{\alpha+1}^0(\mathcal{J}(X))$ .*

In [1] functions in  $\mathcal{F}_2$  are called  $T_1$ . In that paper the equivalence of Theorem 5 is proven in the case when  $\alpha = 2$  and  $f$  is defined on all of  $X$ . The argument presented there works for partial functions as well and mirrors the argument that is presented in this paper for  $\alpha = 3$ .

**Theorem 6.** *Let  $X$  and  $Y$  be Polish. A Borel function  $f: X \rightarrow Y$  is in  $\mathcal{E}_3$  if and only if  $\mathcal{C}(f) \in \Pi_4^0(\mathcal{J}(X))$ .*

Notice that Theorem 6 is not an exact converse since we assume  $f$  is defined on all of  $X$ . Getting an argument proving Theorem 6 for partial functions would probably show how one could prove the general equivalence that is conjectured, this should be clear from the proof of Theorem 5. We have a true converse and more detailed results for the class  $\mathcal{E}_2$ .

**Theorem 7.** *Let  $X, Y$  be Polish,  $Y^*$  be a separable Banach space containing  $Y$  and  $A \subseteq X$  be Borel. The each of the following conditions implies the next for  $f: A \rightarrow Y$ .*

- (i)  $\mathcal{C}(f) \in \Pi_3^0(\mathcal{J}(X))$
  - (ii)  $f \in \mathcal{E}_2$
  - (iii) there is sequence of continuous functions  $\{f_k\}_{k \in \omega}$  from  $A$  into  $Y^*$  such that  $f_k \rightarrow f$  pointwise and for any  $P \in \mathcal{J}(A)$  we have that  $f|P$  is continuous if and only if  $\{f_k|P\}_{k \in \omega}$  is uniformly convergent.
- Moreover, if  $A \in \Pi_2^0(X)$  or  $f$  has  $G_\delta$ -graph in  $X \times Y$ , then the conditions are equivalent.

We conjecture that the above theorems and corollaries may be extended to all  $1 \leq \alpha$  if we assume functions to have  $G_\delta$ -graphs.

Functions that are in  $\mathcal{E}_2$  can be seen as generalizing and unifying the two senses of semicontinuity mentioned in Section 2.

Let  $f: X \rightarrow \mathbb{R}$  be a semicontinuous (in the first sense) function. Now  $f$  is the pointwise limit of an increasing or decreasing sequence of continuous functions. So, by Dini's Theorem,  $f$  satisfies condition (iii) of Theorem 7. Hence, every semicontinuous (in the first sense) function satisfies all the conditions of Theorem 7.

We will show that every semicontinuous (in the second sense) function satisfies the conditions of Theorem 7 in Section 6.

Generally, the conditions of Theorem 7 do not imply semicontinuity in either sense as is shown by the following function which is easily checked to be in  $\mathcal{E}_2$ .

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

A natural question that arises at this point is whether the Borel complexity of  $\mathcal{C}(f)$  is bounded in  $\omega_1$  for functions with  $G_\delta$ -graph in  $X \times Y$ . It is well known that for an uncountable Polish space  $X$  there are functions defined on  $X$  with  $G_\delta$ -graph that have arbitrarily high Borel complexity. So, to find a function  $f$  defined on  $X$  such that  $\mathcal{C}(f)$  is Borel but not  $\Pi_\beta^0$  in  $\mathcal{J}(X)$  it is enough, by Proposition 1, Proposition 2 and Theorem 3, to take any  $f: X \rightarrow Y$  with  $G_\delta$  graph that is not in  $\mathcal{B}_\beta$ .

### 4 Proof of Theorem 3.

Obviously, (i) implies (ii).

We know show that (ii) implies (iii) The next two Propositions are essentially Exercise 23.5(i) of [2].

**Proposition 8.** *Suppose  $\alpha$  is a limit ordinal. If  $\{X_k\}_{k \in \omega}$  are Polish and  $\{A_k\}_{k \in \omega}$  is such that  $A_k$  is  $\Sigma_{\alpha_k}^0$ -complete in  $X_k$  and  $\alpha_k < \alpha$  for all  $k \in \omega$  and*

$\lim_{k \in \omega} \alpha_k = \alpha$  and

$$D = \left\{ \sigma \in \prod_{k \in \omega} X_k : (\exists n \in \omega)(\forall k \geq n)(\sigma(k) \in A_k) \right\},$$

then  $D$  is a  $\Sigma_{\alpha+1}^0$ -complete subset of  $\prod_{k \in \omega} X_k$ .

**Proposition 9.** *Suppose  $\alpha = \beta + 1$ . If  $\{X_k\}_{k \in \omega}$  are Polish and  $\{A_k\}_{k \in \omega}$  is such that  $A_k$  is  $\Sigma_\beta^0$ -complete in  $X_k$  and*

$$D = \left\{ \sigma \in \prod_{k \in \omega} X_k : (\exists n \in \omega)(\forall k \geq n)(\sigma(k) \in A_k) \right\},$$

then  $D$  is a  $\Sigma_{\alpha+1}^0$ -complete subset of  $\prod_{n \in \omega} X_k$ .

**Lemma 10.** *Suppose  $2 \geq \alpha$  is a successor ordinal,  $X$  and  $Y$  are Polish,  $M \subseteq X$  is Borel, and  $f: M \rightarrow Y$  is Borel. If  $\mathcal{C}(f|K) \in \Pi_{\alpha+1}^0(J(X))$  for every compact set  $K \subseteq M$ , then  $f \in \mathcal{E}_\alpha$ .*

PROOF. Let  $\beta + 1 = \alpha$ . Suppose  $f \notin \mathcal{E}_\alpha$ . There is an  $p \in M$  and an  $\epsilon > 0$  such that for every pair of open sets such that  $p \in U$  and  $f(p) \in V$  we have  $\text{osc}(f, G) \geq \epsilon$  for any  $G \in \Pi_\beta^0(M)$  containing  $f^{-1}(V) \cap U \cap M$ .

Let  $\{V_n\}_{n \in \omega}$  be a decreasing sequence of open subsets of  $Y$  such that  $f(p) \in V_n$  and  $H_d(V_n, f(p)) < 1/2^n$  for every  $n \in \omega$ . Let  $\{U_n\}_{n \in \omega}$  be a decreasing sequence of open subsets of  $X$  such that  $H_d(U_n, p) < 1/2^n$  and  $p \in U_n$  for every  $n \in \omega$ . Fix  $n \in \omega$  such that  $1/2^n < \epsilon$ . Since  $\text{osc}(f, G) \geq \epsilon$  for every  $G \in \Pi_\beta^0(M)$  containing  $f^{-1}(V_n) \cap U_n \cap M$ , we conclude that  $f^{-1}(V_n) \cap U_n \cap M$  cannot be separated from the set  $\{x \in U_n \cap M : d(f(x), f(p)) \geq \epsilon/2\}$  by a  $\Pi_\beta^0$ -subset of  $U_n$ . Indeed, if such a set did exist its intersection with  $M$  would be a relative  $\Pi_\beta^0$ -subset of  $M$  and contradict our choice of  $n$ . Since  $M$  is Borel and  $f: M \rightarrow Y$  is Borel, the non-separable sets are Borel in  $X$ . Now, [2, 22.13, 24.20] guarantees that there is a compact set  $K_n \subseteq U_n$  and a  $\Sigma_\beta^0$ -complete  $D_n \subseteq K_n$  such that  $D_n = K_n \cap (f^{-1}(V_n) \cap U_n \cap M)$  and  $K_n \setminus D_n = K_n \cap \{x \in U_n \cap M : d(f(x), f(p)) \geq \epsilon/2\}$ . Notice that  $K_n \subseteq M$ .

Let  $K = \{p\} \cup \bigcup_{n \in \omega} K_n \subseteq M$ . Let  $h: \prod_{n \in \omega} K_n \rightarrow J(K)$  be defined by  $h(\sigma) = \{\sigma(n) : n \in \omega\} \cup \{p\}$ . Notice that  $h$  is continuous and that  $h^{-1}(\mathcal{C}(f|K))$  is precisely the set of all  $\sigma \in \prod_{n \in \omega} K_n$  such that  $\sigma(n) \in D_n$  for all  $n$  sufficiently large. Since each  $D_n$  is  $\Sigma_\beta^0$ -complete, Proposition 9 guarantees that  $h^{-1}(\mathcal{C}(f|K))$  is  $\Sigma_{\beta+2}^0$ -complete. Thus,  $\mathcal{C}(f|K)$  is  $\Sigma_{\alpha+1}^0$ -hard in  $J(K)$ .  $\square$

A similar argument using Proposition 8 establishes the limit case.

**Lemma 11.** *Suppose  $\alpha$  is a limit ordinal,  $X$  and  $Y$  are Polish,  $M \subseteq X$  is Borel, and  $f : M \rightarrow Y$  is Borel. If  $\mathcal{C}(f|K) \in \Pi_{\alpha+1}^0(\mathbf{J}(X))$  for every compact  $K \subseteq M$ , then  $f \in \mathcal{E}_\alpha$ .*

Lemma 10 and Lemma 11 proves that (ii) implies (iii).

### 5 Proof of Theorem 4.

The following lemmas are slight variations of [2, 23.3].

**Lemma 12.** *If  $\{X_n\}_{n \in \omega}$  are Polish spaces and  $\{A_n\}_{n \in \omega}$  is such that  $A_n$  is  $\Sigma_\alpha^0$ -hard in  $X_n$  for every  $n \in \omega$ , then  $\prod_{n \in \omega} A_n$  is  $\Pi_{\alpha+1}^0$ -hard in  $\prod_{n \in \omega} X_n$ .*

**Lemma 13.** *Let  $\alpha$  be a limit ordinal and  $\alpha_n < \alpha$  for every  $n$  and  $\lim_{n \in \omega} \alpha_n = \alpha$ . If  $\{X_n\}_{n \in \omega}$  are Polish spaces and  $\{A_n\}_{n \in \omega}$  is such that  $A_n$  is  $\Sigma_{\alpha_n}^0$ -hard in  $X_n$  for every  $n \in \omega$ . Then  $\prod_{n \in \omega} A_n$  is  $\Pi_\alpha^0$ -hard in  $\prod_{n \in \omega} X_n$ .*

PROOF OF THEOREM 4. The case  $\alpha = 2$  is Lemma 11 of [1]. Notice the argument there is valid for any function in  $\mathcal{F}_2$  defined on any set.

Suppose  $2 < \alpha = \beta + 1$ . Suppose  $f \notin \mathcal{F}_\alpha$ . Let  $x \in X$  be such that for every pair of open sets  $U \subseteq X$  and  $V \subseteq Y$  with  $x \in U$  and  $f(x) \in V$  we have  $f|(f^{-1}(V) \cap U)$  is not  $\mathcal{E}_\beta$ . Let  $\{V_n\}_{n \in \omega}$  be a decreasing sequence of open subsets of  $Y$  such that  $H_d(V_n, f(x)) < 1/2^n$  and  $f(x) \in V_n$  for every  $n \in \omega$ . Let  $\{U_n\}_{n \in \omega}$  be a decreasing sequence of open subsets of  $X$  such that  $H_d(U_n, x) < 1/2^n$  and  $x \in U_n$  for every  $n \in \omega$ . Fix  $n \in \omega$ . Since  $f|(f^{-1}(V_n) \cap U_n)$  is not  $\mathcal{E}_\beta$ , there is by Theorem 3, a compact set  $L_n \subseteq f^{-1}(V_n) \cap U_n$  such that  $\mathcal{C}(f|L_n)$  is  $\Sigma_{\beta+1}^0$ -hard in  $\mathbf{J}(L_n)$ . By our choices of  $L_n$ , the set  $L = \{x\} \cup (\bigcup_{n \in \omega} L_n)$  is compact. Notice that  $\lim_{n \in \omega} f|L_n = (x, f(x))$ . Define  $h : \prod_{n \in \omega} \mathbf{J}(L_n) \rightarrow \mathbf{J}(L)$  by  $h(\sigma) = \{x\} \cup (\bigcup_{n \in \omega} \sigma(n))$ . Notice that  $h$  is continuous and that  $\sigma \in h^{-1}(\mathcal{C}(f|L))$  if and only if  $\sigma \in \prod_{n \in \omega} \mathcal{C}(f|L_n)$ . By Lemma 12,  $\prod_{n \in \omega} \mathcal{C}(f|L_n)$  is  $\Pi_{\beta+2}^0(\mathbf{J}(L))$ -hard. Thus,  $\mathcal{C}(f|L) \notin \Sigma_{\beta+2}^0(\mathbf{J}(X)) = \Sigma_{\alpha+1}^0(\mathbf{J}(X))$ .

A similar argument, using Lemma 13 takes care of the case when  $\alpha$  is a limit ordinal. □

### 6 Proof of Theorem 7 and its Corollaries.

We begin with a combinatorial lemma.

**Lemma 14.** *Let  $M$  be a finite collection of nonempty sets. There exist a partition  $\{\mathcal{M}_0, \dots, \mathcal{M}_n\}$  of the set  $\{L \subseteq M : \bigcap L \neq \emptyset\}$  such that, for every*

$0 \leq l \leq n$  if  $N, K \in \mathcal{M}_l$  are distinct and  $\bigcap(N \cup K) \neq \emptyset$ , then there is a  $j < l$  such that  $N \cup K \in \mathcal{M}_j$ .

PROOF. Let  $\mathcal{M}_0$  be the set of  $\subseteq$ -maximal elements of  $\{N \subseteq M : \bigcap N \neq \emptyset\}$ . Assume we have defined  $\mathcal{M}_k$ . We define  $\mathcal{M}_{k+1}$  to be the collection of all  $\subseteq$ -maximal elements of  $\{N \subseteq M : (\exists L \in \mathcal{M}_k)(N \subsetneq L)\}$ . Clearly, there is a minimal  $n \in \omega$  such that  $\mathcal{M}_{n+1} = \{\emptyset\}$ . We will show that  $\{\mathcal{M}_0, \dots, \mathcal{M}_n\}$  is as desired. We show that  $\{\mathcal{M}_0, \dots, \mathcal{M}_n\}$  partitions  $\{L \subseteq M : \bigcap L \neq \emptyset\}$ . It is clear from the construction that the sets  $\{\mathcal{M}_0, \dots, \mathcal{M}_n\}$  are mutually disjoint. Let  $L \subseteq M$  and  $\bigcap L \neq \emptyset$ . Since  $\bigcap L \neq \emptyset$ , there is a  $R \in \mathcal{M}_0$  such that  $L \subseteq R$ . Let  $k$  be the largest integer such that  $L \subseteq S$  for some  $S \in \mathcal{M}_k$ . Clearly,  $0 \leq k \leq n$ . We claim that  $L = S$ . By way of contradiction assume that  $L \subsetneq S$ . By definition of  $\mathcal{M}_{k+1}$ , there is a  $T \in \mathcal{M}_{k+1}$  such that  $L \subseteq T$  which contradicts our choice of  $k$ .

We now show the other condition holds. By maximality, if  $N, K \in \mathcal{M}_0$  are distinct, then  $\bigcap(N \cup K) = \emptyset$ . Suppose that  $N, K$  are distinct members of  $\mathcal{M}_l$  where  $l > 0$  and  $\bigcap(N \cup K) \neq \emptyset$ . Since  $\{\mathcal{M}_0, \dots, \mathcal{M}_n\}$  partitions  $\{L \subseteq M : \bigcap L \neq \emptyset\}$ , there is a  $0 \leq j \leq n$  such that  $N \cup K \in \mathcal{M}_j$ . Since  $N$  and  $K$  are distinct,  $N \subsetneq (N \cup K)$ . So, there is a  $T_1 \in \mathcal{M}_{j+1}$  such that  $N \subseteq T_1$ . If  $N \subsetneq T_1$ , then there is a  $T_2 \in \mathcal{M}_{j+2}$  such that  $N \subseteq T_2$ . Continuing in this manner we may find, since  $\mathcal{M}_{n+1} = \emptyset$ , a  $0 < k$  such that  $N = T_k \in \mathcal{M}_{j+k}$ . Thus,  $j < j + k = l$ .  $\square$

**Lemma 15.** *Suppose  $Y$  is a Banach space,  $X$  is a separable metric space and  $f: X \rightarrow Y$ . If  $M$  is a finite collection of closed subsets of  $X$  and  $\text{osc}(f, m) \leq \epsilon$  for every  $m \in M$ , then there is a continuous function  $g$  such that for all  $x \in \bigcup M$  we have  $d(g(x), f(x)) \leq \epsilon$ .*

PROOF. Let  $\{\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_p\}$  be a partition of  $M$  as in Lemma 14. For each  $K \in \mathcal{M}_0$  pick  $x_K \in \bigcap K$ . Define  $g$  on  $\bigcap K$  so that  $g[\bigcap K] = \{f(x_K)\}$ . Notice that  $g[\bigcup\{\bigcap K : K \in \mathcal{M}_0\}]$  is continuous. Moreover, since  $f(x_K) = g(x_K)$ , for any  $K \in \mathcal{M}_0$ . We have  $\sup\{d(g(w), f(x)) : x \in \bigcup K \text{ \& } w \in \bigcap K\} \leq \epsilon$ .

Suppose  $1 \leq i + 1 \leq p$  and we have extended  $g$  to  $\bigcup\{\bigcap K : K \in \bigcup_{j \leq i} \mathcal{M}_j\}$  so that  $g$  is continuous and for every  $K \in \bigcup_{j \leq i} \mathcal{M}_j$  we have

$$\sup\{d(g(w), f(x)) : x \in \bigcup K \text{ \& } w \in \bigcap K\} \leq \epsilon. \quad (1)$$

We now show how to extend  $g$  to  $\bigcup\{\bigcap K : K \in \bigcup_{j \leq i+1} \mathcal{M}_j\}$ . We consider two cases.

Suppose  $i + 1 < p$ . Let  $L \in \mathcal{M}_{i+1}$ . Let  $A$  be the closed subset of  $\bigcap L$  where  $g$  is already defined. There are  $K_1, \dots, K_n \in \mathcal{M}_i$  such that  $L \subsetneq K_t$  for



$1 \leq t \leq n$ . Let  $x \in \bigcup L$  and  $w \in A$ . Since  $g(w)$  has been defined,  $w \in \bigcap K_t$  for some  $1 \leq t \leq n$ . Let  $l \in L$  be such that  $x \in l$ . Since  $l \in L \subseteq K_t$ , we have  $x \in \bigcup K_t$ . By (1) and  $K_t \in \mathcal{M}_i$ ,  $d(g(w), f(x)) \leq \epsilon$ . It follows that,

$$\sup \left\{ d(g(w), f(x)) : x \in \bigcup L \ \& \ w \in A \right\} \leq \epsilon.$$

Continuously extend  $g$  from  $A$  to  $\bigcap L$  in such a way that  $g[\bigcap L]$  is contained in the convex hull of  $g[A]$ . Hence,  $\sup\{d(g(w), f(x)) : x \in \bigcup L \ \& \ w \in \bigcap L\} \leq \epsilon$ . Notice we may do this for every  $L \in \mathcal{M}_{i+1}$  and still have  $g$  well defined, since, by Lemma 14 for distinct  $L_1, L_2 \in \mathcal{M}_{i+1}$  we have  $\bigcap L_1 \cap \bigcap L_2 \subseteq \bigcap K$  where  $K \in \bigcup_{j \leq i} \mathcal{M}_j$ . So, now  $g$  is continuous on the set  $\bigcup\{\bigcap K : K \in \bigcup_{j \leq i+1} \mathcal{M}_j\}$ .

Suppose now that  $i + 1 = p$ . In this case,  $\bigcup\{\bigcap K : K \in \bigcup_{j \leq i} \mathcal{M}_j\} = \bigcup M$  and we may take any continuous extension of  $g|M$  to all of  $X$ . It is easily verified that  $g : X \rightarrow Y$  has the desired properties.  $\square$

**Proposition 16.** ([3, 23.1]) Every lower semicontinuous map  $F$  from a metric space  $X$  into the collection of convex closed subsets of a Banach space  $Y$  admits a continuous selector (i.e., a continuous  $f : X \rightarrow Y$  such that  $f(x) \in F(x)$  for all  $x$ ).

**Lemma 17.** Let  $X$  be a metric space and  $Y$  be a Banach space. If  $A \subseteq X$  is closed and there are continuous functions  $f : A \rightarrow Y$  and  $g : X \rightarrow Y$  such that  $d(f|A, g|A) \leq \epsilon$ , then  $f$  may be extended to a continuous function  $h : X \rightarrow Y$  so that  $d(h, g) \leq \epsilon$ .

PROOF. Let  $F$  be the function on  $X$  which assigns to each  $x \in X \setminus A$  the closed  $\epsilon$ -ball about  $g(x)$  in  $Y$  and assigns to each  $x \in A$  the set  $\{f(x)\}$ . It is easily checked that  $F$  is a lower semicontinuous function from  $X$  into the collection of closed convex subsets of  $Y$ . By Proposition 16, there is a continuous  $h : X \rightarrow Y$  such that  $h(x) \in F(x)$  for every  $x \in X$ . Clearly,  $h|A = f|A$  and  $d(h, g) \leq \epsilon$ . So,  $h$  is the desired extension.  $\square$

**Lemma 18.** Let  $X$  be a separable metric space and  $Y$  be a separable Banach space. If  $f : X \rightarrow Y$  is in  $\mathcal{E}_2$ , then  $f$  satisfies condition (iii) of Theorem 7.

PROOF. For every  $x \in X$  there are open sets  $U_x^0 \subseteq X$  and  $V_x^0 \subseteq Y$  such that  $(x, f(x)) \in U_x^0 \times V_x^0$  and  $\text{osc}(f, \text{cl}(f^{-1}(U_x^0) \cap V_x^0)) \leq 1/2^0$ . Let  $\{U_n^0 \times V_n^0\}_{n \in \omega}$  be a countable subcover of  $\{U_x^0 \times V_x^0\}_{x \in X}$ . Suppose now that we have constructed countable covers  $\{\{U_n^i \times V_n^i\}_{n \in \omega}\}_{i < k}$  of  $f$  such that:

- (a)  $\text{osc}(f, \text{cl}(f^{-1}(V_n^i) \cap U_n^i)) \leq 1/2^i$  for all  $n \in \omega$ ,
- (b) for every  $i < j < k$  and  $n \in \omega$  we have  $\text{cl}(f^{-1}(V_n^j) \cap U_n^j) \subseteq f^{-1}(V_n^i) \cap U_n^i$  and  $\text{cl}(U_n^j) \subseteq U_n^i$  for some  $l \leq n$ ,

- (c)  $\text{diam}(U_n^i) < 1/2^i$  for every  $n \in \omega$ ,  
 (d) for every  $i < j < k$  we have  $\text{cl}(U_n^j) \subseteq U_n^i$ .

We show how to construct  $\{U_n^k \times V_n^k\}_{n \in \omega}$ .

Let  $m \in \omega$ . For each  $x \in U_m^{k-1} \cap f^{-1}(V_m^{k-1})$  pick open sets  $U_{x,m} \subseteq U_m^{k-1}$  and  $V_{x,m} \subseteq V_m^{k-1}$  such that:

- (i)  $(x, f(x)) \in U_{x,m} \times V_{x,m}$ ,  $\text{osc}(f, \text{cl}(f^{-1}(V_{x,m}) \cap U_{x,m})) \leq 1/2^k$ ,  
 (ii)  $\text{cl}(f^{-1}(V_{x,m}) \cap U_{x,m}) \subseteq f^{-1}(V_m^{k-1}) \cap U_m^{k-1}$  and  $\text{cl}(U_{x,m}) \subseteq U_m^{k-1}$   
 (iii)  $\text{diam}(U_{x,m}) \leq 1/2^k$ .

Notice that (i) and (ii) follow from  $f \in \mathcal{E}_2$ . Let  $\{U_{m_n} \times V_{m_n}\}_{n \in \omega}$  be a countable subcover of  $\{U_{x,m} \times V_{x,m} : x \in U_m^{k-1} \cap f^{-1}(V_m^{k-1})\}$ .

Let  $\{S_m\}_{m \in \omega}$  be a partition of  $\omega$  into infinite sets such that  $m \leq \min(S_m)$  for all  $m \in \omega$ . Let  $\{U_n^k \times V_n^k\}_{n \in \omega}$  be an enumeration (possibly with repetitions) of  $\{U_{m_n} \times V_{m_n}\}_{m,n \in \omega}$  so that  $m_n \in S_m$  for every  $n, m \in \omega$ . Now for every  $n \in \omega$  there is an  $m \in \omega$  (namely,  $m$  with  $n \in S_m$ ) such that  $m \leq n$  and  $U_n^k \times V_n^k \subseteq U_m^{k-1} \times V_m^{k-1}$  and  $\text{cl}(U_n^k) \subseteq U_m^{k-1}$ .

We show that  $\{U_n^k \times V_n^k\}_{n \in \omega}$  has the desired properties. Clearly, (a), (c), and (d) are satisfied. We show (b). Let  $i < j < k+1$  and  $n \in \omega$ . By inductive hypothesis, we may assume  $j = k$ . By the ordering of  $\{U_n^k \times V_n^k\}_{n \in \omega}$ , there is a  $m \leq n$  such that  $\text{cl}(f^{-1}(V_n^k) \cap U_n^k) \subseteq f^{-1}(V_m^{k-1}) \cap U_m^{k-1}$  and  $\text{cl}(U_n^k) \subseteq U_m^{k-1}$ . So, we are done if  $i = k-1$ . Assume that  $i < k-1$ . By inductive hypothesis, for some  $t \leq m$  we have  $\text{cl}(f^{-1}(V_m^{k-1}) \cap U_m^{k-1}) \subseteq f^{-1}(V_t^i) \cap U_t^i$  and  $\text{cl}(U_m^{k-1}) \subseteq U_t^i$ . Hence,  $\text{cl}(f^{-1}(V_n^k) \cap U_n^k) \subseteq f^{-1}(V_t^i) \cap U_t^i$ ,  $\text{cl}(U_n^k) \subseteq U_t^i$ , and  $t \leq n$ . This completes the inductive construction.

Fix  $n \in \omega$ . By Lemma 15, there is a continuous function  $f_n^0: X \rightarrow Y$  such that  $d(f_n^0(x), f(x)) \leq 1$  for all  $x \in \bigcup_{i \leq n} \text{cl}(f^{-1}(V_i^0) \cap U_i^0)$ . Suppose we have defined  $f_n^k: X \rightarrow Y$  so that for every  $x \in \bigcup_{i \leq n} \text{cl}(f^{-1}(V_i^k) \cap U_i^k)$  we have  $d(f_n^k(x), f(x)) \leq 1/2^k$  and  $d(f_n^k, f_n^{k-1}) \leq 1/2^{k-1} + 1/2^k$ . We show how to construct  $f_n^{k+1}: X \rightarrow Y$ . By (b),  $\bigcup_{i \leq n} \text{cl}(f^{-1}(V_i^{k+1}) \cap U_i^{k+1}) \subseteq \bigcup_{i \leq n} f^{-1}(V_i^k) \cap U_i^k$ . By Lemma 15, there is a continuous partial function  $f_n^{k+1}$  such that  $d(f_n^{k+1}(x), f(x)) \leq 1/2^{k+1}$  for all  $x \in \bigcup_{i \leq n} \text{cl}(f^{-1}(V_i^{k+1}) \cap U_i^{k+1})$ . Since  $\bigcup_{i \leq n} \text{cl}(U_i^{k+1}) \subseteq \bigcup_{i \leq n} U_i^k$ , we can extend  $f_n^{k+1}$  so that  $f_n^{k+1} = f_n^k$  for all  $x \notin \bigcup_{i \leq n} U_i^k$ . Notice that  $d(f_n^{k+1}(x), f_n^k(x)) \leq 1/2^k + 1/2^{k+1}$  for all  $x$  for which  $f_n^{k+1}$  has been defined. By Lemma 17, we may extend  $f_n^{k+1}$  to all of  $X$  in such a way that  $d(f_n^k, f_n^{k+1}) \leq 1/2^k + 1/2^{k+1}$  for all  $x \in X$ . Clearly,  $\{f_n^k\}_{k \in \omega}$  converges uniformly to a continuous function  $f_n$ .

For any  $k \in \omega$  and  $x \in \bigcup_{i \leq n} \text{cl}(f^{-1}(V_i^k) \cap U_i^k)$  we have

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_n^k(x)) + d(f_n^k(x), f(x)) \\ &\leq \left(\sum_{l=k}^{\infty} 1/2^l + 1/2^{l+1}\right) + 1/2^k = 1/2^{k-2}. \end{aligned}$$

Suppose  $f|P$  is continuous. Let  $x \in X$  and  $\epsilon > 0$ . Pick  $k \in \omega$  such that  $1/2^{k-2} < \epsilon$ . Since  $f|P$  is compact and  $f|P \subseteq \bigcup_{i \in \omega} U_i^k \times V_i^k$ , for all  $m \in \omega$  sufficiently large we have  $P \subseteq \bigcup_{i \leq m} \text{cl}(f^{-1}(V_i^k) \cap U_i^k)$ . By the previous paragraph,  $d(f(x), f_m(x)) \leq 1/2^{k-2} < \epsilon$  for all  $x \in P$  and large  $m$ . Thus,  $\{f_n|P\}_{n \in \omega}$  converges to  $f|P$  uniformly.  $\square$

By Theorem 3 and Lemma 18, we have the implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) in Theorem 7.

The next lemma shows that (iii)  $\Rightarrow$  (i) when  $A \in \Pi_2^0(X)$ .

**Lemma 19.** *Let  $X$  and  $Y$  be Polish  $Y$  be contained in a separable Banach space  $Y_1$  and  $A \in \Pi_2^0(X)$ . If  $f: A \rightarrow Y$  is the pointwise limit of continuous functions  $\{f_n\}_{n \in \omega}$  where  $f_n: A \rightarrow Y_1$ , then the collection  $U$  of compact sets  $P$  such that  $\{f_n|P\}_{n \in \omega}$  is uniformly convergent is a  $\Pi_3^0$ -set in  $J(X)$ . In particular, if  $f$  satisfies (iii) of Theorem 7, then  $\mathcal{C}(f) \in \Pi_3^0(J(X))$ .*

PROOF. Let  $M_n^k = \{P \in J(A): (\forall l \geq k)(\forall x \in P)(d(f_k(x), f_l(x)) \leq 1/2^n)\}$ . It is easily verified that  $M_n^k$  is closed in  $J(A)$ . We show that  $U = \bigcap_{n \in \omega} \bigcup_{k \in \omega} M_n^k$ . Suppose  $P \in U$ . For every  $n \in \omega$  there is a  $k \in \omega$  such that for all  $l \geq k$  we have  $d(f(x), f_l(x)) \leq 1/2^{n+1}$  for all  $x \in P$ . Hence,  $d(f_k(x), f_l(x)) \leq 1/2^n$  for all  $l \geq k$  and  $x \in P$ . Thus,  $P \in \bigcap_{n \in \omega} \bigcup_{k \in \omega} M_n^k$ .

Suppose  $P \in \bigcap_{n \in \omega} \bigcup_{k \in \omega} M_n^k$ . Let  $n \in \omega$ . There is a  $k \in \omega$  such that  $P \in M_n^k$ . So, for all  $x \in P$  and  $l \geq k$  we have  $d(f_k(x), f_l(x)) \leq 1/2^n$ . Hence,  $d(f(x), f_l(x)) \leq 1/2^{n-1}$  for all  $x \in P$  and  $l \geq k$ . Since  $n \in \omega$  was arbitrary, we conclude that  $P \in U$ .

Since  $U \in \Pi_3^0(J(A))$  and  $J(A) \in \Pi_2^0(J(X))$ ,  $U \in \Pi_3^0(J(X))$ .  $\square$

The next lemma shows that (iii)  $\Rightarrow$  (i) when  $f$  has  $G_\delta$ -graph in  $X \times Y$ .

**Lemma 20.** *Let  $X$  and  $Y$  be Polish spaces,  $A \subseteq X$  and  $Y$  be contained in a separable Banach space  $Y_1$ . If  $f: A \rightarrow Y$  satisfies condition (iii) of Theorem 7 and has  $G_\delta$  graph in  $X \times Y$ , then  $\mathcal{C}(f) \in \Pi_3^0(J(X))$ .*

PROOF. Let  $X^*$  and  $Y_1^*$  be metric compactifications of  $X$  and  $Y_1$  respectively. Notice that  $f$  has  $G_\delta$ -graph in  $X^* \times Y_1^*$ . Let  $\{W_i\}_{i \in \omega}$  be open subsets of  $X^* \times Y_1^*$  such that  $f = \bigcap_{i \in \omega} W_i$ . Construct covers  $\{\{U_n^i \times V_n^i\}_{n \in \omega}\}_{i \in \omega}$  of  $f$  by open rectangles of  $X^* \times Y_1^*$  such that:

- (a)  $\text{cl}(U_n^i \times V_n^i) \subseteq W_i$  for all  $i, n \in \omega$ ,
- (b)  $\text{diam}(V_n^i) < 1/2^i$  for all  $i, n \in \omega$ .

Let  $\{f_n\}_{n \in \omega}$  be continuous functions  $f_n: A \rightarrow Y_1$  which converge to  $f$  as in (iii) of Theorem 7. We may find a  $G \in \Pi_2^0(X)$  such that each  $f_n$  may be extended to a continuous function on  $G$ . For the remainder of the proof  $f_n$  will mean the extended function. For each  $i, k \in \omega$  and finite  $F \subseteq \omega$  let

$$Z_{i,k}^F = \left\{ P \in J(G) : (\forall l \geq k) \left( f_l|P \subseteq \bigcup_{n \in F} U_n^i \times V_n^i \right) \right\}.$$

For every  $i, k$ , and  $F$  we have  $Z_{i,k}^F \in \Sigma_2^0(J(G))$ . Let

$$Z = \bigcap_{i \in \omega} \bigcup_{k \in \omega} \bigcup \{ Z_{i,k}^F : F \subseteq \omega \text{ is finite} \}.$$

Clearly,  $Z \in \Pi_3^0(J(G))$ . Since  $J(G) \in \Pi_2^0(J(X))$ , we have  $Z \in \Pi_3^0(J(X))$ . We will be done if we show that  $Z = \mathcal{C}(f)$ .

Let  $P \in Z$ . We claim that  $P \subseteq A$ . Let  $x \in P$ . Since  $P \in Z$ , there exist  $k_0 \in \omega$  and  $F_0 \in [\omega]^{<\omega}$  such that for all  $l \geq k_0$  we have  $f_l(x) \in \bigcup_{n \in F_0} U_n^0 \times V_n^0$ . Let  $n_0 \in F_0$  be such that  $f_l(x) \in U_{n_0}^0 \times V_{n_0}^0$  for all  $l$  in some infinite  $B_0 \subseteq \omega$ . Since  $P \in Z$ , there exist  $k \in \omega$  and  $F_1 \in [\omega]^{<\omega}$  such that for all  $l \geq k_1$  we have  $f_l(x) \in \bigcup_{n \in F_1} U_n^1 \times V_n^1$ . Hence, there is an  $n_1 \in F_1$  such that  $f_l(x) \in U_{n_1}^1 \times V_{n_1}^1$  for all  $l$  in some infinite  $B_1 \subseteq B_0$ . We may continue inductively to find  $\{n_i\}_{i \in \omega}$  and  $B_i \in [\omega]^\omega$  such that for every  $i \in \omega$  we have  $B_{i+1} \subseteq B_i$  and  $f_l(x) \in U_{n_i}^i \times V_{n_i}^i$  for all  $l$  in  $B_i$ . For each  $i \in \omega$  pick  $l_i \in B_i$ . By (b),  $d(f_{l_i}(x), f_{l_{i+1}}(x)) \leq 1/2^i$ . Thus,  $\{f_{l_i}(x)\}_{i \in \omega}$  is a Cauchy sequence and converges to some  $y \in Y_1^*$ . Let  $j \in \omega$ . Since  $f_{l_i}(x) \in V_{n_j}^j$  for almost all  $i$ , we have  $(x, y) \in \text{cl}(U_{n_j}^j \times V_{n_j}^j) \subseteq W_j$  by (a). Since  $j$  was arbitrary,  $(x, y) \in f$ . So,  $x \in A$ . Hence,  $P \subseteq A$ .

We claim that  $f|P$  is continuous. Let  $j \in \omega$ . There is a finite  $F_j \subseteq \omega$  and a  $k \in \omega$  such that  $f_l|P \subseteq \bigcup_{n \in F_j} U_n^j \times V_n^j$  for all  $l \geq k$ . Since  $P \subseteq A$ , we have  $f|P \subseteq \bigcup_{n \in F_j} \text{cl}(U_n^j \times V_n^j) \subseteq W_j$ . Thus,

$$f|P \subseteq \bigcap_{j \in \omega} \left( \bigcup_{n \in F_j} \text{cl}(U_n^j \times V_n^j) \right) \subseteq \bigcap_{j \in \omega} W_j = f.$$

So,  $f|P$  is contained in a compact subset of  $f$  which means that  $f|P$  is continuous. Thus,  $P \in \mathcal{C}(f)$ .

Suppose  $P \in \mathcal{C}(f)$ . Let  $j \in \omega$ . Since  $f|P$  is compact, there is a finite  $F \subseteq \omega$  such that  $f|P \subseteq \bigcup_{n \in F} (U_n^j \cap A) \times V_n^j$ . Since  $\{f_n\}_{n \in \omega}$  converges to  $f$  as in (iii) of Theorem 7, we have  $\{f_n|P\}_{n \in \omega}$  converging to  $f|P$  uniformly. It

follows that for some  $k \in \omega$  we have  $f_l|P \subseteq \bigcup_{n \in F} (U_n^j \cap A) \times V_n^j$  for all  $l \geq k$ . Since  $j$  was arbitrary,  $P \in Z$ .  $\square$

**Corollary 21.** *Let  $X$  and  $Y$  be metric spaces. If  $f: X \rightarrow J(Y)$  is semicontinuous in the second sense, then  $f \in \mathcal{E}_2$ .*

PROOF. For the proof we say semicontinuous instead of semicontinuous in the second sense.

Assume that  $f$  is lower semicontinuous. Let  $\epsilon > 0$  and  $x \in X$ . There is a  $\delta > 0$  such that for every  $w \in B_\delta(x)$  and every  $q \in f(x)$  there is a  $p \in f(w)$  such that  $d(p, q) < \epsilon/2$ . Let  $M = \text{cl}(f^{-1}(B_{\epsilon/2}(f(x))) \cap B_{\delta/2}(x))$ . We will show that  $\text{osc}(f, M) \leq \epsilon$ . Let  $w \in M$ . Clearly, if  $w \in f^{-1}(B_{\epsilon/2}(f(x))) \cap B_{\delta/2}(x)$ , then  $H_d(f(x), f(w)) \leq \epsilon/2$ . Suppose  $w \in M \setminus f^{-1}(B_{\epsilon/2}(f(x))) \cap B_{\delta/2}(x)$ . Let  $w_n \in f^{-1}(B_{\epsilon/2}(f(x))) \cap B_{\delta/2}(x)$  be such that  $\lim w_n = w$ . Let  $p \in f(w)$ . There exist  $p_n \in f(w_n)$  such that  $\lim p_n = p$ . Since  $H_d(f(w_n), f(x)) \leq \epsilon/2$  for every  $n$ , we have  $d(p, f(x)) \leq \epsilon/2$ . On the other hand, since  $w \in B_\delta(x)$ , for any  $q \in f(x)$  we have  $d(q, f(w)) \leq \epsilon/2$ . Thus,  $H_d(f(w), f(x)) \leq \epsilon/2$ . We conclude that  $\text{osc}(f, M) \leq \epsilon$ .

Assume that  $f$  is upper semicontinuous. Let  $\epsilon > 0$  and  $x \in X$ . There is a  $\delta > 0$  such that for every  $w \in B_\delta(x)$  and every  $q \in f(w)$  there is a  $p \in f(x)$  such that  $d(p, q) < \epsilon/4$ . Let  $M = \text{cl}(f^{-1}(B_{\epsilon/4}(f(x))) \cap B_{\delta/2}(x))$ . We will show that  $\text{osc}(f, M) \leq \epsilon$ . Let  $w \in M$ . Clearly, if  $w \in f^{-1}(B_{\epsilon/4}(f(x))) \cap B_{\delta/2}(x)$ , then  $H_d(f(x), f(w)) \leq \epsilon/4 \leq \epsilon/2$ . Suppose  $w \in M \setminus f^{-1}(B_{\epsilon/4}(f(x))) \cap B_{\delta/2}(x)$ . Let  $w_n \in f^{-1}(B_{\epsilon/4}(f(x))) \cap B_{\delta/2}(x)$  be such that  $\lim w_n = w$ . Let  $p \in f(w)$ . Since  $d(w, x) < \delta$ , there is a  $q \in f(x)$  such that  $d(p, q) < \epsilon/4 \leq \epsilon$ . Let  $r \in f(x)$ . Since  $w_n \in f^{-1}(B_{\epsilon/4}(f(x)))$  for each  $n$ , we may find  $r_n \in f(w_n)$  such that  $\lim d(r_n, r) \leq \epsilon/4$ . Since  $w_n \in B_{\delta/2}(x)$  for all  $n$ , there are  $t_n \in f(x)$  such that  $d(t_n, r_n) < \epsilon/4$ . So,  $\lim d(t_n, r) \leq \epsilon/2$ . Since  $f(w)$  is compact, we may assume that there is a  $t \in f(w)$  such that  $\lim t_n = t$ . So,  $d(r, t) \leq \epsilon/2$ . Thus,  $H_d(f(x), f(w)) \leq \epsilon/2$ . we conclude that  $\text{osc}(f, M) \leq \epsilon$ .  $\square$

### 7 Proof of Theorem 5.

The case  $\alpha = 2$  is Theorem 4 of [1]. We now consider the case  $\alpha = 3$ . For the remainder of this section we assume that  $X$  and  $Y$  are Polish spaces and  $f$  is a partial function defined on  $X$ .

We define an operation  $M$  on collections of subsets of product spaces as in [1]. Given a collection  $\mathcal{A}$  of subsets of  $X \times Y$ . Define

$$M(\mathcal{A}) = \bigcup_{x \in X} \left( \pi_X^{-1}(\{x\}) \cap \bigcap \{A \in \mathcal{A} : x \in \pi_X[A]\} \right).$$

**Lemma 22.** *If  $\mathcal{A}$  is a finite collection of subsets of  $X \times Y$  such that  $\pi_X[A]$  is closed for every  $A \in \mathcal{A}$  and  $f|(\pi_X[A \cap f])$  is  $\mathcal{E}_2$  for each  $A \in \mathcal{A}$ , then  $f|(\pi_X[M(\mathcal{A}) \cap f])$  is  $\mathcal{E}_2$ .*

PROOF. Let  $x \in \pi_X[M(\mathcal{A}) \cap f]$  and  $\epsilon > 0$ . Since  $\mathcal{A}$  is finite and  $f|(\pi_X[A \cap f])$  is  $\mathcal{E}_2$  for each  $A \in \mathcal{A}$ , there is an open neighborhood  $V$  of  $f(x)$  and a open neighborhood  $U$  of  $x$  such that  $\text{osc}(f, \text{cl}(f^{-1}(V) \cap U) \cap \pi_X[A]) \leq \epsilon$  for any  $A \in \mathcal{A}$  such that  $x \in \pi_X[A \cap f]$ . In particular, we have

$$\text{osc}(f|((\text{cl}(f^{-1}(V) \cap U) \cap \pi_X[M(\mathcal{A}) \cap f])) \leq \epsilon. \quad \square$$

**Lemma 23.** *Let  $f \in \mathcal{F}_3$  and  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be countable bases for  $X$  and  $Y$ , respectively. If  $x \in A \subseteq X$  and  $f|A$  is continuous, then there exist  $B_1 \in \mathcal{B}_1$  and  $B_2 \in \mathcal{B}_2$  such that  $f|(f^{-1}(\text{cl}(B_2)) \cap \text{cl}(B_1))$  is  $\mathcal{E}_2$  and  $f[A \cap \text{cl}(B_1)] \subseteq \text{cl}(B_2)$ .*

PROOF. Since  $f \in \mathcal{F}_3$ , there exist open sets  $U \subseteq X$  and  $V \subseteq Y$  such that  $x \in U$ ,  $f(x) \in V$ , and  $f|(f^{-1}(V) \cap U)$  is  $\mathcal{E}_2$ . Pick  $B_1 \in \mathcal{B}_1$  and  $B_2 \in \mathcal{B}_2$  so that  $\text{cl}(B_1) \subseteq U$ ,  $\text{cl}(B_2) \subseteq V$ ,  $x \in B_1$ , and  $f(x) \in B_2$ . Since  $f^{-1}(\text{cl}(B_2)) \cap \text{cl}(B_1) \subseteq f^{-1}(V) \cap U$ , we have that  $f|(f^{-1}(\text{cl}(B_2)) \cap \text{cl}(B_1))$  is  $\mathcal{E}_2$ . Since  $f|A$  is continuous, we may assume  $B_1$  is small enough that  $f[A \cap \text{cl}(B_1)] \subseteq \text{cl}(B_2)$ .  $\square$

**Proposition 24.** [1, Lemma 13] *If  $\mathcal{A}$  is a finite collection of closed subsets of  $X \times Y$  such that  $\pi_X[A]$  is closed for every  $A \in \mathcal{A}$ , then  $M(\mathcal{A}) \in \Pi_2^0(X \times Y)$ .*

**Lemma 25.** *If  $f \in \mathcal{F}_3$  and  $f$  has  $G_\delta$ -graph in  $X \times Y$ , then  $\mathcal{C}(f) \in \Sigma_4^0(J(X))$ .*

PROOF. Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be countable bases for  $X$  and  $Y$  respectively. Let  $\mathcal{W}$  be the collection of all finite collections  $Z = \{W_0, \dots, W_n\}$  of sets of the form  $W_i = \text{cl}(B_1) \times \text{cl}(B_2)$  (where  $B_1 \in \mathcal{B}_1$  and  $B_2 \in \mathcal{B}_2$ ) such that  $f|(\pi_X[M(Z) \cap f])$  is  $\mathcal{E}_2$ . Let  $Z \in \mathcal{W}$ . By Proposition 24 and the assumption that  $f$  has  $G_\delta$ -graph in  $X \times Y$ ,  $M(Z) \cap f \in \Pi_2^0(X \times Y)$ . Since  $f|(\pi_X[M(Z) \cap f])$  is  $\mathcal{E}_2$  and has  $G_\delta$ -graph in  $X \times Y$ , Theorem 7 implies that  $\mathcal{T} = \bigcup \{\mathcal{C}(f|(\pi_X[M(Z) \cap f])) : Z \in \mathcal{W}\} \in \Sigma_4^0(X)$ .

The proof will be complete if we show that  $\mathcal{C}(f) = \mathcal{T}$ . The containment  $\mathcal{T} \subseteq \mathcal{C}(f)$  is obvious. We work for the opposite containment. Let  $C \in \mathcal{C}(f)$ . We will construct a finite collection  $W = \{W_1, W_2, \dots, W_n\}$  of sets of the form  $W_i = \text{cl}(B_1) \times \text{cl}(B_2)$  where  $B_1 \in \mathcal{B}_1$  and  $B_2 \in \mathcal{B}_2$  such that:

- (a)  $f|C \subseteq \bigcup W$ ,
- (b)  $f|(\pi_X[f \cap W_i])$  is  $\mathcal{E}_2$  for every  $1 \leq i \leq n$ , and
- (c)  $f|(C \cap \pi_X(W_i)) \subseteq W_i$  for every  $1 \leq i \leq n$ .

By Lemma 23, for every  $x \in C$  there exist  $B_1^x \in \mathcal{B}_1$  and  $B_2^x \in \mathcal{B}_2$  such that  $x \in B_1^x$ ,  $f(x) \in B_2^x$ ,  $f|(f^{-1}(\text{cl}(B_2^x)) \cap \text{cl}(B_1^x))$  is  $\mathcal{E}_2$ , and  $f[\text{cl}(B_1^x) \cap C] \subseteq \text{cl}(B_2^x)$ . Since  $f|C$  is compact, we we may find a finite subcover  $W^* = \{W_1^*, \dots, W_n^*\}$

of  $\{B_1^x \times B_2^x : x \in C\}$ . For each  $1 \leq i \leq n$  let  $W_i = \text{cl}(W_i^*)$ . The collection  $W = \{W_1 \dots W_n\}$  clearly satisfies conditions (a), (b), and (c).

By (b), and Lemma 22,  $f|(\pi_X[f \cap M(W)])$  is  $\mathcal{E}_2$ . So  $W \in \mathcal{W}$ . Let  $x \in C$ . By (a), there is some  $W_i \in W$  such that  $\langle x, f(x) \rangle \in W_i$ . By (c), for any  $W_k \in W$  if  $x \in \pi_X(W_k)$ , then  $\langle x, f(x) \rangle \in W_k$ . Thus,  $\langle x, f(x) \rangle \in M(W) \cap f$ . So,  $x \in \pi_X[M(W) \cap f]$ . So,  $C \subseteq \pi_X[M(W) \cap f]$ . Since  $f|C$  is continuous,  $C \in \mathcal{C}(\pi_X[M(W) \cap f]) \subseteq \mathcal{T}$ .  $\square$

The theorem now follows from Theorem 4 and Lemma 25.

## 8 Proof of Theorem 6.

**Lemma 26.** *Let  $f \in \mathcal{E}_3$ ,  $\epsilon > 0$  and  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be countable bases for  $X$  and  $Y$ , respectively. If  $x \in A \subseteq X$  and  $f|A$  is continuous, then there exist  $B_1 \in \mathcal{B}_1$  and  $B_2 \in \mathcal{B}_2$  such that there is a  $G \in \Pi_2^0(X)$  such that  $f^{-1}(\text{cl}(B_2)) \cap \text{cl}(B_1) \subseteq G$  and  $\text{osc}(f, G) \leq \epsilon$  and  $f[A \cap \text{cl}(B_1)] \subseteq \text{cl}(B_2)$ .*

PROOF. Since  $f \in \mathcal{E}_3$ , there exist open sets  $U \subseteq X$  and  $V \subseteq Y$  such that  $x \in U$ ,  $f(x) \in V$ , and there is a  $G \in \Pi_2^0(X)$  such that  $f^{-1}(V) \cap U \subseteq G$  and  $\text{osc}(f, G) \leq \epsilon$ . Pick  $B_1 \in \mathcal{B}_1$  and  $B_2 \in \mathcal{B}_2$  so that  $\text{cl}(B_1) \subseteq U$ ,  $\text{cl}(B_2) \subseteq V$ ,  $x \in B_1$ , and  $f(x) \in B_2$ . Now  $f^{-1}(\text{cl}(B_2)) \cap \text{cl}(B_1) \subseteq f^{-1}(V) \cap U \subseteq G$ , and  $\text{osc}(f, G) \leq \epsilon$ . Since  $f|A$  is continuous, we may assume  $B_1$  is small enough that  $f[A \cap \text{cl}(B_1)] \subseteq \text{cl}(B_2)$ .  $\square$

**Lemma 27.** *Suppose  $f : X \rightarrow Y$ . Let  $\epsilon > 0$  and  $\mathcal{A}$  be a finite collection of closed boxes in  $X \times Y$  such that for every  $A \in \mathcal{A}$ :  $\text{diam}(\pi_Y[A]) \leq \epsilon/5$ , there is a  $G_A \in \Pi_2^0(X)$  such that  $\pi_X[A \cap f]$  is subset of  $G_A$  and  $\text{osc}(f, G_A) \leq \epsilon/5$ . There is a  $G \in \Pi_2^0(X)$  such that  $\pi_X[M(\mathcal{A}) \cap f] \subseteq G$  and  $f|G$  is  $\epsilon$ -continuous.*

PROOF. Let  $G_1 = \bigcup_{A \in \mathcal{A}} (G_A \cap \text{cl}(\pi_X[A \cap f]))$ . Clearly,  $G_1 \in \Pi_2^0(X)$  and  $\pi_X[M(\mathcal{A}) \cap f] \subseteq \bigcup_{A \in \mathcal{A}} \pi_X[A \cap f] \subseteq G_1 \subseteq \bigcup_{A \in \mathcal{A}} \pi_X[A]$ . Let

$$G = G_1 \setminus \left( \bigcup \{ \pi_X[A] \cap \pi_X[B] : A, B \in \mathcal{A} \text{ and } \pi_Y[A] \cap \pi_Y[B] = \emptyset \} \right).$$

Now  $G \in \Pi_2^0(X)$  and  $\pi_X[M(\mathcal{A}) \cap f] \subseteq G \subseteq \pi_X[M(\mathcal{A})]$ . We now show that  $f|G$  is  $\epsilon$ -continuous. Let  $x \in G$  and  $x_n \in G$  be such that  $\lim_{n \in \omega} x_n = x$ . Pick  $A \in \mathcal{A}$  and  $B \in \mathcal{A}$  such that  $x \in G_A \cap \pi_X[A]$  and  $\{x_n : n \in \omega\} \subseteq G_B \cap \pi_X[B]$ . Since  $\pi_X[B]$  is closed,  $x \in \pi_X[B]$ . Since  $x \in \pi_X[M(\mathcal{A})]$ , we have  $A \cap B \neq \emptyset$ . Since  $\text{diam}(\pi_Y[A] \cup \pi_Y[B]) < 2\epsilon/5$  and  $\max\{\text{osc}(f, G_B), \text{osc}(f, G_A)\} < \epsilon/5$ , we have  $\limsup_{n \in \omega} d(f(x_n), f(x)) < 4\epsilon/5$ . Thus,  $f|G$  is  $\epsilon$ -continuous.  $\square$

**Lemma 28.** *If  $f$  is  $\mathcal{E}_3$ , then  $\mathcal{C}(f) \in \Pi_4^0(\mathcal{J}(X))$ .*

PROOF. Let  $n \in \omega$ . Let  $\mathcal{B}_1 = \{B_1^k : k \in \omega\}$  and  $\mathcal{B}_2 = \{B_2^k : k \in \omega\}$  be countable bases for  $X$  and  $Y$  respectively. Let  $\mathcal{W}$  be the collection of all finite collections  $Z = \{W_0, \dots, W_m\}$  of sets of the form  $W_k = \text{cl}(B_1^k) \times \text{cl}(B_2^k)$  such that  $\text{diam}(B_2^k) \leq 1/(2^n \cdot 5)$  and there is a  $G_k \in \Pi_2^0(X)$  such that  $\pi_X[f \cap (\text{cl}(B_1^k) \times \text{cl}(B_2^k))] \subseteq G_k$  and  $\text{osc}(f, G_k) \leq 1/(2^n \cdot 5)$ . Let  $Z \in \mathcal{W}$ . By Lemma 27, there is a  $H_Z \in \Pi_2^0(X)$  such that  $\pi_X[M(Z) \cap f] \subseteq H_Z$  and  $f|_{H_Z}$  is  $1/2^n$ -continuous. Let  $\mathcal{T}_n = \bigcup_{Z \in \mathcal{W}} J(H_Z)$ . Clearly,  $\mathcal{T}_n \in \Sigma_3^0(J(X))$  and  $f|_C$  is  $1/2^n$ -continuous for all  $C \in \mathcal{T}_n$ . The set  $\mathcal{T} = \bigcap_{n \in \omega} \mathcal{T}_n$  is in  $\Pi_4^0(J(X))$ . Since  $C \in \mathcal{T}$  implies that  $f|_C$  is  $1/2^n$  continuous for all  $n$ , we have  $\mathcal{T} \subseteq \mathcal{C}(f)$ .

Let  $C \in \mathcal{C}(f)$  and  $n \in \omega$  be arbitrary. We will construct a finite collection  $W = \{W_1, W_2, \dots, W_m\}$  of sets of the form  $W_i = \text{cl}(B_1^i) \times \text{cl}(B_2^i)$  such that

- (a)  $f|_C \subseteq \bigcup W$ ,
- (b) there is a  $G_i \in \Pi_2^0(X)$  such that  $\pi_X[f \cap W_i] \subseteq G_i$  and  $\text{osc}(f, G_i) < 1/(2^n \cdot 5)$ ,
- (c)  $\text{diam}(B_2^i) < 1/(2^n \cdot 5)$ , and
- (d)  $f|(C \cap \pi_X(W_i)) \subseteq W_i$  for every  $1 \leq i \leq m$ .

By Lemma 26, for every  $x \in C$  there exist  $B_1^x \in \mathcal{B}_1$  and  $B_2^x \in \mathcal{B}_2$  such that  $x \in B_1^x$ ,  $f(x) \in B_2^x$ ,  $\text{diam}(B_2^x) < 1/(2^n \cdot 5)$ ,  $f[\text{cl}(B_1^x) \cap C] \subseteq \text{cl}(B_2^x)$ , and there is a  $G \in \Pi_2^0(X)$  such that  $f^{-1}(\text{cl}(B_2^x)) \cap \text{cl}(B_1^x) \subseteq G$  and  $\text{osc}(f, G) < 1/(2^n \cdot 5)$ . Since  $f|_C$  is compact, we may find a finite subcover  $W^* = \{W_1^*, \dots, W_m^*\}$  of  $\{B_1^x \times B_2^x : x \in C\}$ . For each  $1 \leq i \leq m$  let  $W_i = \text{cl}(W_i^*)$ . The collection  $W = \{W_1 \dots W_m\}$  clearly satisfies conditions (a), (b), (c), and (d).

By (b) and (c),  $W \in \mathcal{W}$ . Let  $x \in C$ . By (a), there is some  $W_i \in W$  such that  $\langle x, f(x) \rangle \in W_i$ . By (d), for any  $W_k \in W$  if  $x \in \pi_X(W_k)$ , then  $\langle x, f(x) \rangle \in W_k$ . Thus,  $\langle x, f(x) \rangle \in M(W) \cap f$ . So,  $x \in \pi_X[M(W) \cap f]$ . Therefore,  $C \subseteq \pi_X[M(W) \cap f]$ . Thus,  $\mathcal{C}(f) \subseteq \mathcal{T}_n$ .

It now follows that  $\mathcal{C}(f) \subseteq \mathcal{T}$ . Thus  $\mathcal{C}(f) \in \Pi_4^0(J(X))$ .  $\square$

The theorem now follows from Lemma 28 and Theorem 3.

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